Passive mode locking in masers with unequally spaced spectra

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It is shown that the shape of a short pulse that passes through a plasma maser with an unequally spaced spectrum can be described in a very simple model by the Haus equations with complex coefficients. A quasilinear interaction between waves and particles is found to be equivalent to a slow saturable absorber in a laser, whereas a cubic nonlinearity turns out to be comparable to a fast saturable absorber. Analytic soliton solutions of the generalized Haus equations corresponding to a pulse train with a high off-duty factor are obtained. The laws governing the fast drift of the wave spectrum within the limits of a single pulse are identified. The theoretical approach discussed here can be used to describe the processes that occur in laboratory cyclotron plasma masers and in certain types of lasers. This approach can also be used to explain the origin of several components of natural electromagnetic radiation in the magnetospheres of the Earth and Jupiter.

1. INTRODUCTION

Passive mode locking, a process in which one or several short pulses pass through a cavity resonator, is widely used in lasers.[1–9] In such systems the frequency dispersion of the refractive index is usually unimportant, and the pulse envelope changes in accordance with the gain and absorption line shapes. In passive mode locking, these effects are offset by the saturable-absorber nonlinearity.

Three theoretical methods for studying passive mode locking have by now been developed. The first is mode-dependent and is used to identify the relationship between the phases of the individual resonator modes. In such a study one can determine the conditions for individual-mode phase locking which corresponds to the pulse that passes through the resonator.[2] The laws governing the generation of a short electromagnetic pulse against a background of emission noise are explained satisfactorily by the fluctuation model.[3,4] The third method, closely related to the stationary-wave theory, postulates the existence of a pulse, so that one can write the equation for a steady pulse-train envelope. As a result, one can determine the envelope shape and study its stability to some extent.[5–9]

The basic difference between plasma masers and laser masers is the use of slower electromagnetic waves, which have appreciable nonlinearities of their own and an unequal spectrum spacing caused by the frequency dispersion of the refractive index.

In the present paper we show that plasma masers can operate in a pulsed mode-locked regime even when the factors indicated above are taken into account. In very simple models, the determination of the shape of a steady pulse-train envelope reduces to finding a soliton solution of the Haus equation[6,7] with complex coefficients. In this paper we solve this equation and identify certain features of the emission spectrum.

The theory discussed in this paper can be used to describe the processes that occur in laboratory cyclotron plasma masers[10] and in certain types of lasers. This theory can also be used to explain the origin of several components of natural electromagnetic radiation in the magnetosphere.

2. GENERAL EQUATION FOR A PULSE-TRAIN ENVELOPE

Let us first assume that the maser parameters correspond to a state below the self-excitation threshold and choose an initial point of entry into the system. In linear approximation, a maser in which the operating modes have the same transverse structure is characterized by a complex transmission coefficient $G = \Gamma + i\Phi$ at each input point. We should point out that the Kramers-Kronig equations that relate the real and imaginary parts of the function $G(\omega)$ can be determined by taking into account the linearity of the system and the causality principle.[11]

We assume that the maser receives a short pulse

$$E_0(t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} E_0(\omega)e^{-i\omega t}\,d\omega, \tag{1}$$

which is much shorter than the period T of group propagation through the resonator. Taking the transmission coefficient into account, we can write the following expression for the change in the electric field at the system's input:

$$E(t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} E_0(\omega)e^{-i\omega t}(1+e^{G}+e^{2G}+\dots)\,d\omega. \tag{2}$$

Using the formula for the sum of a geometric progression, we find for $\Gamma<0$

$$E(t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} E_0(\omega)\,\frac{e^{-i\omega t}}{1-e^{G}}\,d\omega. \tag{3}$$

Assume that the electromagnetic pulse changes little after a single pass through the resonator, i.e., we restrict the analysis to the case in which the gain is small and the phase advance is a nearly linear function of frequency. A sufficiently long time after the arrival of the initial signal, the electromagnetic-radiation spectrum will be concentrated near the frequency ω_0, which corresponds to the maximum of the function $\Gamma(\omega)$ ($\partial\Gamma/\partial\omega=0$ and $\partial^2F/\partial\omega^2<0$ when $\omega=\omega_0$). Accordingly, in determining the asymptotic behavior of the

