Spatial echo in a magnetoactive plasma

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We study the spatial echo in a uniform magnetoactive plasma excited by external perturbations which are nonuniform in the radial direction. We show that this nonuniformity leads to the excitation of additional echo signals. We deduce from an analysis of the amplitudes of the echo signals occurring at different points that it may be possible to use the echo effect for plasma diagnostics.

1. INTRODUCTION

An echo is a nonlinear, nonlocal response to perturbations of a plasma, whereby the information about the latter is transferred by charged particles in the form of modulated beams, the van Kampen waves.[1] When there is no magnetic field the interaction of a particle with a wave excited by an external perturbation is determined by the Cherenkov resonance condition $v = \omega/k$, where v is the particle velocity and k and ω are, respectively, the wave number and frequency of the wave. In a magnetoactive plasma the spectrum of the van Kampen waves is given by the condition that the wave and particle velocities are equal when one takes into account the normal and anomalous Doppler shift:

$$v_{\parallel}=(\omega-n\omega_{B\alpha})/k_{\parallel}, \quad n=0, \pm 1, \pm 2, \dots, \tag{1}$$

where $\omega_{B\alpha}$ is the gyrofrequency of a particle of kind α while $v_{\parallel}$ and $k_{\parallel}$ are, respectively, the particle velocity and wavevector components along the magnetic field.

It was shown in Ref. 2 that for transverse electromagnetic waves propagating along the magnetic field the broadening of the spectrum of van Kampen waves in magnetoactive plasma leads to the possibility of the occurrence of an echo at the sum frequency. Moreover, as the Doppler shift is different for different kinds of particle the echo caused by the electron and ion plasma components may occur in different points of space.

Results of an experimental and theoretical study of the echo effect in a plasma in the lower-hybrid frequency band were given in Refs. 3 to 6. Two external potential perturbations with frequencies ω_1 and ω_2 excited an echo signal at the difference frequency $\omega_- = \omega_2 - \omega_1$ as in the case when there is no magnetic field. The theoretical considerations were based upon a model of a uniform collisionless magnetoactive plasma described by a kinetic equation with a self-consistent electrostatic field.[4] In performing the calculations one used the assumption that the applied potential (of the external perturbations) across the magnetic field was uniform.

We show in the present paper that nonuniformity of the external perturbations in a direction at right angles to the magnetic field leads to the occurrence of a sequence of electrostatic echo signals not only at the difference frequency ω_-, but also at the sum frequency $\omega_+ = \omega_2 + \omega_1$. We evaluate the maximum values of the amplitudes of the echo signals occurring at different points and indicate the possibility for plasma diagnostics (e.g., the determination of the electron temperature) using echo effects.

2. GENERAL RELATIONS

We consider a uniform plasma in a constant magnetic field $\mathbf{B}_0 = B_0\mathbf{z}$. We restrict ourselves to the case of potential oscillations and assume the external perturbations to be small and we then solve the set of Vlasov-Poisson equations using the method of successive approximations writing the distribution function f_α of charged particles of kind α and the electrostatic potential ψ in the form[7]

$$f_\alpha=f_{0\alpha}+f_\alpha^{(1)}+f_\alpha^{(2)}+\dots, \quad \psi=\psi^{(1)}+\psi^{(2)}+\dots.$$

We use the Maxwell function for the unperturbed distribution function $f_{0\alpha}$; $f_\alpha^{(1)}$ and $f_\alpha^{(2)}$ (or $\psi^{(1)}$ and $\psi^{(2)}$) are the corrections which are linear and quadratic in the external perturbation.

In the linear approximation we get for the Fourier transforms of the distribution function and the potential the following expressions:

$$f_\alpha^{(1)}(\mathbf{k},\omega)=-\frac{e_\alpha}{m_\alpha}\psi^{(1)}(\mathbf{k},\omega)\sum_{n=-\infty}^{\infty}\left(k_{\parallel}\frac{\partial f_{0\alpha}}{\partial v_{\parallel}}+\frac{n\omega_{B\alpha}}{v_\perp}\frac{\partial f_{0\alpha}}{\partial v_\perp}\right) \tag{2}$$

$$\times J_n\left(\frac{k_\perp v_\perp}{\omega_{B\alpha}}\right)(\omega-k_{\parallel}v_{\parallel}-n\omega_{B\alpha})^{-1}\exp\left[-i\left(\frac{k_\perp v_\perp}{\omega_{B\alpha}}\right)\sin\varphi+in\varphi\right],$$

