Contribution to the hydrodynamics of an anisotropic superfluid

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The hydrodynamics of an anisotropic superfluid with the symmetry of the A phase of ^{3}He is discussed. The equations of nonlinear hydrodynamics formulated in the present paper do not reduce to the Hu-Saslov hydrodynamic equations, differing also from the Khalatnikov-Lebedev canonical equations by the assumption of a more general dependence of the spontaneous intrinsic orbital angular momentum on the density and the temperature. The gauge wheel effect is considered on the basis of the formulated theory. It is shown that this effect disappears at $T=0$ upon the elimination of the arbitrariness connected with the choice of the definition of the chemical potential. It is impossible to obtain this result from the Hu-Saslov theory, since it is impossible to go over to the $T\rightarrow 0, \rho_n \rightarrow 0$ limit in it: the equation for the superfluid velocity is incompatible with the equation for the momentum conservation law. The thermorotation effect, which is connected with the existence in an anisotropic superfluid of an intrinsic orbital angular momentum and the dependence of its magnitude on the temperature, is also considered.

I. INTRODUCTION

The hydrodynamics of the A phase of superfluid ^{3}He, despite the large number of papers that have been published on it (see Refs. 1–12 and the papers cited therein), still remains a subject of discussions and debate. The difficulties encountered in the formulation of the hydrodynamic equations are connected, on the one hand, with the serious discrepancies among the results obtained in the computations of the A phase's spontaneous intrinsic orbital angular momentum,[13–19] which exists because the Cooper pairs in the A phase occur in a quantum state with a definite orbital angular momentum oriented parallel to the unit orbital vector **l**. On the other hand, the spontaneous intrinsic angular momentum makes the liquid anisotropic, which gives rise to the problem of how to make the correct allowance for the angular momentum conservation law, a problem which was encountered in the theory of liquid crystals long before the discovery of the superfluid phases of ^{3}He (Ref. 20).

Two methods of approach can be used in formulating the hydrodynamic equations. The first method is purely phenomenological, and amounts to the search for that form of the equations which would correspond to the symmetry of the liquid, and satisfy the conservation laws. This approach turned out to be extremely fruitful for He II (Ref. 21). But in this method we cannot avoid arbitrariness, which is especially important in the hydrodynamics of an anisotropic superfluid. Thus, the nonlinear hydrodynamic equations formulated by Hu and Saslov[3] and Pleiner and Brand[11] are, despite the fairly large number of parameters, by no means the most general hydrodynamic equations satisfying the symmetry and conservation-law requirements. It is impossible to obtain from these equations the hydrodynamic equations obtained by Khalatnikov and Lebedev[2] with the aid of the canonical formalism. According to Ref. 2, the arbitrariness in the formulation of the hydrodynamic equations results from the addition to the canonical equations of corrections necessary for the consideration of dissipation. Then there also appear reaction terms that do not contribute to the dissipation, but alter the form of the hydrodynamic equations. The system of hydrodynamic equations proposed in the present paper also cannot be obtained from the equations of Hu and Saslov[3] by any choice of the parameters of their theory, but coincides at $T=0$ with the Khalatnikov-Lebedev canonical equations,[2] differing from them at $T>0$ by a more general dependence of the intrinsic orbital angular momentum on the temperature and the pressure. We shall, for simplicity, limit ourselves to the exposition of a hydrodynamic theory that neglects dissipation, since the introduction of dissipative terms into the hydrodynamic equations offers no difficulty and is noncontroversial.

Another, more rigorous but correspondingly more complicated approach consists in the direct derivation of the hydrodynamic equations from the microscopic theory. For a Fermi liquid it might be expected that such a derivation would be carried out in two limiting cases. The first is the weak-coupling case, in which the BCS theory is applicable. The second is the strong-coupling case, in which two fermions form a molecule with dimension much smaller than the interparticle separation. Such a molecule can be considered to be a point boson, and we arrive at the model of a slightly nonideal Bose gas whose particles possess nonzero spin and orbital angular momentum. Superfluid ^{3}He corresponds to the first case, and is described by the BCS theory. But it is significantly more complicated to derive hydrodynamics from the weak-coupling theory than from the theory of the nonideal Bose gas because of the difficulties encountered in the determination of the intrinsic angular momentum of the Cooper pairs. Therefore, lately, researchers, in analyzing the hydrodynamics of the A phase of ^{3}He, have often turned to the theory of the point-boson gas.[7,18] Although the latter theory does not bear a direct relation to the A phase, it allows us to understand certain general characteristics of the hydrodynamics of an anisotropic superfluid, and, furthermore, in prospect, it can find application in future in the description of other anisotropic superfluids, e.g., molecular hydrogen.[22] The derivation of hydrodynamics from the theory of point bosons with intrinsic angular mo-

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mentum has not been published before,[1)] and in the present paper such a derivation is presented (Appendix A) for the simple $T=0$ case on the basis of a modified Gross-Pitaevskiĭ theory[23,24]; the difference between it and the original theory is that the wave function of the condensate bosons is not a scalar, but a three-dimensional vector, which reflects the possession of intrinsic angular momentum by the bosons.

The phenomenological system of equations formulated in the present paper agrees with the Gross-Pitaevskiĭ hydrodynamic theory at $T=0$ if the intrinsic angular momentum L figuring in the phenomenology as an arbitrary function of the density ρ and entropy S tends, as $T\to 0$, to the maximum value (the saturation angular momentum) $\rho\hbar/M$ (M is the boson mass). Notice that the Khalatnikov-Lebedev canonical equations[2] can be obtained from our system of equations if the equality $L=\rho\hbar/M$ is satisfied at any temperature. According to Khalatnikov and Lebedev, we can obtain hydrodynamics with arbitrary L, like, incidentally, any other variant of hydrodynamic theory that is noncontradictory and satisfies the symmetry and conservation-law requirements, through an appropriate choice of reaction terms.

