Nonlinear current-voltage characteristic of a metallic film located in an external magnetic field

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A new phenomenon—the magnetodynamic nonlinearity of the low-temperature static current—voltage characteristic (CVC) of a thin metallic plate—is predicted. An asymptotically exact non-linear-CVC theory is constructed with allowance made for an external magnetic field oriented parallel to the plane of the sample and perpendicularly to the direction of the current. The mechanism underlying the nonlinearity is connected with the trapping of electrons in the potential well produced by the intrinsic magnetic field of the current. This mechanism is unique among the known nonoverheating nonlinear mechanisms obtaining in metals. Numerical computations indicate the feasibility of experimental detection of the investigated effects.

I. INTRODUCTION

As is well known,[1] the static electrical conductivity of a thin metallic film with diffuse boundaries is described in the linear regime by the expression

$$\sigma_l = \frac{3}{4}\sigma_0 \frac{d}{l} \ln\frac{l}{d}, \qquad (1.1)$$

where σ_0 is the conductivity of the bulk sample, d is the plate thickness, and l is the mean free path of the electrons. The formula (1.1) is valid in the Knudsen limiting case $d \ll l$. The conductivity σ_l is due to a small group of drifting electrons that move almost parallel to the plate surface, and do not collide with the boundaries during the entire mean free time. The relative number of drifting electrons is, in order of magnitude, equal to d/l.

At low temperatures the dominant nonlinear effect involved in the conduction in the metal is connected with the influence of the intrinsic magnetic field of the current on the motion of the conduction electrons. The constant electric current I flowing along the film produces within it a constant magnetic field that is nonuniform over the thickness. It is asymmetrically distributed, i.e., it is equal to zero in the middle of the plate, and assumes at the opposite surfaces values H and $-H$ that are equal in magnitude, but opposite in sign, with

$$H = 2\pi I / cD, \qquad (1.2)$$

where D is the horizontal dimension of the film in the direction perpendicular to the current and c is the velocity of light.

It must be emphasized that even a relatively weak intrinsic magnetic field of the current, i.e., one satisfying the conditions

$$d \ll (Rd)^{1/2} \ll l, \qquad (1.3)$$

alters appreciably the conductivity and the shape of the current-voltage characteristic (CVC) of the sample (here R is the characteristic radius of curvature of the electron trajectory in the field H). Owing to the fact that the magnetic field inside the plate is sign-variable, there appears a group of so-called trapped electrons that move without colliding with the boundaries along trajectories winding in the vicinity of the plane of inversion of the sign of the magnetic field. The relative number of such trapped electrons is, in order of magnitude, equal to $(d/R)^{1/2}$, and their conductivity in the region (1.3) of variation of the parameters of the problem is characterized by the formula

$$\sigma_{tr} \sim \sigma_0 (d/R)^{1/2} \propto I^{1/2}. \qquad (1.4)$$

It is clear from a comparison of (1.1) and (1.4) that, when the inequalities (1.3) are satisfied, the voltage drop across the sample is determined by the conductivity of the trapped electrons. As a result, the influence of the intrinsic magnetic field of the current gives rise to a nonlinear current-voltage characteristic: the voltage drop V is proportional to the square root $I^{1/2}$ of the current. Let us point out that, for the conditions (1.3) to be fulfilled, it is not necessary that the characteristic radius R of curvature in the intrinsic magnetic field of the current be small compared to the mean free path l of the electrons.

In an external magnetic field $\mathbf{h}_0$ parallel to the plate and perpendicular to the direction of the current, the plane of inversion of the sign of the resultant magnetic field in the sample shifts toward one of the surfaces, and disappears entirely when $h_0 > H$. Thus, the external field h_0 alters the shape of the nonlinear CVC. In particular, we should expect a sharp change in the slope of the current-voltage characteristic in the vicinity of that current value at which $H = h_0$. The proposed mechanism of the deviation from Ohm's law is, as far as we know, the only CVC-nonlinearity mechanism of the nonoverheating type in metals.

In the present paper we construct an asymptotically exact theory of the current-voltage characteristic of a thin metallic film, and analyze the shape of the characteristic as a function of the external magnetic field h_0. In the second section we formulate the problem, investigate the dynamics of the electrons in the nonuniform magnetic field, and give general formulas, obtained from the kinetic Boltzmann equation, for the distributions of the trapped- and untrapped-electron current densities. In the third section we find an

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asymptotic expression for the current under the conditions (1.3), and obtain an explicit solution, which gives the magnetic-field distribution in the sample and the shape of the CVC, to the corresponding magnetostatics problem. In the fourth, concluding section we discuss the dependence of the voltage V on the current I for different values of the external field h_0, as well as the h_0 dependence of V at a fixed current strength I. The distinctive features of the CVC are due to the competition between the contributions from the drifting and trapped electrons to the current. Numerical estimates demonstrate the feasibility of experimental detection of the predicted nonlinear effects.