$$\psi^{(1)}(\mathbf{k},\omega)=\frac{4\pi}{k^2\varepsilon(\omega,k_\perp,k_{\parallel})}\rho_{\rm ext}(\mathbf{k},\omega), \tag{3}$$

where $\rho_{\rm ext}(\mathbf{r},t)$ is the density of the external charges, $\varepsilon(\omega,k_\perp,k_{\parallel})$ is the longitudinal permittivity describing the electrostatic field in a magnetoactive plasma and the indices $\perp$ and $\parallel$ indicate, respectively, vector components perpendicular and parallel to the magnetic field, while φ is the azimuthal angle in velocity space and J_n a Bessel function of the first kind. We have written Eqs. (2), (3) in a system of coordinates in which $k_y = 0$, i.e., $\mathbf{k} = (k_\perp, 0, k_{\parallel})$.

In the second approximation of perturbation theory the Fourier transform of the electrostatic potential has the form

$$\psi^{(2)}(\mathbf{k},\omega)=-\frac{4\pi}{k^2\varepsilon(\omega,k_\perp,k_{\parallel})}R(\mathbf{k},\omega), \tag{4}$$

where

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$$R(\mathbf{k},\omega)=\frac{1}{(2\pi)^4}\sum_\alpha\frac{e_\alpha^3}{m_\alpha^2}\int d\mathbf{v}\int d\mathbf{k}'\,d\omega'\psi^{(1)}$$

$$\times(\mathbf{k}-\mathbf{k}',\omega-\omega')\psi^{(1)}(\mathbf{k}',\omega')$$

$$\times k_\parallel(k_\parallel-k_\parallel')\sum_{m,n=-\infty}^{\infty}[\omega-k_\parallel v_\parallel-(m+n)\omega_{B\alpha}]^{-2}$$

$$\times(\omega'-k_\parallel'v_\parallel-n\omega_{B\alpha})^{-1}$$

$$\times\left(k_\parallel\frac{\partial f_{0\alpha}}{\partial v_\parallel}+\frac{n\omega_{B\alpha}}{v_\perp}\frac{\partial f_{0\alpha}}{\partial v_\perp}\right)J_{m+n}\left(\frac{k_\perp v_\perp}{\omega_{B\alpha}}\right)$$

$$\times J_m\left(\frac{(k_\perp-k_\perp')v_\perp}{\omega_{B\alpha}}\right)J_n\left(\frac{k_\perp'v_\perp}{\omega_{B\alpha}}\right).$$

In deriving (4) we used the convolution theorem and Eq. (2) for the distribution function in the first approximation.

According to (3) the perturbations are given in the form of external charges the density of which we write in the following model form:

$$\rho_{\text{ext}}(\mathbf{r},t)=\rho_1\delta\left(\frac{z}{\Delta}\right)\sin(k_1x-\omega_1t)+\rho_2\delta\left(\frac{z-l}{\Delta}\right)\sin(k_2x-\omega_2t),\tag{5}$$

i.e., the external perturbations are localized in the points $z=0,\ l$ and have, respectively, frequencies ω_1 and ω_2 (we assume for the sake of argument $\omega_2>\omega_1$) and the wave numbers k_1 and k_2 are in directions at right angles to the magnetic field direction. The quantity Δ which has the dimensions of a length characterizes the depth to which the external perturbations penetrate into the plasma.

Substituting Eqs. (3) and (5) into (4) and performing the inverse Fourier transformation we retain in the expression for the nonlinear potential only the cross terms proportional to the product $\rho_1\rho_2$. Just those terms describe the echo signals. The integration over $k_\parallel'$ and $k_\parallel$ is performed using the Cauchy theorem, closing the integration contour by a semicircle of infinite radius in the upper half-plane and neglecting the contribution from the poles of the dielectric functions $\varepsilon(\omega,k_\perp,k_\parallel)$, $\varepsilon(\omega-\omega',k_\perp-k_\perp',k_\parallel-k_\parallel')$, $\varepsilon(\omega',k_\perp',k_\parallel')$ in comparison with the contribution from the kinetic poles $\omega'-k_\parallel'v-n\omega_{B\alpha}+i0=0$ and $[\omega-k_\parallel v-(m+n)\omega_{B\alpha}+i0]^2=0$ (here $i0$ indicates the rule for going around the pole).[8] As a result we obtain the following expression for the nonlinear electrostatic potential (the indices + and − refer, respectively, to oscillations at the sum and the difference frequencies):