The main purpose of the present paper is to consider effects peculiar to anisotropic superfluids, including the gauge-wheel effect, in particular, its magnitude at $T\to 0$. This effect was first pointed out by Liu and Cross.[5] It is due to the term proportional to ($\mathbf{l}\cdot\operatorname{curl}\mathbf{v}_n$) in the equation for the phase (the Josephson equation). It would appear that, when in the limit as $T\to 0$ the normal component also disappears, this effect should vanish, but the term ($\mathbf{l}\cdot\operatorname{curl}\mathbf{v}_n$) in the Josephson equation vanishes in this limit not for any manner of defining the chemical potential,[6] and different authors have, depending on their choice of the chemical potential, arrived at different conclusions concerning the magnitude of the gauge-wheel effect.[6,19] The dependence of the magnitude of the effect on the choice of the chemical potential means that this effect cannot be estimated from equations containing the chemical potential, since it is not directly observable, and cannot be uniquely determined. Therefore, we preferred to eliminate the chemical potential from the Josephson equation by expressing it with the aid of the Gibbs-Duhem relation in terms of observable hydrodynamic variables, including pressure and temperature. As a result, it turned out that the term ($\mathbf{l}\cdot\operatorname{curl}\mathbf{v}_n$) enters into the Josephson equation with a factor that tends to zero in their limit as $T\to 0$, provided the angular momentum L in this limit tends to the saturation angular momentum $\rho\hbar/M$. Thus, the gauge-wheel effect vanishes at $T=0$, which, as it seems to us, is necessary for any reasonable hydrodynamic theory.

According to the Hu-Saslov theory,[3] the gauge-wheel effect does not vanish at $T=0$, since the term ($\mathbf{l}\cdot\operatorname{curl}\mathbf{v}_n$) remains in the Josephson equation after the elimination of the chemical potential from this equation. The disagreement with our result here stems from the different definition of pressure. In this connection, in Appendix B we discuss the nontrivial—for anisotropic superfluids—question whether indeed the experimentally determined pressure should coincide with the pressure that enters into our hydrodynamic equations. Let us also note that it is impossible to obtain from the Hu-Saslov hydrodynamic theory hydrodynamics at $T=0$ with a vanishing normal component, since for this case the Hu-Saslov equation for the superfluid velocity is not consistent with the equation for the momentum conservation law.

In the paper we consider another phenomenon characteristic of an anisotropic superfluid: the thermorotation effect. It consists in the fact that temperature oscillations in a freely suspended vessel containing a superfluid should give rise to torsional oscillations of vessl. The magnitude of this effect, like that of the gauge wheel effect, depends on the value of the intrinsic orbital angular momentum. Estimates show that the thermorotation effect is entirely accessible to experimental observation.

2. EQUILIBRIUM. THERMODYNAMIC RELATIONS

Let us choose as the hydrodynamic variables determining the state of the liquid the density ρ, the entropy S, the total flux $\mathbf{j}$, the superfluid velocity $\mathbf{v}_s$, and the orbital vector $\mathbf{l}$. To these let us also add the magnitude L of the intrinsic orbital angular momentum (the modulus of the intrinsic angular momentum vector $\mathbf{L}=L\,\mathbf{l}$), which, in hydrodynamics, is not an independent variable, being a function of the other variables; but we shall introduce the corresponding relation later.

The expression for the change in the energy density (the Gibbs relation) has the following form:

$$dE=\mu\,d\rho+T\,dS+\mathbf{v}_n\,d\mathbf{j}+\mathbf{j}_{sn}\,d\mathbf{v}_s+\mathbf{h}\,d\mathbf{l}+\omega_L\,dL+\nabla_j\left(\frac{\partial E}{\partial\nabla_j a_\alpha}da_\alpha\right). \tag{1}$$

We take into account the dependence of the energy on not only the eleven thermodynamic variables (ρ, S, $\mathbf{j}$, $\mathbf{v}_s$, and $\mathbf{L}=L\,\mathbf{l}$), but also their space derivatives, since this dependence can be important for a liquid with intrinsic angular momentum.[18] Therefore, the factors in front of the differentials in the relation (1) are functional derivatives of the energy:

$$\mu=\frac{\delta E}{\delta\rho}=\frac{\partial E}{\partial\rho}-\nabla_j\left(\frac{\partial E}{\partial\nabla_j\rho}\right),$$

$$\mathbf{v}_n=\frac{\delta E}{\delta\mathbf{v}_n}=\frac{\partial E}{\partial\mathbf{v}_n}-\nabla_j\left(\frac{\partial E}{\partial\nabla_j v_n}\right),\ \text{etc.} \tag{2}$$

We have also retained in (1) divergence-type terms, which reduce to a surface energy (the sum over all the eleven thermodynamic variables a_α, where $\alpha=1,2,\ldots,11$). To allow for them in the case of the thermodynamic theory would, apparently, be exceeding the accuracy of the theory, and they can be discarded right away. But we shall, for reasons that will become clear after we have given the definition of pressure, retain them for the time being.

Since the intrinsic orbital angular momentum $\mathbf{L}=L\,\mathbf{l}$ is treated as a separate variable, the vortex current $\frac{1}{2}\operatorname{curl}\mathbf{L}$ corresponding to it is not included in either the total current $\mathbf{j}$ or the superfluid current $\mathbf{j}_{sn}=\mathbf{j}-\rho\mathbf{v}_n$ defined in the coordinate system moving with the normal velocity $\mathbf{v}_n$. Let us also note that the vector $\mathbf{l}$ and the superfluid velocity $\mathbf{v}_s$ are not entirely independent variables, since they are connected by the Mermin-Ho relations.[25] These relations are a consequence of the fact that the phase of the order parameter is at the same time the angle of rotation about the vector **l**. Since the three-dimensional rotation group is noncommutative, the phase Φ is not globally defined: only small changes in it are well defined, and the operations of differentiation, as applied to it, do not commute. We have

$$\delta_1\delta_2\Phi-\delta_2\delta_1\Phi=[\delta_1\mathbf{l}\times\delta_2\mathbf{l}]\,\mathbf{l}. \tag{3}$$

As a result, we obtain for the change in the superfluid velocity $\mathbf{v}_s=(\hbar/M)\nabla\Phi$ when $\delta_1\to d$ and $\delta_i\to\nabla_i$ the expression

$$d(v_s)_i=\frac{\hbar}{M}\nabla_i\,d\Phi+\frac{\hbar}{M}[\nabla_i\mathbf{l}\times\mathbf{l}]\,d\mathbf{l}. \tag{4}$$