2. FORMULATION OF THE PROBLEM, ELECTRON DYNAMICS, AND CURRENT DENSITY

1. Let us consider a metallic film of thickness d along the y axis of which flows a current I. Let us orient the x axis perpendicularly to the plate surfaces, and let us choose the $x = 0$ plane in the middle of the plate. A constant and uniform external magnetic field $\mathbf{h}_0$ is applied along the z axis (Fig. 1). The intrinsic magnetic field $\mathbf{H}(x)$ of the current is parallel to $\mathbf{h}_0$ (the z axis). Then the strength of the resultant magnetic field in the sample is

$$\mathscr{H}(x)=h_0+H(x). \tag{2.1}$$

The equation of magnetostatics has the form

$$-\frac{d\mathscr{H}}{dx}=\frac{4\pi}{c}j(x), \tag{2.2}$$

where $j(x)$ is the current density. The boundary conditions to Eq. (2.2) are as follows:

$$\mathscr{H}(d/2)=h_0-H, \quad \mathscr{H}(-d/2)=h_0+H. \tag{2.3}$$

The connection between the strength H of the intrinsic magnetic field at the surface and the total current I is given by the expression (1.2).

To find the dependence of the current density $j(x)$ on the electric field E, we must solve the kinetic Boltzmann equation. It follows from the Maxwell equation curl $\mathbf{E} = 0$ that the field $E_y = E$ inside the sample is uniform. As to the transverse component of the electric field $E_x(x)$, it can be neglected in the current $j(x)$ expressed in terms of the small parameters d/R and d/l. In this respect, the situation is similar to the anomalous skin effect, in which the role of d is played by the skin depth δ. The requirement of diffuse electron scattering at both plate boundaries serves as the boundary conditions to the kinetic equation.

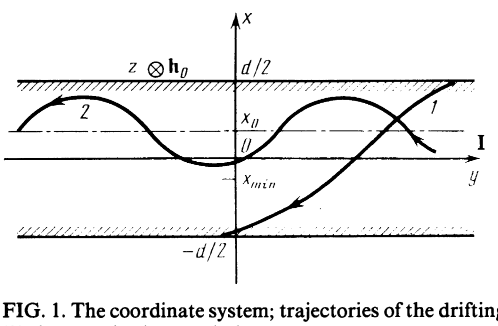

FIG. 1. The coordinate system; trajectories of the drifting (1) and trapped (2) electrons in the metal plate.

The kinetic equation can be linearized in the electric field E, and the nonlinearity is due to the magnetic field $\mathscr{H}(x)$, being contained in the Lorentz force, which governs the dynamics of the electron motion. This means that the relation between $j(x)$ and E is linear, and that the conductivity $\sigma(x)$ does not depend on the field E and is a functional of the resultant magnetic field $\mathscr{H}(x)$. In other words, the nonlinearity manifests itself in the presence of a complicated functional dependence of the conductivity on the current density. This magnetodynamic nonlinearity plays the dominant role if the force eE_x exerted by the electric field is weak compared to the x component of the Lorentz force $ev\mathscr{H}/c$ ($-e$ is the charge and v is the electron Fermi velocity). Assuming $E_x \lesssim E$, and estimating the magnetic field from Eq. (2.2) to be $\mathscr{H} \sim 4\pi\sigma_{tr}Ed/c$, we obtain the inequality

$$\frac{c^2}{4\pi\sigma_0 d^2}\frac{(Rd)^{1/2}}{v} \ll 1, \tag{2.4}$$

which, together with the requirement (1.3), determines the region of existence of the magnetodynamic nonlinearity. The condition (2.4) places a lower bound on the sample thickness, but it is easily fulfilled in a good conductor if $d \gtrsim (c^2/4\pi\sigma_0\nu)^{1/2} \approx 10^{-5}$ cm (ν is the electron-relaxation rate). In this case the right inequality in (1.3) guarantees the fulfillment of (2.4).

2. Let us proceed to discuss the electron dynamics in the nonuniform magnetic field $\mathscr{H}(x) = h_0 + H(x)$. Let us choose the gauge of the vector potential of the resultant field in the form

$$\mathbf{A}=\{0, A(x), 0\}, \quad A(x)=\int^{x} dx'\mathscr{H}(x'). \tag{2.5}$$

It is convenient to choose the lower limit of integration in (2.5) differently, depending on whether or not there exists inside the plate a plane $x = x_0$ at which the magnetic field $\mathscr{H}(x)$ reverses sign ($\mathscr{H}(x_0) = 0$ at $0 \leqslant x_0 \leqslant d/2$). Such a plane exists when $h_0 < H$ (see (2.3)), and, as the lower limit, we shall choose x_0. In this case the vector potential $A(x)$ in the sample is negative, and attains its maximum value, equal to zero, at the point $x = x_0$, i.e., $A(x_0) = 0$. If, on the other hand, the field $\mathscr{H}(x)$ is of constant sign ($h_0 > H$), then we set the lower limit equal to $d/2$: the vector potential, being also negative, vanishes at the upper plate boundary, i.e., $A(d/2) = 0$.

The integrals of the electron motion in the field $\mathscr{H}(x)$ are the total energy (which we shall assume to be the Fermi energy) and the generalized momenta $p_z = mv_z$ and $p_y = mv_y - eA(x)/c$ (m is the electron mass). The electron trajectory in the plane perpendicular to the magnetic field is determined by the velocities $v_x(x)$ and $v_y(x)$. For a Fermi sphere of radius $p_F = mv$ we have

$$|v_x(x)|=(v_\perp^2-v_y^2)^{1/2}, \quad v_\perp=(v^2-v_z^2)^{1/2},$$
$$v_y(x)=(p_y+eA(x)/c)/m. \tag{2.6}$$

The classically admissible regions for the electron motion along the x axis can be found from the inequalities

$$-p_y-mv_\perp \leqslant eA(x)/c \leqslant -p_y+mv_\perp. \tag{2.7}$$

They insure the positiveness of the radicand in the formula (2.6) for $|v_x(x)|$.