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field at the system's input, we can substitute an integral of Eq. (3) the transmission coefficient

$$G=g+i\varphi+iT\Omega-\Delta\Omega^2, \tag{4}$$

where

$$g=\Gamma(\omega=\omega_0),\quad \varphi=\Phi(\omega=\omega_0),\quad T=\partial\Phi/\partial\omega,$$

$$\Delta=-\frac{1}{2}\left(\frac{\partial^2\Gamma}{\partial\omega^2}+i\frac{\partial^2\Phi}{\partial\omega^2}\right),\ \text{and}\ \Omega=\omega-\omega_0;$$

here all derivatives are taken at $\omega=\omega_0$. Since the gain is small ($|g|\ll 1$), and since the second derivatives of the transmission coefficient with respect to the frequency are also small ($|\Delta|\Omega^2\ll 1$), we can use (3) and (4) to find the following expression for the pulse-train envelope:

$$\mathscr{E}(t)=E(t)e^{i\omega_0 t}$$

$$=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}E_0(\omega_0+\Omega)\frac{e^{-i\Omega t}}{1-(1+g-\Delta\Omega^2)e^{i(\varphi+\Omega T)}}d\Omega. \tag{5}$$

If we can differentiate within the integral with respect to time, Eq. (5) reduces to the differential-difference equation

$$\left(1+g+\Delta\frac{\partial^2}{\partial t^2}\right)\mathscr{E}(t)-e^{-i\varphi}\mathscr{E}(t+T)=-e^{-i(\varphi+\Omega T)}\mathscr{E}_0(t), \tag{6}$$

where the envelope $\mathscr{E}_0$ corresponds to the field E_0. The initial pulse (1) attenuates completely in a time much longer than the period of the pulse train, and the right side of (6) actually vanishes in the asymptotic regime we are concerned with. The steady pulse-train envelope therefore satisfies the equation

$$\left(1+g+\Delta\frac{\partial^2}{\partial t^2}\right)\mathscr{E}(t)-e^{-i\varphi}\mathscr{E}(t+T)=0. \tag{7}$$

Of importance in the explanation of the experimental results are the periodic solutions of Eq. (7). We can write the condition for the emission periodicity to within the initial rf-carrier phase as

$$\mathscr{E}(t+T_p)=e^{i(\psi+\varphi)}\mathscr{E}(t), \tag{8}$$

where T_p is the period of the pulse train, and ψ is a constant. Equations (7) and (8) do not contain the initial emission level and can, in principle, be written by analyzing a single, rather than multiple passes of the wave packet through the resonator. A more detailed calculation is justifiable because the particular features of the approximations used here can be traced in more detail by using this approach. Equation (6) may also prove to be useful in analyzing the behavior of a system operating in the amplification regime.

We restrict the analysis to a pulse train with a large off-duty factor. Among other advantages, this allows us to disregard the overlap of direct and reflected pulses in the resonator. In the case at hand, this problem can be reduced, following Ref. 6, to determining the shape of a single pulse. To this end we must take account of the fact that the period of a pulse train, T_p, is approximately equal to the period of group propagation, T, if the frequency dispersion is weak; i.e., $T=T_p-\delta$, where $|\delta|\ll T$. Accordingly, we can write[1)]

$$\mathscr{E}(t+T)=\mathscr{E}(t+T_p)-\delta(\partial\mathscr{E}/\partial t)|_{t+T_p}. \tag{9}$$

According to Eqs. (7)–(9), the shape of the envelope of a single pulse from a steady periodic pulse train with a large off-duty factor thus satisfies the following equation approximately

$$\Delta\frac{\partial^2\mathscr{E}}{\partial t^2}+\delta e^{i\psi}\frac{\partial\mathscr{E}}{\partial t}+(1+g-e^{i\psi})\mathscr{E}=0. \tag{10}$$

This equation must be solved with the stipulation that the fields fall off rapidly as $t\to\pm\infty$.

There can obviously be no steady periodic pulse train in a linear system in which there is absorption and frequency dispersion. For this reason, Eq. (10) does not have a soliton solution. Real maser systems are nonlinear systems, and a nonlinear analog of Eq. (10) can have soliton solutions. In some cases, allowance for a slight nonlinearity reduces to replacement of the gain g by an $\mathscr{E}$-dependent operator which takes into account the nonlinear corrections to the gain and to the frequency. This substitution is of course justifiable only if $\Gamma(t\to\pm\infty)<0$.