We can derive the relation between curl $\mathbf{v}_s$ and the vector **l** in the same fashion:

$$\operatorname{rot}\mathbf{v}_s=\frac{\hbar}{2M}\varepsilon_{ijk}[\nabla l_j\times\nabla l_k]\,l_i. \tag{5}$$

Let us now derive an expression for the pressure P, which, in thermodynamics, is defined as the volume derivative of the energy, the total mass, the entropy, the momentum, and the angular momentum contained in the volume remaining unchanged when the volume is changed, i.e.,

$$\rho V=\text{const},\quad SV=\text{const},\quad \mathbf{j}V=\text{const},\quad ([\mathbf{r}\times\mathbf{j}]+\mathbf{L})\,V=\text{const},$$

where V is the small liquid volume under consideration. Hence

$$\frac{\partial\rho}{\partial V}=-\frac{\rho}{V},\quad \frac{\partial S}{\partial V}=-\frac{S}{V},\quad \frac{\partial\mathbf{j}}{\partial V}=-\frac{\mathbf{j}}{V},\quad \frac{\partial L}{\partial V}=-\frac{L}{V}, \tag{6}$$

and for the pressure

$$P=-\frac{\partial}{\partial V}(EV)=-E-\frac{\partial E}{\partial V}V \tag{7}$$

we obtain the expression

$$P=\mu\rho+TS+\mathbf{v}_n\mathbf{j}+\omega_L L-E. \tag{8}$$

Differentiating (8), and using the relation (1), we obtain the Gibbs-Duhem relation

$$dP=S\,dT+\rho\,d\mu+\mathbf{j}\,d\mathbf{v}_n+L\,d\omega_L-\mathbf{j}_{sn}\,d\mathbf{v}_s-\mathbf{h}\,d\mathbf{l}-\nabla_j\left(\frac{\partial E}{\partial\nabla_j a_\alpha}da_\alpha\right). \tag{9}$$

We could, if we remained within the scope of thermodynamics, discard the divergence-type terms both in the relation (1) and in (9). But P is not only a thermodynamic, but also a dynamical, quantity, which enters into the hydrodynamic equations (into the momentum-flux tensor). When it serves in this capacity, the divergence-type terms are important for it, and, in particular, these terms are important for the gauge-wheel effect under discussion in the present paper. We could discard the divergence-type terms in the relation (1) for the energy, but then, in order to retain them in the Gibbs-Duhem relation (9), we would have to postulate that the pressure entering into our hydrodynamic theory does not coincide with the thermodynamic pressure defined by the formula (8).

In discussing the gauge-wheel effect, we gave much attention to the question which chemical potential is the true one, i.e., remains constant for the equilibrium state.[6,8] The equilibrium state of the liquid is determined from the condition for the minimum of the energy for given total mass, entropy, momentum, and angular momentum, i.e., from the condition for a minimum of the following functional:

$$\int dV\{E-\mu_0\rho-T_0S-\mathbf{v}_0\mathbf{j}-\boldsymbol{\Omega}([\mathbf{r}\times\mathbf{j}]+\mathbf{L})\}, \tag{10}$$

where μ_0, T_0, $\mathbf{v}_0$, and $\boldsymbol{\Omega}$ are Lagrangian multipliers. Varying this functional with respect to ρ, S, the phase Φ, and the vector **l**, and taking account of the Mermin-Ho relation (4), we obtain the conditions for equilibrium:

$$\begin{gathered}\mu=\mu_0,\quad T=T_0,\quad \mathbf{v}_n=\mathbf{v}_0+[\boldsymbol{\Omega}\times\mathbf{r}],\\ \omega_L=\boldsymbol{\Omega}\mathbf{l},\quad \operatorname{div}\mathbf{j}_{sn}=0,\quad \mathbf{h}+\frac{\hbar}{M}[(\mathbf{j}_{sn}\nabla)\mathbf{l}\times\mathbf{l}]=L\boldsymbol{\Omega}.\end{gathered} \tag{11}$$

Thus, the chemical potential introduced above indeed determines the state of equilibrium.

As has already been mentioned, in hydrodynamics, the angular momentum L is not an independent hydrodynamic variable, and equilibrium is fairly rapidly established with respect to it. Therefore, in accordance with (11), we should everywhere set the quantity ω_L equal to

$$\omega_L=\boldsymbol{\Omega}\mathbf{l}={}^1/_2(\mathbf{l}\operatorname{rot}\mathbf{v}_n). \tag{12}$$

From this we can determine the dependence of L on the remaining thermodynamic variables. We shall, for simplicity, assume that the angular momentum L is a function of only ρ and S, i.e., that

$$dL=\frac{dL}{d\rho}d\rho+\frac{dL}{dS}dS. \tag{13}$$

Eliminating dL from the relation (1) with the aid of (13), we obtain a new relation containing the following renormalized values of the chemical potential and the temperature:

$$\begin{gathered}\tilde\mu=\mu+{}^1/_2(\mathbf{l}\operatorname{rot}\mathbf{v}_n)\,dL/d\rho,\\ \tilde T=T+{}^1/_2(\mathbf{l}\operatorname{rot}\mathbf{v}_n)\,dL/dS.\end{gathered} \tag{14}$$

But the conditions (11) for equilibrium contain renormalized μ and T values defined at constant L, and it is precisely these values that are therefore the true chemical potential and temperature. This is due to the fact that, having distinguished the angular momentum L as an independent variable in our phenomenological scheme, we need an independent angular-momentum conservation law besides the momentum conservation law. To obtain the thermodynamic relation not containing dL, but at the same time containing the true μ and T values, we must renormalize the total and superfluid currents by including the vortex current $\frac{1}{2}\operatorname{rot}\mathbf{L}$ in them, i.e., in place of the currents **j** and $\mathbf{j}_{sn}$ we must introduce the currents

$$\mathbf{g}=\mathbf{j}+{}^1/_2\operatorname{rot}\mathbf{L},\quad \boldsymbol{\lambda}=\mathbf{j}_{sn}+{}^1/_2\operatorname{rot}\mathbf{L}. \tag{15}$$

For such a choice of the variables the angular momentum conservation law does not come out as an independent conservation law, but is a direct consequence of the law of conservation of momentum with density **g**.