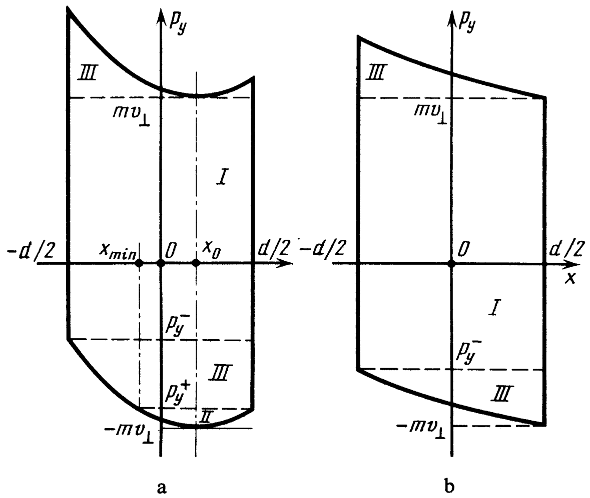

FIG. 2. The (x, p_y) phase plane and the regions of existence of the various electron groups in a) a sign-variable $(h_0 < H)$ resultant magnetic field and b) a resultant magnetic field of constant sign $(H < h_0)$: I) the region of existence of the untrapped electrons, II) that of the trapped electrons, and III) that of the surface electrons.

Figure 2 shows schematically of the regions of motion of an electron in the (x, p_y) phase plane in two cases: when there exists inside the plate a plane $x = x_0$ of inversion of the sign of the field $\mathscr{H}(x)$ (Fig. 2a) and when there is no such plane (Fig. 2b). The upper boundary for the phase plane is described by the curve $p_y = mv_\perp - eA(x)/c$; the lower boundary, by the curve $p_y = -mv_\perp - eA(x)/c$. It can be seen from the figure that the electrons naturally split up into groups according to the magnitude and sign of the integral p_y of the motion.

a) Untrapped electrons

$$p_y^- \equiv -mv_\perp - eA(-d/2)/c \leqslant p_y \leqslant mv_\perp, \quad |x| \leqslant d/2. \tag{2.8}$$

These electrons collide during their motion with both surfaces of the plate. Their trajectories are almost not bent by the magnetic field because of the fact that $d \ll R$. There are always untrapped electrons, whether or not the metal contains the plane $x = x_0$ (i.e., no matter what the relation between h_0 and H is).

b) Trapped electrons

They occur only in the case when $h_0 < H$ and the magnetic field $\mathscr{H}(x)$ inside the sample passes through zero. Their states are confined to the region (see Fig. 2a)

$$-mv_\perp \leqslant p_y \leqslant p_y^+ \equiv -mv_\perp - eA(d/2)/c, \quad x_{min} \leqslant x \leqslant d/2. \tag{2.9}$$

The coordinate $x_{\min}$ of the lower boundary of the region of existence of the trapped electrons can be found from the equation $A(x_{\min}) = A(d/2)$.

The trajectories of the trapped electrons are almost two-dimensional oscillating curves because of the periodic motion in the direction of the x axis and the uniform motion along the y and z axes. The period of the oscillations about the $x = x_0$ plane is equal to $2T(p_y)$, where

$$T = \int_{x_1}^{x_2} \frac{dx}{|v_x(x)|}. \tag{2.10}$$

The turning points x_1 and x_2 $(x_1 < x_0 < x_2)$ are the roots of the equation

$$eA(x_{1,2})/c = -mv_\perp - p_y. \tag{2.11}$$

In that part of the plate where the trapped electrons occur, i.e., in the region $x_{\min} \leqslant x \leqslant d/2$, the vector potential $A(x)$ is an even function of $x - x_0$. Therefore,

$$x_1 + x_2 = 2x_0, \quad x_{min} = 2x_0 - d/2. \tag{2.12}$$

c) Surface electrons

They undergo collisions with only one of the boundaries of the film. Their contribution to the conductivity of the metal determines the insignificant numerical constant, which is of the order of unity, in the argument of the logarithm contained in the asymptotic form of the expression for the conductivity due to the drifting electrons. Therefore, below we shall, in computing the current density, neglect the surface electrons.