3. NONLINEAR INTERACTION IN AN ACTIVE MEDIUM

We take into account the nonlinearities of a maser system in a very simple model, assuming that the interaction between the waves and particles occurs in a narrow region. In a plasma cyclotron maser, the effective interaction of the extraordinary electromagnetic waves (whistlers) with the electrons usually occurs near the magnetic-field minimum at reasonably high energies of the epithermal particles.[12] This is also true of the interaction of Alfvén waves with protons.

Thus far we have written equations for an arbitrary entry point into a maser. We can now assume, however, that this is precisely the location where the waves interact with the particles. If the particles do not move an appreciable distance in a magnetic-confinement system in the time it takes a short electromagnetic pulse to pass through the interaction region, then calculations can be carried out without consideration of dispersion. In the opposite limiting case, we can use equations that are averaged over the bounce oscillations of particles between the mirror points. If the quasilinear approximation is correct, the distribution function $f(t,\mathbf{v})$ of the energetic particles near the magnetic-field minimum inside the electromagnetic pulse will satisfy in both cases the following equation[12]:

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial\mathbf{v}}\left(\hat{D}|\mathscr{E}|^2\frac{\partial f}{\partial\mathbf{v}}\right), \tag{11}$$

where $\hat{D}(\mathbf{v})$ is a tensor that determines the nature of the diffusion in the velocity space. In turn, if the electromagnetic-emission spectrum is narrow and the mean-frequency shift is relatively small, the buildup of the waves after a single pass through the resonator is given by

$$g_s=g+\int\mathbf{K}\frac{\partial(f-f_0)}{\partial\mathbf{v}}d^3v, \tag{12}$$

where $\mathbf{K}(\mathbf{v})$ is a vector, and g and $f_0(\mathbf{v})$ correspond to the state of the system long before the arrival of the next pulse. If a pulse train has a large off-duty factor, the distribution function is restored during the time between the pulses because of the dispersal of the particles along the magnetic bottle and because of their drift across it. Restricting ourselves therefore to the effect of a single electromagnetic pulse on the distribution function, we introduce

$$\tau=\int_{-\infty}^{t}|\mathscr{E}|^2\,dt.$$

Expanding the distribution function in eigenfunctions of a self-adjoint operator

$$\frac{\partial}{\partial \mathbf{v}}\left(\hat{D}\frac{\partial Z_n}{\partial \mathbf{v}}\right)=-\lambda_n Z_n, \tag{13}$$

which satisfies the necessary boundary conditions, we can write the solution of Eq. (11) as

$$f=\sum_{n=0}^{\infty} f_n Z_n \exp(-\lambda_n\tau). \tag{14}$$

If τ is small enough, i.e., if the energy in the pulse is relatively low, we can simplify Fourier series (14) by expanding now the exponential functions accurate to τ^2:

$$\exp(-\lambda_n\tau)\approx 1-\lambda_n\tau+{}^1/_2\lambda_n{}^2\tau^2.$$

Using (11) and (13), we find in this approximation the following distribution function which corresponds to (14):

$$f=f_0+\tau\frac{\partial}{\partial \mathbf{v}}\left(\hat{D}\frac{\partial f_0}{\partial \mathbf{v}}\right)+\frac{\tau^2}{2}\frac{\partial}{\partial \mathbf{v}}\left(\hat{D}\frac{\partial^2}{\partial \mathbf{v}^2}\hat{D}\frac{\partial f_0}{\partial \mathbf{v}}\right) \tag{15}$$

According to (12), quasilinear interaction is therefore a cumulative process that is slow in comparison with the duration of the electromagnetic pulse, since

$$g_S=g+\alpha\tau-\beta\tau^2, \tag{16}$$

where α and β are constants. We see immediately from Eqs. (11) and (12) that the gain (16) depends solely on τ. These calculations had to be carried out, because Eqs. (12) and (15) determine the values of the coefficients α and β, which have an appreciable effect on the results obtained below.

Let us define the conditions necessary for the existence of a pulse train and for its stability in a maser with a nonlinear gain (16). In order for g_s to reach positive values and be negative in the limit $t\to\pm\infty$, the following conditions must hold:

$$\begin{gathered}\alpha>0,\quad \beta>0,\quad g<0,\\ -\alpha^2/4\beta<g<-\alpha\tau+\beta\tau^2.\end{gathered} \tag{17}$$

With these relations between the parameters, the bunching of the emission into short clusters reduces the losses in the medium and is desirable from the energy standpoint.[4] An important point in the discussion below is that Bespalov and Koval'[13] have shown by analyzing a quasilinear system of equations that the waves at the leading edge of an electromagnetic pulse can grow if this pulse is long compared to the period of the bounce oscillations of particles between the mirror points. In other words, the conditions $\alpha>0$ and $\beta>0$ can be satisfied. We will confirm this conclusion in the Appendix, where we use a specific example of cyclotron interaction between waves and particles in a plasma magnetic bottle to determine the signs of α and β. In the case discussed here, the approach of the self-excitation threshold and the absorption saturation are both caused by the interaction at cyclotron resonance with epithermal particles having a nonequilibrium distribution function.