The transition from the currents **j** and $\mathbf{j}_{sn}$ to the currents **g** and $\boldsymbol{\lambda}$ is effected through the following formal transformation of the form of the energy density:

$$\begin{gathered}\tilde E(\mathbf{g},\ldots)=E(\mathbf{g}-{}^1/_2\operatorname{rot}\mathbf{L},\ldots)+\Delta E,\\ \Delta E={}^1/_2\operatorname{rot}\mathbf{L}\cdot\mathbf{v}_s-\boldsymbol{\omega}(\mathbf{l})\mathbf{L},\end{gathered} \tag{16}$$

where $E(\mathbf{j},\ldots)$ is the previous expression given for the energy in terms of the current $\mathbf{j}$ and $\boldsymbol{\omega}(\mathbf{l})$ is the right-hand side of the Mermin-Ho relation (the formula (5)). The correction ΔE reduces to a surface energy, but it alters the form of the $\mathbf{v}_s$ and $\mathbf{l}$ derivatives and their gradients, as a result of which the functional derivative $\delta\hat{E}/\delta v_s$ gives the current $\boldsymbol{\lambda}$, and not the current $\mathbf{j}_{sn}$. The principal thermodynamic relation assumes the following form instead of the form (1):

$$d\hat{E}=\mu\, d\rho+T\, dS+\mathbf{v}_n\, d\mathbf{g}+\boldsymbol{\lambda}\, d\mathbf{v}_s+\boldsymbol{\psi}\, d\mathbf{l}+\nabla_j\left(\frac{\partial\hat{E}}{\partial\nabla_j a_\alpha}\, da_\alpha\right). \quad (17)$$

Let us give the relations between the old and the new derivatives of the energy for those cases in which they differ from each other (the relation between $\boldsymbol{\lambda}=\delta\hat{E}/\delta\mathbf{v}_s$ and $\mathbf{j}_{sn}=\delta E/\delta\mathbf{v}_s$ has already been given in (15)):

$$\boldsymbol{\psi}=\mathbf{h}-{}^1/_2\mathbf{L}\,\mathrm{rot}\,\mathbf{v}_n+\frac{\hbar}{2M}[\mathbf{l}\times(\mathrm{rot}\, L\nabla)\mathbf{l}],$$

$$\delta\hat{E}/\delta L=\omega_L-{}^1/_2(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n)=0. \quad (18)$$

The transition to the new currents also changes the expression for the pressure and the Gibbs-Duhem relation

$$P=\mu\rho+TS+\mathbf{v}_n\mathbf{g}-\hat{E}+{}^1/_2\,\mathrm{div}\,[\mathbf{v}_n\mathbf{L}], \quad (19)$$

$$dP=S\, dT+\rho\, d\mu+\mathbf{g}\, d\mathbf{v}_n-\boldsymbol{\lambda}\, d\mathbf{v}_s-\boldsymbol{\psi}\, d\mathbf{l}-\nabla_j\left(\frac{\partial E}{\partial\nabla_j a_\alpha}\, da_\alpha\right)$$
$$+d\{{}^1/_2\,\mathrm{div}[\mathbf{v}_n\times\mathbf{L}]\}. \quad (20)$$

Thus, when we go over to the currents $\mathbf{g}$ and $\boldsymbol{\lambda}$, the differential dL disappears from the thermodynamic relation for the change in the energy (the formula (17)); the relation then contains the true values of the chemical potential μ and the temperature T. There appears in the formula (19) for the pressure and, consequently, in the Gibbs-Duhem relation (20) the term $\frac{1}{2}\,\mathrm{div}\,[\mathbf{v}_n\times\mathbf{L}]$, which does not arise in the case when, following Hu and Saslov,[3] we use the thermodynamic expresison for P in the new variables, including the currents $\mathbf{g}$ and $\boldsymbol{\lambda}$. This is precisely the term that leads to the situation in which the terms giving the gauge-wheel effect in the Josephson equation from which the chemical potential has been eliminated have different magnitudes in the Hu-Saslov and our theories. In Appendix B we indicate the conditions under which the experimentally measured pressure should coincide with the quantity that we have introduced into the theory as pressure. In terms of the old variables, which treat the angular momentum L as an independent variable, our pressure coincides with the thermodynamic pressure, but after the elimination of L from the set of variables the pressure differs from the thermodynamic pressure by a quantity of the divergence type.

The thermodynamic relations presented above are Galilean-invariant. They also possess the correct transformation properties for the transformation into the system of coordinates rotating with angular velocity $\boldsymbol{\Omega}$. The energy E' in the rotating coordinate system is connected with the energy E in the laboratory coordinate system by the following relation[26]:

$$E'=E-\boldsymbol{\Omega}([\mathbf{r}\times\mathbf{j}]+\mathbf{L})=E-\mathbf{w}\mathbf{j}-(\mathbf{l}\boldsymbol{\Omega})L, \quad (21)$$

where $\mathbf{w}=\boldsymbol{\Omega}\times\mathbf{r}$. The formulas (1), (8), and (9) preserve their form in the rotating coordinate system if we take into account the fact that the functional derivatives of the energy in the two reference frames are connected by the following relations:

$$\mu'=\mu+\mathbf{v}_n\mathbf{w}-w^2, \quad \mathbf{h}'=\mathbf{h}-L\boldsymbol{\Omega}, \quad \omega_L'=\omega_L-(\boldsymbol{\Omega}\mathbf{l}). \quad (22)$$

When we go over to the currents that include the intrinsic-angular-momentum generated vortex current, then the relation between the energies $\hat{E}'$ and $\hat{E}$ in the rotating and laboratory coordinate systems

$$\hat{E}'=\hat{E}-\mathbf{w}\mathbf{g}-{}^1/_2\,\mathrm{div}\,[\mathbf{w}\times\mathbf{L}] \quad (23)$$

contains a divergence-type surface term that we cannot drop if we want the formula (19), which also contains a divergence-type term (the last term), to be valid in the rotating coordinate system as well.