$$\psi_\pm^{(2)}(x,z,t)$$

$$=\pm\sum_\alpha\sum_{m,n=-\infty}^{\infty}P_\alpha(z,m)[\omega_\pm-(m+n)\omega_{B\alpha}]\exp[i(k_\pm x-\omega_\pm t)]\tag{6}$$

$$\times\int_0^\infty dv_\perp v_\perp\exp\left(-\frac{v_\perp^2}{2v_{T\alpha}^2}\right)J_{m+n}\left(\frac{k_\pm v_\perp}{\omega_{B\alpha}}\right)$$

$$\times J_m\left(\frac{k_2v_\perp}{\omega_{B\alpha}}\right)J_n\left(\pm\frac{k_1v_\perp}{\omega_{B\alpha}}\right)$$

$$\times\int_0^\infty\frac{dv_\parallel}{v_\parallel^5}\frac{\exp\{-(v_\parallel^2/2v_{T\alpha}^2)+i(z-l_{\alpha mn}^\mp)[\omega_\pm-(m+n)\omega_{B\alpha}]/v_\parallel\}}{\zeta_n(\pm\omega_1,\pm k_1,v_\parallel)\zeta_m(\omega_2,k_2,v_\parallel)\zeta_{m+n}(\omega_\pm,k_\pm,v_\parallel)}$$

$$+\text{c.c.},$$

where

$$l_{\alpha mn}^\pm=l(\omega_2-m\omega_{B\alpha})[\omega_\pm-(m+n)\omega_{B\alpha}]^{-1},$$
$$P_\alpha(z,m)=4i(2\pi)^{1/2}\omega_1\rho_1\rho_2\Delta^2(z-l)$$
$$\times(\omega_2-m\omega_{B\alpha})(e_\alpha\omega_{p\alpha}^2/m_\alpha v_{T\alpha}^5),$$
$$\zeta_r(\omega_j,k_j,v_\parallel)=[k_j^2+(\omega_j-r\omega_{B\alpha})^2/v_\parallel^2]\,\varepsilon(\omega_j,k_j,(\omega_j-r\omega_{B\alpha})/v_\parallel),$$
$$\omega_{p\alpha}=(4\pi e_\alpha^2n_0/m_\alpha)^{1/2},\quad v_{T\alpha}=(T_\alpha/m_\alpha)^{1/2},\quad k_\pm=k_2\pm k_1.\tag{7}$$

The integral over $v_\parallel$ which occurs in (6) vanishes for $z\neq l_{\alpha mn}^\pm$ due to the presence in the integrand of a fast oscillating function. Hence, the nonlinear echo potential (6) is nonzero only in the vicinity of the points (7) which thus determine the location of its maxima.[9]

When $k_{1,2}=0$ the arguments of the Bessel functions in (6) vanish i.e., in the sum over m and n only one term with $m=n=0$ is nonzero. It follows from Eq. (7) that in that case the echo occurs only at the difference frequency in the unique point $z=l\omega_2/\omega_-$ which corresponds to the results of Refs. 3 to 6.

As the integers m and n occurring in (7) can take on any values from $-\infty$ to $+\infty$ it would appear that the echo can be found in an infinite number of points. However from an analysis of the integral over $v_\perp$ which occurs in (6) one sees easily that there is a restriction on the values of m and n. To this end we write this integral in dimensionless variables:

$$\int_0^\infty dx\,x\exp(-x^2)J_{m+n}(\delta_\pm x)J_m(\delta_2x)J_n(\pm\delta_1x),\tag{8}$$

where

$$x=v_\perp/v_{T\alpha}\sqrt{2},\quad \delta_j=k_jv_{T\alpha}\sqrt{2}/\omega_{B\alpha}\quad(j=1,2,\pm).$$

One can estimate the integral (8) assuming that $\delta_j\ll1$.[10] Thanks to the presence in the integrand of the fast decreasing factor $\exp(-x^2)$ we can then neglect the contribution from the integration region $x>\delta_j^{-1}$. After that writing the Bessel functions as series in powers of their arguments we show easily that the value of the integral (8) is proportional to the quantity

$$\delta_1^{|n|}\delta_2^{|m|}\delta_\pm^{|m+n|}.$$

Hence it follows that those echo signals for which the quantity $\lambda=|n|+|m|+|m+n|$ is a minimum have the largest amplitude. With increasing λ the amplitude of the echo signal decreases according to a power law.