Besides quasilinear interaction, an electromagnetic pulse may be affected by the nonlinearity of the main plasma. If the cubic nonlinearity of the medium appears only locally (primarily near the magnetic-field minimum at low temperature), then such a fast nonlinearity, according to Galeev and Sagdeev,[14] can be taken into account by replacing the gain g in (10) by

$$g_F=g+\zeta|\mathscr{E}|^2, \tag{18}$$

where ζ can be complex. This expression allows us to introduce a nonlinear correction in both the gain and the frequency.

The conditions necessary for the existence of a pulse train and for its stability [conditions similar to those in (17)] in a maser with a nonlinear gain (18) can be written as

$$-\zeta_R|\mathscr{E}_p|^2<g<0,\quad \zeta_R=\operatorname{Re}\zeta,\quad |\mathscr{E}_p|^2=\max|\mathscr{E}|^2. \tag{19}$$

In the next sections we will use Eqs. (16) and (18) to calculate the characteristics of pulse trains.

4. SOLITON SOLUTION OF THE HAUS EQUATION WITH COMPLEX COEFFICIENTS IN THE CASE OF A SLOW NONLINEARITY

Let us assume that expression (16) is valid. We can then reduce the determination of the shape of a pulse-train envelope to the solution of the generalized nonlinear Eq. (10):

$$\Delta\frac{\partial^2\mathscr{E}}{\partial t^2}+\delta e^{i\psi}\frac{\partial\mathscr{E}}{\partial t}+[1+g+\alpha\tau-\beta\tau^2-e^{i\psi}]\mathscr{E}=0. \tag{20}$$

We recall that here $\Delta=\Delta_R+i\Delta_I$ is a known complex quantity and that g, α, and β are known real values. We can choose the real parameters δ and ψ in such a way that the equation will have a soliton solution. In this case, δ determines the pulse repetition period and ψ the phase shift between the pulses. If all coefficients in Eq. (20) were real, then this equation would be identical to the Haus equation for a laser with a slow, saturable absorber.[6]

Let us now analyze the Haus equation with complex coefficients. In keeping with the premises on which the derivation of Eq. (20) is based, we are concerned with soliton solutions that decay rapidly at infinity. A direct check clearly shows that we can seek with success a soliton solution in the form

$$\mathscr{E}=\mathscr{E}_p[\operatorname{ch}(t/t_p)]^{ia-1}\exp\{i(bt/t_p)\}, \tag{21}$$

where $\mathscr{E}_p$, t_p, a, and b are constants that must be found. Substituting (21) into (20), equating the coefficients of identical powers of $\tanh(t/t_p)$, and separating the real and imaginary parts, we find an algebraic system of equations from which we must determine the characteristics of the pulse train:

$$\begin{gathered}a^2+3(\Delta_R/\Delta_I)a-2=0,\\ \beta(\mathscr{E}_pt_p)^4=\Delta_R(2-a^2)+3\Delta_Ia,\\ 2\Delta_Iab+2\Delta_Rb-t_pa\delta\cos\psi+t_p\delta\sin\psi=0,\\ \alpha(\mathscr{E}_pt_p)^2t_p-2\Delta_Rab+2\Delta_Ib-t_pa\delta\sin\psi-t_p\delta\cos\psi=2\beta(\mathscr{E}_pt_p)^4,\\ \Delta_Ib^2-t_pb\delta\cos\psi+t_p{}^2\sin\psi=\Delta_Ra-\Delta_I,\\ \alpha(\mathscr{E}_pt_p)^2t_p-\Delta_Rb^2-t_pb\delta\sin\psi-t_p{}^2\cos\psi+t_p{}^2(1+g)\\ =\Delta_R+\Delta_Ia+\beta(\mathscr{E}_pt_p)^4.\end{gathered} \tag{22}$$

Here the unknown quantities are $\mathscr{E}_p$, t_p, a, b, δ, and ψ. Since the number of equations equals the number of unknowns, the equations can be solved for appropriate parameters of the plasma maser. The easiest to analyze are the first two equations in (22). We see from these equations that

$$(\mathscr{E}_p t_p)^4=\frac{3a}{\beta\Delta_I}(\Delta_R^2+\Delta_I^2),$$
$$a=\frac{1}{2\Delta_I}[-3\Delta_R+(9\Delta_R^2+8\Delta_I^2)^{1/2}],\tag{23}$$

where it is taken into account that $\beta>0$ according to (17).