3. The current density for the trapped and untrapped particles is quite easily derived with the aid of the standard methods of solving the kinetic Boltzmann equation. Let us omit these computations, and give the result:

$$j_{\text{untr}}(x) = \frac{3\sigma_0}{2\pi l} E \int_0^v \frac{dv_z}{v} \int_{p_y^-}^{mv_\perp} \frac{dp_y}{p_F} \frac{v_y(x)}{|v_x(x)|} \int_{-d/2}^{d/2} dx' \frac{v_y(x')}{|v_x(x')|} \times \exp(-\nu|\tau(x, x')|), \tag{2.13}$$

$$j_{\text{tr}}(x) = \frac{3\sigma_0}{\pi l} E \int_0^v \frac{dv_z}{v} \int_{-mv_\perp - (e/c)A(x)}^{p_y^+} \frac{dp_y}{p_F} \frac{v_y(x)}{|v_x(x)|} \times \left\{ \frac{\operatorname{ch} \nu\tau(x_1, x)}{\operatorname{sh} \nu T} \int_{x_1}^{x_2} dx' \frac{v_y(x')}{|v_x(x')|} \operatorname{ch} \nu\tau(x', x_2) + \int_{x_1}^{x} dx' \frac{v_y(x')}{|v_x(x')|} \operatorname{sh} \nu\tau(x, x') \right\} \theta(x - x_{min}). \tag{2.14}$$

In these formulas

$$\tau(x, x') = \int_x^{x'} \frac{dx''}{|v_x(x'')|} \tag{2.15}$$

is the time it takes an electron to move from the point x to the point x' and $\theta(x)$ is the unit step function, equal to unity when $x > 0$ and zero when $x < 0$.

3. ASYMPTOTIC EXPRESSIONS FOR THE CURRENT DENSITIES; SOLUTION OF THE MAGNETOSTATIC EQUATION

1. The expressions for the current densities of the untrapped and trapped electrons are simpler when the inequalities (1.3), in which the quantity R is determined by the characteristic value of the resultant magnetic field in the plate, are satisfied. The right-hand side of (1.3) corresponds to the condition $v|\tau(x, x')| \lesssim (Rd)^{1/2}/l \ll 1$, and therefore we can limit ourselves in the formulas (2.13) and (2.14) to the consideration of the leading approximation in $v\tau$. Further, on account of the left inequality in (1.3), we have $e|A(x)|/$

$mv_\perp c \lesssim d/R \ll 1$. This means that the dominant contribution to the p_y integrals in (2.13) and (2.14) is made by the neighborhoods of those points at which the velocity $|v_x(x)|$ is small. These points are the ends of the integration intervals, i.e., the ends of the regions of existence of the individual groups of electrons.

Since the trajectories of the untrapped electrons are only slightly bent by the magnetic field, we can ignore this bending, setting $p_y^- = -mv_\perp$ and $v_y = p_y/m$ in the expression (2.13). But after this the p_y integral in (2.13) diverges logarithmically at the limits of integration. Its evaluation with logarithmic accuracy leads to the following asymptotic expression for the current density of the drifting electrons:

$$j_{untr} = \frac{3}{8}\sigma_0 E \frac{d}{l} \ln\frac{R_+}{d}, \qquad R_\pm = \frac{cp_F}{e|h_0 \pm H|}. \tag{3.1}$$

Notice that the asymptotic form (3.1) can be obtained from (1.1) by replacing the mean free path l under the logarithm sign by the effective mean free path $(R_+ d)^{1/2}$ of the untrapped electrons. The current density (3.1) does not depend on l, and is determined by the electron scattering at the sample boundaries.

To find the asymptotic form of the current density $j_{tr}(x)$ of the trapped electrons, we can neglect the second term in the curly brackets in (2.14), replace the product $v_y(x)v_y(x')$ by $v_\perp^2$, and expand $|v_x(x)|$ about $p_y = -mv_\perp$, using the fact that $e|A(x)|/c$ is small compared to $mv_\perp$. After this the integrations are elementary, and, carrying them out, we obtain

$$j_{tr}(x) = \frac{36\pi^{1/2}}{5\Gamma^2(1/4)} \sigma_0 E \left\{\frac{e}{cp_F}\left[A(x) - A\left(\frac{d}{2}\right)\right]\right\}^{1/2} \times \theta(x - x_{min}). \tag{3.2}$$

It is remarkable that the relation between $j_{tr}(x)$ and the vector potential of the magnetic field is local in terms of the coordinate x. This circumstance enables us to obtain an analytic solution to the magnetostatics problem. The current (3.2) due to the trapped electrons is proportional to their mean free path. On account of the fact that the states of these electrons at the Fermi surface are located in a narrow band of relative width $(d/R)^{1/2} \ll 1$, their relaxation rate ν is determined not by the transport, but by the total, scattering cross section, i.e., only the "departure" term in the collision integral operates.

2. Let us proceed to solve the magnetostatic equation (2.2) with the currents (3.1) and (3.2) and the boundary conditions (2.3).

Let us begin with the $h_0 < H$ case, when there exists in the film a plane $x = x_0$ of inversion of the sign of the resultant magnetic field $\mathscr{H}(x)$. Here we must take into account not only the large current due to the trapped electrons, but also the small—with respect to the parameter $(R_\pm d)^{1/2}/l \ll 1$—current due to the drifting particles. If we neglect the current (3.1), then we cannot satisfy the boundary conditions (2.3).