3. THE HYDRODYNAMIC EQUATIONS

The system of hydrodynamic equations obeying all the conservation laws, and assuming the existence of an intrinsic angular momentum as an independent variable, can be written in the following form (the dissipation is neglected):

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\,\mathbf{j}=0, \quad (24)$$

$$\frac{\partial S}{\partial t}+\mathrm{div}\, S\mathbf{v}_n=0, \quad (25)$$

$$\frac{\hbar}{M}\frac{\partial\Phi}{\partial t}+\mu+\mathbf{v}_n\mathbf{v}_s+\frac{\hbar}{2M}\omega_L=0, \quad (26)$$

$$\frac{\partial j_i}{\partial t}+\nabla_j\Pi_{ij}=0, \quad (27)$$

$$\frac{\partial\mathbf{l}}{\partial t}+\left(\left(\mathbf{v}_n+\frac{\hbar\mathbf{j}_{sn}}{ML}\right)\nabla\right)\mathbf{l}=-\frac{1}{L}[\mathbf{l}\times\mathbf{h}], \quad (28)$$

$$\frac{\partial L}{\partial t}+\mathrm{div}\left(L\mathbf{v}_n+\frac{\hbar}{M}\mathbf{j}_{sn}\right)=G. \quad (29)$$

The momentum flux tensor is given by the expression

$$\Pi_{ij}=P\delta_{ij}+j_i(v_n)_j+(v_s)_i(j_{sn})_j+\nabla_i a_\alpha\frac{\partial E}{\partial\nabla_j a_\alpha}+\frac{1}{2}\varepsilon_{ijk}l_kG. \quad (30)$$

The Euler equation for the superfluid velocity $\mathbf{v}_s=(\hbar/M)\nabla\Phi$ is obtained through space differentiation of the equation (26) for the phase. In doing this we must take into account the noncommutativity of the operators of differentiation as applied to the phase (see the formula (4)):

$$\frac{\partial(v_s)_i}{\partial t}+\nabla_i\left(\mu+\mathbf{v}_s\mathbf{v}_n+\frac{\hbar}{2M}\omega_L\right)-\left[\frac{\partial\mathbf{l}}{\partial t}\times\nabla_i\mathbf{l}\right]\mathbf{l}. \quad (31)$$

From Eqs. (27)–(30) we obtain the balance equations for the intrinsic orbital angular momentum $\mathbf{L}$, the extrinsic orbital angular momentum $\mathbf{r}\times\mathbf{j}$, and the total orbital angular momentum $\mathbf{r}\times\mathbf{j}+\mathbf{L}$:

$$\frac{\partial L_i}{\partial t}+\nabla_j\left(L_i(v_n)_j+l_i(j_{sn})_j+\left[\mathbf{l}\times\frac{\partial E}{\partial\nabla_j\mathbf{l}}\right]_i\right)$$
$$=-\left[\mathbf{l}\times\frac{\partial E}{\partial\mathbf{l}}\right]_i-\left[\nabla_k\mathbf{l}\times\frac{\partial E}{\partial\nabla_k\mathbf{l}}\right]_i+Gl_i, \quad (32)$$

$$\frac{\partial[\mathbf{r}\mathbf{j}]_i}{\partial t}+\nabla_j\left\{\varepsilon_{ikl}r_k\Pi_{lj}+\left[\mathbf{j}\times\frac{\partial E}{\partial\nabla_j\mathbf{j}}\right]_i+\left[\mathbf{v}_s\times\frac{\partial E}{\partial\nabla_j\mathbf{v}_s}\right]_i\right\}$$

$$=-\left[\mathbf{j}\times\frac{\partial E}{\partial \mathbf{j}}\right]_i -\left[\mathbf{v}_s\times\frac{\partial E}{\partial \mathbf{v}_s}\right]_i-\left[\nabla_k\mathbf{j}\times\frac{\partial E}{\partial \nabla_k\mathbf{j}}\right]_i -\left[\nabla_k\mathbf{v}_s\times\frac{\partial E}{\partial \nabla_k\mathbf{v}_s}\right]_i-\left[\nabla a_\alpha\times\frac{\partial E}{\partial \nabla a_\alpha}\right]_i-Gl_i, \tag{33}$$

$$\frac{\partial([\mathbf{r}\times\mathbf{j}]_i+L_i)}{\partial t} +\nabla_j\left\{L_i(v_n)_j+l_i(j_{sn})_j+\left[\mathbf{b}_\beta\times\frac{\partial E}{\partial\nabla_j\mathbf{b}_\beta}\right]_i+\varepsilon_{ikl}r_k\Pi_{lj}\right\} =-\left[\mathbf{b}_\beta\times\frac{\partial E}{\partial \mathbf{b}_\beta}\right]_i-\left[\nabla_k\mathbf{b}_\beta\times\frac{\partial E}{\partial \nabla_k\mathbf{b}_\beta}\right]_i-\left[\nabla a_\alpha\times\frac{\partial E}{\partial \nabla a_\alpha}\right]_i, \tag{34}$$

where the $\mathbf{b}_\beta$ are the hydrodynamic vector variables $\mathbf{l}$, $\mathbf{v}_s$, and $\mathbf{j}$ ($\beta = 1,2,3$) and the a_α are all the variables, including the set of vector components ($\alpha = 1,2,\ldots,11$). The right-hand side of (34) is the total derivative of the energy with respect to the angle of rotation about the ith coordinate axis. It is clear that the invariance under such rotations is guaranteed by the total-angular-momentum conservation law.