3. THE ECHO IN THE LOWER-HYBRID FREQUENCY BAND

We turn to the analysis of echo effects in the case when the frequencies of the external perturbations lie in the lower-hybrid frequency band. Under the condition $\omega_{pe} < \omega_{Be}$ this band includes the frequencies $\omega_{pi} < \omega_{1,2} < \omega_{pe}$. It is clear from Eq. (7) that the echo signals caused by the electron and ion components of the plasma may in principle be excited at different points. However, in the case considered $\omega_{Bi} \ll \omega_{pi} < \omega_{1,2}$ and it is clear from Eq. (7) that the echo caused by the ion component occurs in the point $l^{-}_{i00} = l\omega_2/\omega_-$ ($l^{+}_{i00} < l$). In the same point there occurs an echo caused by the electron component with an amplitude which is larger by a factor $(m_i/m_e)^{3/2}(T_i/T_e)^{1/2}$. The "ion" echo in this case is therefore indiscernible against the "electron" background.

Taking into account the condition for a minimum λ obtained in the previous section we restrict our considerations to only a few echo signals caused by the electron plasma component with the largest amplitudes. These signals occur in the following points [see (7)]:

$$l_1 = l\omega_2/\omega_- \quad (m=n=0), \quad l_2 \approx 2l \quad (m=\pm 2, n=\mp 1),$$
$$l_3 \approx {}^3/_2 l \quad (m=\pm 3, n=\mp 1), \quad l_4 \approx l\omega_{Be}/\omega_+ \quad (m=-1, n=1). \tag{9}$$

We note that in the point l_1 an echo occurs at the difference frequency, and in the point l_4 at the sum frequency, whereas in the points $l_{2,3}$ they occur at both the difference and the sum frequencies.

To estimate the integral over $v_\parallel$ in (6) we use approximate expressions for the dielectric functions $\zeta_r(\omega_j, k_j, v_\parallel)$ obtained assuming that $k_{1,2} v_{Te} \ll \omega_{1,2} \ll \omega_{Be}$:

$$\zeta_0(\omega_j, k_j, v_\parallel) \approx 1 + \frac{\omega_{pe}^2}{\omega_j^2}\frac{v_\parallel^2}{2v_{Te}^2}\left[1 - J_+\left(\frac{v_\parallel}{2^{1/2} v_{Te}}\right)\right],$$
$$\zeta_r(\omega_j, k_j, v_\parallel) \approx \omega_{pe}^2/v_{Te}^2 + (r\omega_{Be})^2/v_\parallel^2, \quad r \neq 0, \tag{10}$$

where

$$J_+(x) = x \exp\left(-\frac{x^2}{2}\right) \int_{i\infty}^{x} d\tau \exp\left(\frac{\tau^2}{2}\right).$$

As a result we obtain the following expressions for the nonlinear echo potential in the points where it is a maximum:

$$\psi_-^{(2)}(x, z=l_1, t) \approx \beta(2/\omega_1\omega_2\omega_-^2)\sin(k_- x - \omega_- t),$$
$$\psi_\pm^{(2)}(x, z=l_2, t) \approx \mp(\beta/2\omega_{Be}^4) k_1 k_2^2 k_\pm \rho_L^4 \sin(k_\pm x - \omega_\pm t),$$
$$\psi_\pm^{(2)}(x, z=l_3, t) \approx \mp(\beta/16\omega_{Be}^4) k_1 k_2^3 k_\pm^2 \rho_L^6 \sin(k_\pm x - \omega_\pm t), \tag{11}$$
$$\psi_+^{(2)}(x, z=l_4, t) \approx \beta \frac{\omega_{Be} - \omega_+}{\omega_+^2 \omega_{Be}^3} k_1 k_2 \rho_L^2 \sin(k_+ x - \omega_+ t),$$

where ρ_L is the electron Larmor radius and

$$\beta = -(2\pi)^{3/2} l \Delta^2 \rho_1 \rho_2 \omega_1 (4e\omega_{pe}^2/m_e v_{Te}).$$