Let us introduce the dimensionless vector potential and transverse coordinate

$$a(2x/d) = A(x)/A(d/2), \qquad \xi = 2x/d. \tag{3.3}$$

In the dimensionless quantities the magnetostatic equation (2.2) has the following form:

$$a''(\xi) = \frac{3\alpha}{4}\left\{[1 - a(\xi)]^{1/2}\theta(\xi - \xi_{min}) + \frac{4\beta}{3}\right\}. \tag{3.4}$$

It should be solved in the region $-1 \leqslant \xi \leqslant 1$ with the boundary conditions

$$a'(1) = \frac{d}{2}\frac{h_0 - H}{A(d/2)}, \qquad a'(-1) = \frac{d}{2}\frac{h_0 + H}{A(d/2)}, \qquad a(1) = 1. \tag{3.5}$$

Here the prime denotes differentiation with respect to ξ, the quantity $\xi_{min} = 2x_{min}/d$ is the dimensionless coordinate of the lower boundary of the region of existence of the trapped particles, $a(\xi_{min}) = 1$,

$$\alpha = \frac{48\pi^{3/2}}{5\Gamma^2(1/4)} \frac{\sigma_0 E d^2}{c|A(d/2)|^{1/2}} \left(\frac{e}{cp_F}\right)^{1/2},$$
$$\beta = \frac{5\Gamma^2(1/4)}{64}\left[\frac{cp_F d^2}{4\pi e|A(d/2)|}\right]^{1/2} \frac{\ln(R_+/d)}{l}. \tag{3.6}$$

The parameter β is, in order magnitude, equal to the ratio of the untrapped-electron current to the trapped-electron current.

In the interval $\xi_{min} \leqslant \xi \leqslant 1$ the solution to Eq. (3.4) is symmetric about the point $\xi_0 = (1 + \xi_{min})/2$ at which the "vector potential" $a(\xi)$ has its minimum value, which is equal to zero: $a(\xi_0) = a'(\xi_0) = 0$. It is given by the following formula:

$$|\xi - \xi_0| = \alpha^{-1/2} \int_0^{a(\xi)} dt[1 - (1-t)^{3/2} + 2\beta t]^{-1/2}. \tag{3.7}$$

This relation implicitly gives $a(\xi)$.

In the remaining part of the plate, where we have only drifting electrons, i.e., in the region $-1 \leqslant \xi \leqslant \xi_{min}$, the solution to Eq. (3.4) is given by the expression

$$a(\xi) = 1 - \alpha^{1/2}(1 + 2\beta)^{1/2}(\xi - \xi_{min}) + \frac{\alpha\beta}{2}(\xi - \xi_{min})^2. \tag{3.8}$$

The two solutions (3.7) and (3.8), together with their derivatives join at the point $\xi = \xi_{min}$.

The boundary conditions (3.5) determine the value of $A(d/2)$, the location $x_0 = \xi_0 d/2$ of the zero of the resultant magnetic field, and the E dependence of the intrinsic field H, i.e., the current-voltage characteristic of the plate. Let us, for convenience of the subsequent exposition, introduce the notation

$$h = \frac{25\Gamma^4(5/4)\, b^3}{32\pi}\frac{cp_F d}{el^2} \approx 0.86\frac{cp_F d}{el^2},$$
$$\mathscr{E} = \frac{4chl}{3\pi\sigma_0 d^2}, \qquad b = 2\int_0^1 \frac{x\,dx}{(1 - x^3)^{1/2}} = \pi^{1/2}\frac{\Gamma(5/3)}{\Gamma(7/6)} \approx 1.724. \tag{3.9}$$

The parameters h and $\mathscr{E}$ are characteristics of the sample in question, and, as will be seen below, serve as the natural measurement scales for the magnetic and electric fields in the case of strong nonlinearity. The quantity h is the characteristic strength of the magnetic field in which the path traversed by a trapped electron over a period of the motion is of the order of the mean free path, i.e., $(Rd)^{1/2} \sim l$.

If we combine term by term the first two boundary conditions in (3.5), then we obtain with the aid of (3.6)–(3.8) the following expression for the relative shift of the zero of the resultant magnetic field:

$$\xi_0 = \frac{2x_0}{d} = \frac{ch_0}{2\pi d j_{\text{untr}}} = \frac{h_0}{h} \frac{\mathscr{E}}{E \ln(R_+/d)}. \tag{3.10}$$

According to (3.10), the position of the $x = x_0$ plane is determined by the quantities h_0 and j_{untr} no matter what the relation between the current densities of the drifting and trapped particles (i.e., the value of the parameter β) is.

The formula (3.10) indicates quite a curious effect-whereby the position of the $x = x_0$ plane inside the plate is unstable, an effect which should occur in the regime of prescribed voltage (prescribed electric field E). In this case the current in the film and the intrinsic magnetic field H of this current are functions of h_0: $H = H(h_0)$. Let initially $h_0 = 0$ and the conditions (1.3) be fulfilled, i.e., let $H(0)$ be due to the trapped-electron current and $H(0) \gg h$. The plane of inversion of the sign of the magnetic field is located exactly in the middle of the sample (i.e., $x_0 = 0$). When the external field h_0 is switched on, this plane undergoes a relative shift $\xi_0 = h_0/H_{\text{untr}}$, where

$$H_{\text{untr}} = 2\pi d j_{\text{untr}}/c \sim [H(0)h]^{1/2} \ll H(0), \tag{3.11}$$