Equations (23) determine the link between the amplitude and duration of a pulse and also regulate the frequency variation of the emission. The mean frequency $\langle\omega(t)\rangle$ of a dynamic emission spectrum varies roughly as the derivative of the field phase taken with opposite sign. According to (5) and (21), we have

$$E=\mathscr{E}_p\left(\operatorname{ch}\frac{t}{t_p}\right)^{-1}\exp\left[-i\left(\omega_0 t-b\frac{t}{t_p}-a\ln\operatorname{ch}\frac{t}{t_p}\right)\right],$$

so that the frequency variation within the limits of the pulse is

$$\langle\omega\rangle=\omega_0-b/t_p-(a/t_p)\operatorname{th}(t/t_p).\tag{24}$$

It is worthwhile noting that at $t=0$ the frequency differs from ω_0 at the intensity peak and that the alternating component of the frequency depends on the parameter a.

These results must, of course, satisfy conditions (17), which we can write for the pulse (21) as

$$\alpha>0,\quad \beta>0,\quad g<0,$$
$$-\alpha^2/4\beta<g<-2\alpha\mathscr{E}_p^2 t_p+4\beta\mathscr{E}_p^4 t_p^2.\tag{25}$$

To determine all characteristics of a pulse train we must, according to (22) and (23), solve four algebraic equations with four unknowns. The simplest way to solve these equations is numerically. The effect of a slight frequency dispersion on passive mode locking can easily be studied analytically. Such an analysis is possible, because when $\Delta_I=0$ one of the solutions of the system (22) has $a=0$, $b=0$, and $\psi=2\pi n$ (where $n=0,\pm1,\pm2,\ldots$), while the remaining unknowns are found by analogy with Ref. 6:

$$\mathscr{E}_p=\frac{\alpha}{3\beta}\left(\frac{\beta}{2\Delta_R}\right)^{1/4}(1+\Lambda),$$
$$t_p=-\frac{\alpha}{2g}\left(\frac{2\Delta_R}{\beta}\right)^{1/2}(1-\Lambda),\quad \delta=\frac{\alpha}{3}\left(\frac{2\Delta_R}{\beta}\right)^{1/2}(1-2\Lambda),\tag{26}$$

where $\Lambda=(1+6g\beta/\alpha^2)^{1/2}$; we used the inequalities (25) to find the root of the quadratic equation. Let us assume that Δ_I is small but nonzero. To determine the characteristics of the pulse train, we then need only find the corrections linear in Δ_I to the solution. Some simple calculations show that quantities (26) have no corrections linear in Δ_I. An approximate solution of the system (22), however, yields

$$a=\frac{2\Delta_I}{3\Delta_R},\quad b=\frac{\Delta_I\delta(2t_p^2+\Delta_R)}{3\Delta_R t_p(\delta^2+2\Delta_R)},$$
$$\psi=\frac{2\Delta_I(t_p^2\delta^2-\Delta_R^2)}{3\Delta_R t_p^2(\delta^2+2\Delta_R)}+2\pi n.\tag{27}$$

Here are some other examples of the way in which the system in (22) can be used. We can easily show with these equations that even if the frequency spectrum is equally spaced ($\Delta_I=0$), there are two other solutions in addition to the pulse solution (26). One of these solutions has $b=0$ and $\psi=\pi+2\pi n$ and the other $b\neq0$ and $\delta^2\cos\psi+2\Delta_R=0$. We can also draw some definite conclusions about systems with a strong frequency dispersion by setting $\Delta_R=0$. In this case, we would have $a=2^{1/2}\operatorname{sgn}\Delta_I$ and $(\mathscr{E}_p t_p)^4=3\Delta_I a/\beta$.