The energy balance equation has the following form:

$$\frac{\partial E}{\partial t}+\operatorname{div}\mathbf{Q}=\left(\omega_L-\frac{1}{2}(\mathbf{l}\operatorname{rot}\mathbf{v}_n)\right)G, \tag{35}$$

where the energy flux is equal to

$$\mathbf{Q}=\mu\mathbf{j}+ST\mathbf{v}_n+(\mathbf{j}\mathbf{v}_n)\mathbf{v}_n+(\mathbf{v}_n\mathbf{v}_s)\mathbf{j}_{sn}+\omega_L\left(L\mathbf{v}_n+\frac{\hbar}{M}\mathbf{j}_{sn}\right) +\frac{G}{2}[\mathbf{l}\times\mathbf{v}_n]-\frac{\partial a_\alpha}{\partial t}\frac{\partial E}{\partial\nabla a_\alpha}. \tag{36}$$

The energy conservation law is fulfilled in those cases when the intrinsic and extrinsic angular momenta cannot be transformed into each other, i.e., when $G = 0$, or when equilibrium is established between them, and the equality (12) is satisfied. In the latter case the expression for G can be obtained from the consistency condition for the hydrodynamic equations (24), (25), and (29) with the expression (13) for the differential dL:

$$G=\left(\frac{\hbar}{M}-\frac{dL}{d\rho}\right)\operatorname{div}\mathbf{j}_{sn}+\left(L-\rho\frac{dL}{d\rho}-S\frac{dL}{dS}\right)\operatorname{div}\mathbf{v}_n. \tag{37}$$

After the dependence of the angular momentum $L(\rho,S)$ on ρ and S has been specified, Eq. (29) ceases to be independent and can be dropped. If, following Khalatnikov and Lebedev,[2] we assume that $L = (\hbar/M)\rho$ at any temperature, then our equations become identical to their equations; in that case $G = 0$, and Eq. (29) is identical to the equation of continuity (24).

Let us now go over from the currents $\mathbf{j}$ and $\mathbf{j}_{sn}$ to the currents $\mathbf{g}$ and $\boldsymbol{\lambda}$, which include the vortex current (see (15)). As a result, the equation (28) for $\mathbf{l}$, as well as the equation expressing the law of conservation of momentum, changes:

$$\frac{\partial\mathbf{l}}{\partial t}+\left(\left(\mathbf{v}_n+\frac{\hbar\lambda}{ML}\right)\nabla\right)\mathbf{l}-\frac{1}{2}[\operatorname{rot}\mathbf{v}_n\times\mathbf{l}]=-\frac{1}{\mathbf{L}}[\mathbf{l}\times\boldsymbol{\psi}], \tag{38}$$

$$\frac{\partial g_i}{\partial t}+\nabla_j\sigma_{ij}=0, \tag{39}$$

$$\sigma_{ij}=P\delta_{ij}+g_i(v_n)_j+(v_s)_i\lambda_j+\nabla_i a_\alpha\frac{\partial E}{\partial\nabla_j a_\alpha} -\frac{1}{2}(\operatorname{rot}\mathbf{L})_i(v_n)_j+\frac{1}{2}[\nabla_i\mathbf{L}\times v_n]_j +\frac{1}{2}\varepsilon_{ijp}\nabla_k(L_p(v_n)_k+l_p\lambda_k)+\frac{1}{2}(l_i\psi_j-l_j\psi_i) +\frac{1}{4}(L_i(\operatorname{rot}\mathbf{v}_n)_j-L_j(\operatorname{rot}\mathbf{v}_n)_i). \tag{40}$$

The relation between $\mathbf{h}$ and $\boldsymbol{\psi}$ is given in (18).

We can now compare our system of hydrodynamic equations with Hu and Saslov's equations[3] (see also Ref. 11). We can, ignoring the difference in notation and the fact that we make a more complete allowance for the gradient terms in the expression for the energy (Hu and Saslov took only the $\nabla\mathbf{l}$ dependence of the energy into consideration), obtain Hu and Saslov's equations by setting $L = 0$ in the expression (19) for the pressure, the Gibbs-Duhem relation (20), and the expression (40) for the momentum flux tensor, retaining, however, L in the equation (38) for $\mathbf{l}$ (the parameter β in the Hu-Saslov equations in $\hbar^2/M^2L$), and adding to the expression for the momentum flux tensor terms proportional to the parameters γ^1, γ^2, and γ^3 figuring in Hu and Saslov's expression for the momentum flux tensor. These terms should go to zero in the limit as $T\rightarrow 0$ and $\rho_n\rightarrow 0$ (see the following section).

4. HYDRODYNAMICS AT T = 0. THE GAUGE-WHEEL EFFECT

For the discussion of the hydrodynamics at $T = 0$, we shall need an explicit expression for the gradient energy, i.e., for that part of the free energy which depends on the gradients of the hydrodynamic quantities, including $\mathbf{v}_s = (\hbar/M)\nabla\Phi$, $\nabla\mathbf{l}$, as well as $\nabla\rho$ and ∇S. It is a generalization of the kinetic energy of the isotropic superfluid. But here we shall assume that the terms containing the gradients $\nabla\rho$ and ∇S can be collected into the ∇L-dependent term introduced into the expression for ∇E (see (16)). Then the gradient energy has the following form:

$$E_B={}^1/_2\rho_s v_s^2-{}^1/_2\rho_0(\mathbf{l}\mathbf{v}_s)^2+C\mathbf{v}_s\operatorname{rot}\mathbf{l}-C_0(\mathbf{l}\mathbf{v}_s)(\mathbf{l}\operatorname{rot}\mathbf{l}) +\mathrm{E}_l(\nabla\mathbf{l})+\Delta E, \tag{41}$$

where $E_l(\nabla\mathbf{l})$ is the $\nabla\mathbf{l}$-dependent energy. The $\mathbf{v}_s$ derivative of this energy gives in the laboratory system the superfluid current, which includes the vortex current $\frac{1}{2}\operatorname{rot}\mathbf{L}$:

$$\mathbf{g}_s=\partial E_B/\partial\mathbf{v}_s=\rho_s\mathbf{v}_s-\rho_0(\mathbf{l}\mathbf{v}_s)\mathbf{l}+C\operatorname{rot}\mathbf{l}-C_0(\mathbf{l}\operatorname{rot}\mathbf{l})\mathbf{l}+{}^1/_2\operatorname{rot}\mathbf{L}. \tag{42}$$

The current $\frac{1}{2}\operatorname{rot}\mathbf{L}$ includes the currents $\nabla\rho\times\mathbf{l}$ and $\nabla S\times\mathbf{l}$. Mermin and Muzikar,[18] who introduced the term $\sim\nabla\rho\times\mathbf{l}$ into the expression for the superfluid current of the A phase, also consider it to be part of the current $\frac{1}{2}$ curl $\mathbf{L}$.