from the middle of the plate towards the upper surface. It can be seen from this that the $x = x_0$ plane exists (i.e., that $x_0 < d/2$) only in the case when the external field h_0 does not exceed the intrinsic magnetic field H_{untr} of the current due to the drifting particles, which is much lower than the intrinsic magnetic field $H(0)$ of the current due to the trapped electrons when $h_0 = 0$. As h_0 increases, because of the displacement of the sign-inversion plane $x = x_0$ (and, as a consequence, the decrease of the conductivity due to the trapped electrons), the intrinsic field $H(h_0)$ decreases rapidly. At the instant h_0 becomes equal to H_{untr}, the conductivity σ_{tr} becomes equal to σ_{untr}, the $x = x_0$ plane coincides with the upper plate surface ($x_0 = d/2$), the resultant field $\mathscr{H}(x) = h_0 + H(x)$ assumes a constant sign, and the group of trapped electrons disappears. Therefore, when $h_0 \gg H_{\text{untr}}$, the intrinsic magnetic field $H = H_{\text{untr}}$, i.e., is entirely determined by the untrapped particles. Thus, in the regime of prescribed potential difference the magnetodynamic nonlinearity is controlled by the external field h_0, whose "critical" value is determined by the weak current j_{untr}, and not by the strong current j_{tr}.

In the regime of prescribed current I (the intrinsic field is prescribed) the resultant magnetic field $\mathscr{H}(x) = h_0 + H(x)$ is sign-variable ($x_0 < d/2$) when $h_0 < H$ (see (2.3)). As will be seen from the analysis of the CVC, in this case, as the external field strength h_0 increases, the electric field E and, hence, j_{untr} and H_{untr} increase. Therefore, the relative shift ξ_0 of the $x = x_0$ plane changes little as h_0 increases, and the transition from the high conductivity σ_{tr} to the low σ_{untr} occurs over a broad range of h_0 values. Thus, there obtains between the trapped and untrapped particles in the regime of prescribed current a distinctive negative feedback, owing to which the stability of the location of the zero of the resultant magnetic field in the sample is ensured: as h_0 increases, the group of trapped electrons does not disappear, and ensures the nonlinearity of the CVC in a broad range of values of $h_0 \leqslant H$ because of the fact that it, by becoming small, increases the untrapped-particle current and thereby maintains its own current at the prescribed level ($I_{\text{tr}} = I$).

Let us now turn to the derivation of the current-voltage characteristic. From (3.7) we find $a(1)$ and $a'(1)$; from (3.8), $a'(-1)$. We substitute the expressions obtained into the boundary conditions (3.5). These relations contain the quantities α, β, and $|A(d/2)|$. Using (3.6), we eliminate α and $|A(d/2)|$ from them. After a series of identity transformations with account taken of the notation introduced in (3.9) and the explicit form (3.10) for ξ_0, we arrive at the following relation between the electric field E and the intrinsic magnetic field H (i.e., the current I):

$$\left(\frac{E}{\mathscr{E}}\right)^2 \left[1 - \frac{h_0}{h} \frac{\mathscr{E}}{E \ln(R_+/d)}\right]^3$$
$$= \frac{H-h_0}{h}(1+2\beta)^{-1/2}\left(\frac{1}{b}\int_0^1 dt[1-(1-t)^{3/2}+2\beta t]^{-1/2}\right)^3. \tag{3.12}$$

The parameter β in the relation (3.12) characterizes the ratio of the untrapped- and trapped-electron currents, and is defined as the real positive root of the cubic equation:

$$\frac{\beta^3}{1+2\beta} = b^{-3}\left(\frac{h}{H-h_0}\right)^2 \frac{E}{\mathscr{E}} \ln^3 \frac{R_+}{d}. \tag{3.13}$$

It is easy to solve this equation explicitly in the cases when the values of the right member of (3.13) are small and large compared to unity. Therefore, let us consider the limiting cases of small and large β separately.

In the strong-nonlinearity limit, when $\beta \ll 1$, or when

$$1 \ll \beta^{-1} = b\left[1 - \frac{h_0}{h} \frac{\mathscr{E}}{E \ln(R_+/d)}\right]^2 \frac{E}{\mathscr{E} \ln(R_+/d)}, \tag{3.14}$$

we can set the parameter β equal to zero in the equation (3.12) for the CVC. As a result, we have

$$\frac{H-h_0}{h} = \left(\frac{E}{\mathscr{E}}\right)^2 \left[1 - \frac{h_0}{h} \frac{\mathscr{E}}{E \ln(R_+/d)}\right]^3. \tag{3.15}$$

The expression in the square brackets in (3.15) is none other than $1 - \xi_0$ (see (3.10)). If $\xi_0 \ll 1$, then the formula (3.15) describes the parabolic section of the nonlinear CVC: the voltage drop V is proportional to the square root of the current I ($H \propto E^2$). The deviation, connected with the finite quantity ξ_0, from the parabolic law is due to important role played by the weak untrapped-particle current. Thus, the formula (3.15) for the current-voltage characteristics illustrates the above-noted feedback effect existing between the trapped and untrapped electrons.

The physical meaning of the inequality (3.14), which determines the region of applicability of the result (3.15), becomes quite clear when the inequality is represented in the form

$$1-\xi_0 \gg \left(\frac{h}{H}\right)^{1/2} \frac{\ln(R_+/d)}{b^{1/2}} \sim \frac{(Rd)^{1/2}}{l} \ln\frac{R_+}{d}. \tag{3.16}$$

It can be seen from (3.16) that the plane $x = x_0$ of inversion of the sign of the resultant magnetic field should not be too close to the upper plate surface $x = d/2$. In fact, in the external field h_0 the relative number of trapped electrons is, in order of magnitude, equal to $[d(1-\xi_0)/R_-]^{1/2}$, and the conductivity σ_{tr} will be significantly higher than σ_{untr} only when the condition (3.16) is fulfilled.