5. SOLITON SOLUTION OF THE HAUS EQUATION IN THE CASE OF A FAST NONLINEARITY

We assume now that the maser nonlinearity is described by Eq. (18). The pulse envelope will then satisfy the equation derived from (10) by substituting g_F for g:

$$\Delta\frac{\partial^2\mathscr{E}}{\partial t^2}+\delta e^{i\psi}\frac{\partial\mathscr{E}}{\partial t}+(1+g+\zeta|\mathscr{E}|^2-e^{i\psi})\mathscr{E}=0.\tag{28}$$

If the coefficients are real, Eq. (28) can be reduced to the Haus equation for a laser with a fast saturable absorber.[7]

We will seek a soliton solution of Eq. (28) with the complex coefficients $\Delta=\Delta_R+i\Delta_I$ and $\zeta=\zeta_R+i\zeta_I$ in the form in (21), which we have already tested. Substituting (21) into (28), we find the following algebraic system of equations for the characteristics of the pulse train:

$$\zeta_I(\mathscr{E}_p t_p)^2+\Delta_I(a^2-2)+3\Delta_R a=0,$$
$$\zeta_R(\mathscr{E}_p t_p)^2+\Delta_R(a^2-2)-3\Delta_I a=0,$$
$$2\Delta_R b+t_p\delta\sin\psi=0,\quad 2\Delta_I b-t_p\delta\cos\psi=0,\tag{29}$$
$$\zeta_I(\mathscr{E}_p t_p)^2+\Delta_R a-\Delta_I b^2+t_p b\delta\cos\psi-t_p^2\sin\psi=\Delta_I,$$
$$\zeta_R(\mathscr{E}_p t_p)^2-\Delta_I a-\Delta_R b^2-t_p b\delta\sin\psi-t_p^2\cos\psi+t_p^2(1+g)=\Delta_R.$$

Here $\mathscr{E}_p$ and t_p are the amplitude and duration of a pulse, a and b are the parameters used in the solution of (21), and δ and ψ determine the repetition period of the pulses and the phase shift between them [see Eqs. (8) and (9)].

If $b\neq0$, we can write a general solution of the equations in (29):

$$a=\frac{3(\Delta_R\zeta_R+\Delta_I\zeta_I)-[9(\Delta_R\zeta_R+\Delta_I\zeta_I)^2+8(\Delta_R\zeta_I-\Delta_I\zeta_R)^2]^{1/2}}{2(\Delta_R\zeta_I-\Delta_I\zeta_R)}$$
$$\psi=-\operatorname{arctg}(\Delta_R/\Delta_I)+\pi n,$$
$$t_p=\left[\frac{2\Delta_R a(\Delta_R^2+\Delta_I^2)}{\Delta_R\Delta_I(1+g)+(\Delta_R^2+\Delta_I^2)\sin\psi}\right]^{1/2},\tag{30}$$
$$\mathscr{E}_p=\frac{1}{t_p}\left[\frac{3a(\Delta_R^2+\Delta_I^2)}{\Delta_I\zeta_R-\Delta_R\zeta_I}\right]^{1/2},$$
$$b^2=1-\frac{1}{\Delta_I}[\zeta_I(\mathscr{E}_p t_p)^2+\Delta_R a-t_p^2\sin\psi],\quad \delta=-2\Delta_R b/t_p\sin\psi.$$

Equations (30) are not the only possible solutions of Eqs. (29). There are also solutions in which $b=0$, $\delta=0$, a and $\mathscr{E}_p t_p$ are determined by Eqs. (29), while the correction to the phase shift ψ and the pulse length t_p can be found from the simple system

$$\frac{\sin\psi}{\cos\psi-(1+g)}=\frac{\Delta_I(a^2-1)+2\Delta_R a}{\Delta_R(a^2-1)-2\Delta_I a},$$
$$t_p^2=\frac{2\Delta_I a-\Delta_R(a^2-1)}{\cos\psi-(1+g)}\tag{31}$$

In the solution of Eq. (30), we assumed that the determinant

$$\begin{vmatrix} \Delta_R & \Delta_I \\ \zeta_R & \zeta_I \end{vmatrix} \neq 0.$$

If the coefficients in Eqs. (29) correspond to a situation in which this determinant vanishes, we can find from (29) and (31) the following expression, instead of (30):

$$\begin{gathered} a=0,\ b=0,\ \delta=0, \\ \psi=\arcsin\frac{\Delta_R}{(\Delta_R^2+\Delta_I^2)^{1/2}}-\arcsin\frac{\Delta_I(1+g)}{(\Delta_R^2+\Delta_I^2)^{1/2}}+2\pi n, \\ t_p=\left[\frac{\Delta_R}{\cos\psi-(1+g)}\right]^{1/2},\quad \mathscr{E}_p=\frac{1}{t_p}\left(\frac{2\Delta_R}{\zeta_R}\right)^{1/2}. \end{gathered} \tag{32}$$