Let us now consider the passage to the limit $T\rightarrow 0$, assuming that the normal excitations disappear in the process and $\rho_n\rightarrow 0$. A Galilean-invariant self-consistent theory is obtained in this limit when the following conditions are fulfilled:

$$\rho_s\to\rho,\quad \rho_0\to 0,\quad C\to 0,\quad C_0\to 0,\quad L\to(\hbar/M)\rho. \tag{43}$$

The limitation imposed by the conditions (43) on the magnitude of the intrinsic angular momentum L stems from the fact that the gas of normal excitations, which is the reservior with which the superfluid component could exchange angular momentum at $T\neq 0$, disappears at $T=0$. The role of such a reservior cannot be played by the extrinsic orbital angular momentum $\mathbf{r}\times\rho\mathbf{v}_s$, since the latter is not independent of the intrinsic angular momentum at $T=0$ because of the connection, imposed by the Mermin-Ho relation (5), between curl $\mathbf{v}_s$ and $\mathbf{l}$. Therefore, the angular momentum conservation law is obeyed only for a definite choice of the quantity $L=(\hbar/M)\rho$.

The fulfillment of the conditions (43) does not lead to the elimination of the velocity $\mathbf{v}_n$ from the hydrodynamic equations; this can be achieved only by redefining the chemical potential, which is equal to $\delta e/\delta\rho$ (Refs. 6, 8, 9, and 27), as well as the quantity $\delta E/\delta\mathbf{l}$. These functional derivatives were defined above for a given total current $\mathbf{g}$ and a given velocity $\mathbf{v}_s$, which cease to be independent variables because of the reduction in the number of degrees of freedom of the liquid at $T=0$. Therefore, although the thus introduced derivatives of the energy are constants for the equilibrium state (Ho and Mermin accordingly called μ the true chemical potential[6]), we find it convenient to use for the specification of $\delta E/\delta\rho$ and $\delta E/\delta\mathbf{l}$ at $T=0$ another procedure in which $\mathbf{v}_n$, the drift velocity of the nonexistent normal excitations, is eliminated from the hydrodynamic equations. But in this case $\mathbf{v}_n$ will appear in the conditions for equilibrium; but his should not be surprising, since here $\mathbf{v}_n$ is a parameter characterizing the coordinate system in which equilibrium obtains.

Proceeding to discuss the gauge-wheel effect, we, in order to get rid of the arbitrariness connected with the choice of the chemical potential, eliminate the latter from the Euler equation (31) with the aid of the Gibbs-Duhem relation (20). In the linear theory for the homogeneous texture, i.e., for $\mathbf{l}=$ const, we obtain

$$\frac{\partial\mathbf{v}_s}{\partial t}+\frac{1}{\rho}\nabla P-\frac{S}{\rho}\nabla T+\frac{1}{2}\left(\frac{\hbar}{M}-\frac{L}{\rho}\right)\nabla(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n)=0. \tag{44}$$

The term $\sim\nabla(\mathbf{l}\cdot\mathrm{curl}\,\mathbf{v}_n)$, which is responsible for the gauge-wheel effect, is absent when $L=(\hbar/M)\rho$. Therefore, according to the conditions (43), this effect disappears at $T=0$. This assertion presupposes that the pressure introduced by us in the phenomenological theory corresponds to the experimentally measurable pressure. This presupposition is discussed in Appendix. B. The Hu-Saslov theory[3] yields, after the elimination of the chemical potential from the Euler equation, Eq. (44) which $L=0$. The difference is due to the absence in the corresponding equations obtained by Hu and Saslov of the last term with L in Eqs. (19) and (20).

The total disappearance of the normal component at $T=0$ leads to a situation in which the equation for the total current $\mathbf{g}$ (the momentum conservation law) ceases to be independent, and should derive from the other hydrodynamic equations (for $\mathbf{v}_s$, ρ, and $\mathbf{l}$). In our hydrodynamics this is realized because of the conditions (43). But in the hydrodynamics of Hu and Saslov such a consistency condition is not fulfilled even in the linear theory. From their equation for the total current at $T=0$ in the case in which $\rho_n=0$ and $l=$ const, when $\mathbf{g}=\rho\mathbf{v}_s$, we obtain in the linear theory the equation

$$\begin{aligned}\frac{\partial\mathbf{v}_s}{\partial t}+\frac{1}{\rho}\{\nabla P+\gamma_\perp^{(2)}[\mathbf{l}\times\Delta\mathbf{v}_n]+(\gamma_\parallel^{(2)}-\gamma_\perp^{2)})\,[\mathbf{l}\times(\mathbf{l}\nabla)(\mathbf{l}\nabla)\mathbf{v}_n]\\ +\gamma_\perp^{(3)}([\mathbf{l}\times\nabla\,\mathrm{div}\,\mathbf{v}_n]\\ -\nabla(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n))+(\gamma_\parallel^{(3)}-\gamma_\perp^{(3)})\\ \times([\mathbf{l},\times\nabla(\mathbf{l}\nabla)(\mathbf{l}\mathbf{v}_n)]-\mathbf{l}(\mathbf{l}\nabla)(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n))\}=0.\end{aligned} \tag{45}$$

This equation does not agree with the Euler equation in the same theory (Eq. (44) with $T=0$ and $L=0$) for any choice of the parameters $\gamma_\parallel^2=\gamma_\parallel^3$, $\gamma_\perp^2$, and $\gamma_\perp^3$. Notice that terms with these parameters can, for generality, be added to our hydrodynamic theory, but for the condition, under discussion here, for consistency of the Euler equation and the momentum conservation law to be fulfilled, they should vanish at $T=0$.