Let us point out that the results obtained in this section are based on the asymptotic expression (3.2) for j_{tr}, which, strictly speaking, is applicable only when the condition (3.16) is fulfilled, i.e., when $\beta \ll 1$. But in the slightly nonlinear regime, when $\beta \gg 1$, Eq. (3.12) again yields the correct result for the CVC. Therefore, we can assume that the formula (3.12) is a good interpolation of the true CVC in the region of intermediate β values, i.e., in the region $\beta \sim 1$, as well. These arguments demonstrate that the relations (3.12) and (3.13) indeed describe the current-voltage characteristic for an arbitrary value of the parameter β.

In the region of weak nonlinearity, i.e., for $\beta \gg 1$, or

$$\beta^{-2} = \frac{b^3}{2}\left[1 - \frac{h_0}{h}\frac{\mathscr{E}}{E\ln(R_+/d)}\right]^2 \frac{E}{\mathscr{E}\ln(R_+/d)} \ll 1, \quad (3.17)$$

we obtain from (3.12) and (3.13) the relation

$$\frac{H}{h} = \frac{E}{\mathscr{E}}\ln\frac{R_+}{d}. \quad (3.18)$$

In this case the nonlinearity of the current-voltage characteristic stems from the weak logarithmic dependence of the untrapped-electron current on the intrinsic magnetic field $H = H_{untr}$. The $x = x_0$ plane is then flush with the upper plate surface $x = d/2$, and the intrinsic field H of the current is practically equal to the external field h_0:

$$1-\xi_0 = \frac{H-h_0}{H} \ll \frac{(Rd)^{1/2}}{l}\ln\frac{R_+}{d} \ll 1. \quad (3.19)$$

3. Let us now discuss the current-voltage characteristic in the region of stronger external fields, i.e., in the region $h_0 > H$. Here the resultant magnetic field in the sample does not change sign, there are no trapped electrons, and the conductivity of the plate is due to the untrapped electrons. When the right inequality in (1.3) with the quantity R replaced by R_+ is satisfied, the current-voltage characteristic is given, in accordance with the asymptotic expression (3.1), by Eq. (3.18). The formula (3.18) shows that the slightly nonlinear CVC becomes linear in the limit when $H \ll h_0$, since the cyclotron radius R_+ tends to $R_0 = cp_F/eh_0$ in this limit.

4. In conclusion, let us note that the slightly nonlinear CVC (3.18) is not realized at all in the region of sufficiently weak external fields, i.e., when $h_0 < h$, or

$$l < (R_0 d)^{1/2}. \quad (3.20)$$

As the current I (the intrinsic field H) decreases, the parabolic current-voltage characteristic (3.15) goes over in the region $H \lesssim h$ into a linear dependence whose slope is determined by the conductivity σ_l (1.1):

$$\frac{H}{h} = 2\frac{E}{\mathscr{E}}\ln\frac{l}{d}. \quad (3.21)$$

4. THE CVC CALCULATIONS AND A DISCUSSION OF THE RESULTS

Figure 3 shows the results obtained in a numerical calculation of the current-voltage characteristic of a thin metallic plate with thickness $d = 10^{-3}$ cm and electron mean free path $l = 0.1$ cm for different values of the external-field strength h_0. According to (3.9), for $p_F = 10^{-19}$ gm · cm/sec, the magnitudes of the scaling units h and $\mathscr{E}$ are respectively equal to

$$h = 0.54\,\text{Oe}, \quad \mathscr{E} = 3.1\cdot 10^{-5}\ \text{V/cm}. \quad (4.1)$$

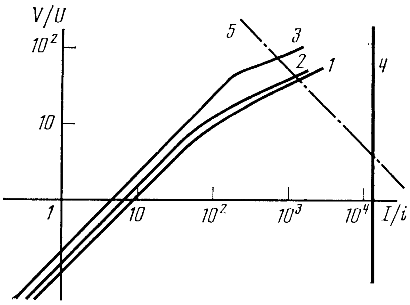

FIG. 3. The current-voltage characteristic, depicted on a double logarithmic scale, of a metallic plate for different values of the parameter h_0/h: the curve 1 corresponds to $h_0/h \leqslant 1$; the curve 2, to $h_0/h = 10$; the curve 3, to $h_0/h = 100$. On the vertical straight line 4 the radius $R = 10^{-3}$ cm and the dot-dash straight line corresponds to $IV/LD = 1$ W/cm² ($i = 0.86$ A and $U = 3.1 \times 10^{-3}$ V).