Equations (32) can be used to determine the link between the new results and the existing solutions of Eq. (28) with real coefficients. For example, if the transition from $\Delta_I = 0$, $\zeta_I = 0$ to small but finite quantities results from the solution (30), this transition is hard. A fast frequency drift (characterized by a) appears in this case in the dynamic spectrum of the emission, and the phase shifts immediately by $\pi/2$. We recall by way of comparison that according to (26), in a maser with a slow, saturable absorber a small inequality in the spacing of the spectrum produced a soft change in the characteristics of the pulse train.

In the solutions (30)–(32), we must take into account the conditions (19) necessary for the existence of the pulse train and for its stability. In particular, if $g<0$, the inequalities necessary for the solution (32) are satisfied.

6. DISCUSSION

We have found that there are analogs of slow and fast nonlinear saturable absorbers in cyclotron plasma masers. After some simplifications, the Haus equations with complex coefficients can be used to study passive mode locking and to determine the shape of the electromagnetic pulse that oscillates between resonator ends. In this paper we found soliton solutions of these equations and determined the characteristics of pulse trains with a large off-duty factor, including the frequency variation of the electromagnetic emission spectrum which does not correspond to the linear dispersion equation. These results are probably of interest in the research not only into plasma masers but also into certain types of neodymium-glass[15] and dye lasers. The effect of an unequally spaced frequency spectrum on active and passive mode locking in lasers was studied in Refs. 16 and 17. These studies used a mode-dependent approach that did not determine the shape of the electromagnetic pulse.

The theory we developed can be used to accurately describe the processes occurring in laboratory maser systems considered in Ref. 10. Passive mode locking in open magnetic bottles can be used to form an electromagnetic pulse train with a scale considerably smaller than that of besters detected in Ref. 18. In such experimental studies, it is advisable to focus attention on the phase shift of electromagnetic signals observed near the coupled ends of a magnetic bottle. The fact that the signals are out of phase indicates in this case that passive mode locking has been established.

The theory of cyclotron instability in the Van Allen radiation belts is an important area of application of the results obtained by us. An important problem of this theory, which has so far not been solved completely, is how to interpret the excitation of the fine "chorus" structure of the whistlers. Chorus emissions usually have several time scales. A quasiperiodicity with a time scale of several tenths of a second seems to be caused by the bounce-resonance effects.[19] On the other hand, a quasiperiodicity with a time scale of several seconds, which roughly corresponds to the period of group propagation of whistler waves along a resonator, is attributable to the processes we have analyzed in this paper. In the geomagnetic-fluctuation range, this study may prove to be useful in refining the theory of pearl production.[20] This fluctuation range probably corresponds to pulses of Alfvén waves that propagate through the proton radiation belt that lies between the coupled regions of the ionosphere.

Wave propagation in Jupiter's radiation belts may be another interesting area of application of our theory. The density of background plasma in the Jupiter's magnetosphere is relatively high only in the equatorial region because of the planet's high angular velocity.[21] This circumstance automatically leads to a small region of cyclotron interaction between whistlers and energetic electrons. Under these conditions, the electromagnetic emission can be modulated near the self-excitation threshold with a period approximately equal to the period of group propagation of the whistlers. Near the Io plasma torus, for example, this period is ~ 100 s. Thus far, there have been no continuous spectral measurements carried out in the Jovian magnetosphere. Only some isolated experiments have been carried out at particular moments during the flight of space probes. These experiments have revealed the presence of a fine chorus structure in the emissions.[22] This structure seems to correspond to the bounce oscillations of energetic electrons.

We wish to point out in conclusion that the analogy between the operating regime of masers and lasers is not restricted to the process considered here. Cyclotron plasma masers, just as lasers, can operate even under slower quasiperiodic regimes, in which the electromagnetic radiation in the resonator is distributed approximately uniformly. Even in this case, however, the processes occurring in cyclotron plasma masers have unique features that are discussed in Refs. 12 and 13.