A comparison of the echo signals occurring at different points can be used for the purpose of plasma diagnostics. For instance, one gets easily from (11) an expression for the electron temperature

$$T_e = \frac{8m_e\omega_{Be}^2}{k_2 k_-}\left|\frac{\psi_-^{(2)}(x, z=l_3, t)}{\psi_-^{(2)}(x, z=l_2, t)}\right|, \tag{12a}$$

or

$$T_e = \frac{2m_e}{k_1 k_2}\frac{\omega_+^2 \omega_{Be}^5}{\omega_1\omega_2\omega_-^2(\omega_{Be} - \omega_+)}\left|\frac{\psi_+^{(2)}(x, z=l_4, t)}{\psi_-^{(2)}(x, z=l_1, t)}\right|. \tag{12b}$$

We note that in Eq. (12) neither the amplitudes of the external perturbations nor their characteristic penetration depth occur. Moreover, we do not need for the determination of T_e the absolute values of the amplitudes of the echo signals; it is sufficient to measure merely their ratio.

4. HIGH-FREQUENCY EXTERNAL PERTURBATIONS

We consider the situation when the frequencies of the external perturbations of the echo oscillations exceed the electron cyclotron frequency. If $\omega_j^2 - \omega_{Be}^2 \gg \omega_{pe}^2$ ($j = 1, 2, \pm$) the permittivity $\varepsilon(\omega_j, k_\perp, k_\parallel) \approx 1$, i.e., the echo potential (6) is independent of the dispersion properties of the plasma. Hence, in the case considered the echo has a ballistic character.[11]

In what follows we restrict ourselves to echo signals for which $\lambda \leqslant 2$. One sees easily from Eq. (7) that then there are no echo signals at the sum frequency ω_+ ($l_{emn} < l$). We showed in the previous section that the echo at the sum frequency occurs for $\lambda \geqslant 4$.

It follows from (7) that in the case considered the echo at the difference frequency occurs in the points

$$L_{1\pm} = l(\omega_2 \pm \omega_{Be})/(\omega_- \pm \omega_{Be}) \quad (m=\mp 1,\ n=0),$$
$$L_{2\pm} = l\omega_2/(\omega_- \pm \omega_{Be}) \quad (m=0,\ n=\mp 1), \tag{13}$$
$$L_{3\pm} = l(\omega_2 \pm \omega_{Be})/\omega_- \quad (m=\mp 1,\ n=\pm 1).$$

Moreover, an echo signal is excited at the point l_1 [see (9)] the existence of which is independent of the presence of a constant magnetic field in the plasma. The potential of this signal is given by Eq. (11).

We get for the nonlinear echo potential in the points (13) the following expressions:

$$\psi_-^{(2)}(x, z=L_{1\pm}, t) \approx \beta \frac{k_2 k_- \rho_L^2}{\omega_1(\omega_2 \pm \omega_{Be})(\omega_- \pm \omega_{Be})^2}\sin(k_- x - \omega_- t),$$
$$\psi_-^{(2)}(x, z=L_{2\pm}, t) \approx -\beta \frac{k_1 k_- \rho_L^2}{(\omega_1 \mp \omega_{Be})\omega_2(\omega_- \pm \omega_{Be})^2}\sin(k_- x - \omega_- t)$$
$$\psi_-^{(2)}(x, z=L_{3\pm}, t) \approx \beta \frac{k_1 k_2 \rho_L^2}{(\omega_1 \pm \omega_{Be})(\omega_2 \pm \omega_{Be})\omega_-^2}\sin(k_- x - \omega_- t). \tag{14}$$

Comparing the amplitudes (14) with the amplitude of the echo signal in the point l_1 we easily get expressions for the electron temperature which are similar to (12).

5. CONCLUSION

The inhomogeneity of the external perturbations in the direction at right angles to the magnetic field leads to a qualitative change in the echo picture. Instead of one signal at the difference frequency a sequence of echo pulses of the electrostatic potential both at the difference and at the sum frequencies is excited. The position of these pulses is determined by the ratio of the frequencies of the external perturbations and the cyclotron frequency. A study of the echo effect in a magnetoactive plasma can be used for its diagnostics. For instance, by determining the ratio of the amplitudes of the echo signals excited in the appropriate points and knowing $k_{1,2}$ and B_0 one can use Eq. (12) to determine the electron temperature.

As in a magnetized plasma charged particles move along the magnetic field lines the excitation of an echo is possible under conditions when the sources of the external perturbations are positioned on a single field line. Thus, in principle, one can obtain the radial distribution of the electron temperature by shifting point sources of external perturbations at right angles to the magnetic field.

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