The unit i of measurement of the current is related to the characteristic magnetic field h by the formula (1.2) and the voltage scale $U = \mathscr{E}L$, where L is the distance between the potential contacts in the direction of the current. The dot-dash inclined straight line 5 in Fig. 3 is described by the equation

$$\frac{I}{i}\frac{V}{U} = 1\frac{\text{W}}{\text{cm}^2}\cdot\frac{2\pi}{c\mathscr{E}h} = 3.8\cdot 10^4. \quad (4.2)$$

When the dimensions $L = D = 1$ cm, the straight line 5 corresponds to the power density $IV/LD = 1$ W/cm², which can easily be detoured in the liquid helium so that the sample will not warm up. For these values of L and D the quantity $i = 0.86$ A and $U = 3.8 \times 10^{-5}$ V. The vertical straight line 4 is described by the equation

$$\frac{R}{d} = \frac{cp_F}{ehd}\frac{i}{I} = 1.2\cdot 10^4\frac{i}{I} = 1. \quad (4.3)$$

It limits from above the current I flowing through the sample, and determines the boundary of the region where the left inequality in (1.3) is satisfied.

The curve 1 corresponds to the values of $h_0 \leqslant h$ at which we can neglect the second term in the square brackets in (3.15) in the nonlinear region. (Since the CVC's in Fig. 3 are plotted in log-log scale, the variation of the slopes of the curves at high currents corresponds to the nonlinearity region.) For the curve 1, Ohm's law is valid in the region $I/i \ll 5$, and is described by a formula corresponding to (3.21),

$$I/i = 2(V/U)\ln(l/d). \quad (4.4)$$

A smooth transition from the linear to parabolic CVC occurs in the $5 < I/i < 50$ current range. The parabolic CVC is realized in the region $I/i > 50$, and is given by the equation

$$I/i = (V/U)^2. \quad (4.5)$$

Let us emphasize that the nonlinearity is noticeable even at

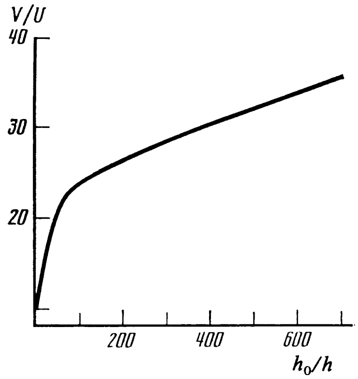

FIG. 4. Dependence of the voltage drop across the sample on the external magnetic field for $I/i = 100$.

the fairly low currents $I \gtrsim 5i \approx 4$A. The region where the transition from the limiting law (4.4) to (4.5) occurs is broad because of the low values of the external-field intensity h_0.

The dependence of the current-voltage characteristic on the external field h_0 is manifested in the region $h_0 \gg h$. Here, in the first place, the linear section of the CVC in the region of low currents is determined not by the relation (4.4), but by the expression

$$I/i = (V/U)\ln(R_0/d). \tag{4.6}$$

This is precisely the cause of the rise of the curves 2 and 3 above the curve 1 in the region of low currents. In the second place, there appears a broad region of weak nonlinearity where

$$\frac{I}{i} = \frac{V}{U}\ln\left[\frac{cp_F}{ehd}\left(\frac{h_0}{h} + \frac{I}{i}\right)^{-1}\right]. \tag{4.7}$$

Finally, in the third place, the second term in the square brackets in (3.15) plays an important role in the strong-nonlinearity regime.

On the curve 2, which corresponds to $h_0 = 10h = 5.4$ Oe, the CVC section (4.7) is realized in the region of current strengths $I/i < 12$–13; subsequently, there occurs a transition to the dependence (3.15). The CVC tends in the strong-nonlinearity region to the parabola (4.5), but this is impeded by $\ln(R_+/d)$, which decreases with increasing current.

On the curve 3, which corresponds to $h_0 = 100h = 54$ Oe, the strong-nonlinearity region (3.14) occurs in the region of higher currents $I/i > 120$–130. Although in this case the absolute width of the region where the transition from (4.7) to (3.15) occurs is greater, the relative width is smaller, and, on the logarithmic scale, the modification of the CVC is visually much more abrupt.

It is of interest to analyze the dependence of the voltage V on the constant magnetic field h_0. Figure 4 illustrates such a dependence for the case $I/i = 100$, in which the strong-nonlinearity regime is realized. At low h_0/h values the formula (3.15) is valid, and there exists in the sample a group of trapped electrons, the current due to which is quite sensitive to changes in h_0. Therefore, the voltage V increases sharply when the external-field intensity is increased slightly (see the comments on the formula (3.10)). Near $h_0/h \approx I/i$, the group of trapped electrons disappears, and the growth of the voltage $V(h_0)$ in accordance with (4.7) occurs only because the logarithm $\ln(R_+/d)$ decreases. Let us note that the function $V(h_0)$ for $|h_0| \to 0$ is nonanalytic: the derivative $\partial V/\partial h_0$ undergoes a jump as the external field h_0 passes through zero. The magnitude of this jump is

$$2\left|\frac{\partial V}{\partial h_0}\right| = \frac{U}{h}\left|3\ln^{-1}\left(\frac{cp_F}{ehd}\frac{i}{I}\right) - \left(\frac{i}{I}\right)^{1/2}\right|. \tag{4.8}$$

The appearance of nonanalyticity is due to the fact that we are dealing with the CVC in the strong-nonlinearity regime, when there exists in the sample the strong intrinsic magnetic field H of the current.

It seems to us that, although the detection of the predicted magnetodynamic-nonlinearity effects in the conductivity of thin metallic samples will meet with certain difficulties, it is entirely within reach of present-day experiment.

[1] K. Fuchs, Proc. Camb. Philos. Soc. 34, 100 (1938).

Translated by A. K. Agyei