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APPENDIX

Absorption saturation in the case of a cyclotron interaction between whistlers and energetic electrons

The cyclotron interaction between whistlers and energetic electrons and the mathematical description of this interaction depend in many respects on the ratio of the period of bounce oscillations of particles between the mirror points, $T_b = 2\pi l_B/v_\perp$ (l_B is the scale of the longitudinal nonuniformity of the magnetic field) to the period of group propagation of whistlers along the resonator, $T = kl/\omega$ (l is the length of a magnetic tube of force). We assume that the distribution function anisotropy is moderately large, so that, according to Ref. 12, the dominant whistlers are those whose frequencies ω are lower than the gyrofrequency ω_B. Using the cyclotron-resonance condition $\omega - k_{\parallel} v_{\parallel} = \omega_B$, we easily see that

$$T_b \approx (2\pi l_B \omega / l \omega_B) T \ll T. \tag{A1}$$

This ratio of time scales allows us to use an expression for whistler-wave enhancement, averaged over the particle-bounce oscillation periods.[12] For low-energy pulses, we can use expression (16), in which the linear gain is given by

$$g = -\frac{|\ln R|}{T} + \int_0^\infty \int_0^1 K \frac{\partial f_0}{\partial \varkappa} d\varkappa\, dv, \tag{A2}$$

where R is the effective coefficient of reflection of waves from the magnetic-bottle ends; this coefficient also takes other linear losses into account. The nonlinear corrections to the gain, in accordance with (12), (15), and (16), are given by

$$\alpha = \int_0^\infty \int_0^1 K \frac{\partial^2}{\partial \varkappa^2} \left(D \frac{\partial f_0}{\partial \varkappa} \right) d\varkappa\, dv, \tag{A3}$$

$$\beta = -\frac{1}{2} \int_0^\infty \int_0^1 K \frac{\partial^2}{\partial \varkappa^2} \left[D \frac{\partial^2}{\partial \varkappa^2} \left(D \frac{\partial f_0}{\partial \varkappa} \right) \right] d\varkappa\, dv. \tag{A4}$$

Equations (A2)–(A4) automatically take account of the fact that in the case of waves of relatively low frequencies, the particles diffuse in the velocity space not in accordance with the velocity modulus v, but primarily at pitch angles characterized by $\varkappa = |v_{\parallel}|/v$, at the center of the magnetic bottle.

The distribution may be arbitrary in many respects if a system is operated below the self-excitation threshold. To shorten the equations, we assume that the distribution is characterized by a small velocity spread and that it is nonzero only when $\varkappa_c < \varkappa < \varkappa_m$, where $\varkappa_c = (B_{\min}/B_{\max})^{1/2}$ corresponds to the loss-cone limit in the velocity space, and $\varkappa_m \ll 1$. In this case, the smooth positive functions, which appear in (A2)–(A4) as the coefficients

$$D(\varkappa, v) = D_0 \varkappa, \quad K(\varkappa, v) = K_0 \varkappa^2, \tag{A5}$$

according to the conclusions of Ref. 12, can be simplified considerably.

Our goal in this Appendix will be reached if we can give an example of a distribution in which α and β are positive. Furthermore, according to (A2), the self-excitation threshold can be approached at a reasonably high resonator Q if the contribution to cyclotron linear gain (A2), characterized by the integral

$$J = \int K (\partial f_0 / \partial \varkappa)\, d\varkappa\, dv,$$

is positive. Since quasilinear relaxation may be appreciable in the magnetic tubes of force adjacent to the tube under study, we will slightly complicate our problem by requiring that the distribution function satisfy the boundary conditions: $f_0 = 0$ for $\varkappa = \varkappa_c$ and $D\partial f_0 / \partial \varkappa = 0$ for $\varkappa = \varkappa_m$.

We can write the distribution function with the properties of interest in the form

$$f_0 = A\xi [(1-\xi)^2 (P - \xi^2) + Q(2-\xi)] E\left(\frac{1}{\xi} - 1 \right) \delta(v - v_0), \tag{6}$$

where E and δ are the unit-step function and the delta-function; $\xi = (\varkappa - \varkappa_c)/(\varkappa_m - \varkappa_c)$; and A, P, and Q are positive constants, with $P > 1$. The distribution function (A6) is positive, localized in the chosen part of the velocity space, and satisfies the given boundary conditions. We can now easily show that this function can ensure positive values of α, β, and the integral J with the coefficients (A5) that we introduced above. If, for example, the loss cone is sufficiently narrow, we will have $P = 4$ and $Q = 3$.

[1)]The second derivative with respect to time can also be taken into account in this expansion essentially without complicating the calculations that follow.

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