A number of microscopic calculations performed in the BCS theory for the A phase (but not all such calculations (see the discussion and comparison in Refs. 7 and 18)) indicate that the parameter C_0 in the expression (42) for the current does not, in contravention of the conditions (43), go to zero as $T\to 0$. Having accepted this result, we can no longer use our hydrodynamics to describe the A phase with arbitrary $\mathbf{l}$ texture at $T=0$. Perhaps for this purpose we need to generalize the theory further by introducing for the order parameter an additional degree of freedom that will eliminate the limitations imposed by the conditions (43). Volovik and Mineev[7] followed another course, assuming that the normal component in the A phase with an inhomogeneous $\mathbf{l}$ texture does not disappear even at $T=0$. Further discussion of this problem should involve a microscopic analysis, and is beyond the scope of the present paper. Let us only note that the complete phenomenological theory should certainly describe the case in which $\rho_n=0$ as well, since this case is realized in a liquid of Bose molecules. All the specific effects are considered by us for a homogeneous texture, i.e., for $\mathbf{l}=$ const, and the question of the magnitude of the term $\sim\mathbf{l}(\mathbf{l}\cdot\mathrm{curl}\,\mathbf{l})$ in the expression for the current has, in the present case, no effect on the results of the analysis.

The gauge-wheel effect, which is determined by the last term in the left member of Eq. (44) consists in the following: the nonuniform rotation of the normal component (curl $\mathbf{v}_n\neq 0$) gives rise to a force that acts on the superfluid component. According to Ho and Mermin,[6] this effect can be observed if two rotating cylindrical vessels (or two such vessels executing axial oscillations with different amplitudes) are connected by a tube along their common axis. We can assume that the normal component in a vessel of sufficiently small height will, owing to the viscosity, be rigidly fixed to the vessel, so that the vessel will rotate as a rigid body, with the vector $\mathbf{l}$ everywhere parallel to the axis of the vessel. To observe the effect, we can confine ourselves to one vessel: its axial oscillations should lead to pressure and temperature oscillations whose amplitudes will depend on the conditions at the walls.

5. THE THERMOROTATION EFFECT

In an anisotropic superfluid there should also occur an effect that is, in a sense, the inverse of the gauge-wheel effect: temperature or pressure oscillations in a freely suspended cylindrical vessel should give rise to axial oscillations of the vessel. We shall consider only temperature oscillations, and call the corresponding effect a thermorotation effect. Qualitatively, it can be explained in the following manner. A change in temperature causes a change in the intrinsic orbital angular momentum. Owing to the total-angular-momentum conservation law, the extrinsic orbital angular momentum, which is determined by the normal component, and the related angular momentum of the vessel should then change. Let the vessel have a sufficiently small height, so that the normal component is rigidly fixed to the vessel and $\mathbf{l} = \text{const}$, i.e., the intrinsic angular momentum varies only in absolute value. The torque acting on the system "vessel + normal component" is equal to

$$K=\int dV\frac{\partial L}{\partial t}=\frac{\partial L}{\partial T}\int dV\frac{\partial T}{\partial t}. \tag{46}$$

Let us determine the temperature dependence of L, using the expression $L = (\hbar/M)\rho_s$ obtained in Ref. 28. Let the temperature oscillations be in resonance with the torsional oscillations of the vessel, and let the moment of inertia of the vessel be small compared to that of the liquid dragged by the vessel. We have then for the amplitude of the angular oscillations of the vessel that arise when the temperature-oscillation amplitude is equal to ΔT the following estimate:

$$\Delta\varphi\sim\frac{h}{MR^2}\frac{1}{\delta\omega_0}\frac{\Delta T}{\rho}\frac{d\rho_s}{dT}. \tag{47}$$

For a vessel radius $R \sim 1$ cm, an oscillation frequency $\omega_0 \sim 1$ rad/sec, a torsional-oscillation damping constant $\delta \sim 10^{-3}$, and $(\Delta T/\rho) d\rho_s/dT \sim 10^{-2}$, we obtain the entirely observable value $\Delta\varphi \sim 10^{-2}$.

In the more general approach the torque that acts on the vessel when the temperature or another hydrodynamic quantity is varied should be determined with the use of the equation for the total current $\mathbf{g}$. The torque K in (46) is the moment of a force equal to the time derivative of the vortex current $\frac{1}{2}$ curl $\mathbf{L}$, which is part of the total current $\mathbf{g}$. Generally speaking, any term of the type $[\Delta\rho\times\mathbf{l}]$ or $[\Delta\overline{S}\times\mathbf{l}]$ in the expression for the current will make a contribution to the torque. The formula (46) is based on the assumption made in the preceding section that all such terms have been brought together in the vortex current, which, for $\mathbf{l} = \text{const}$, is equal to $\frac{1}{2}\nabla L\times\mathbf{l}$.

6. CONCLUSION

In the present paper we have presented a hydrodynamic theory for an anisotropic superfluid possessing the same symmetry as the A phase of ^{3}He. We have considered the gauge-wheel and thermorotation effects, in which the anisotropy of, and the existence of spontaneous intrinsic orbital angular momentum in, the liquid manifest themselves. Particular attention has been given to the passage to the limit $T\to 0$, $\rho_n\to 0$. In this limit, because of the disappearance of the normal excitations, the liquid has a smaller number of degrees of freedom, and the Euler equation for the velocity $\mathbf{v}_s$ and the equation for the total current (the momentum conservation law) cease to be independent. Therefore, we must make sure the consistency condition for these equations is fulfilled. Although the A phase does not exist at $T = 0$, the fulfillment of this condition (in the Hu-Saslov hydrodynamics it is not fulfilled even in the linear theory) is important for the correct determination of the temperature dependence of the effects considered in the present paper. In particular, it follows from our phenomenological theory, as it does from Nagai's microscopic theory,[9] that the gauge-wheel effect disappears as $T\to 0$. The investigated effects were considered in that simple geometry in which a homogeneous texture $\mathbf{l} = \text{const}$ should exist. Therefore, the results of the analysis are not affected by the difficulty connected with the appearance in certain microscopic theories for the A phase of a term $\sim\mathbf{l}(\mathbf{l}\cdot\text{curl}\,\mathbf{l})$ in the current at $T = 0$. To overcome this difficulty, Volovik and Mineev[7] introduced at $T = 0$ a finite normal mass connected with the inhomogeneities of the $\mathbf{l}$ texture. It is clear that such a finite normal mass can, if necessary, be introduced into our theory as well.

The thermorotation effect investigated in the present paper manifests itself in the appearance of a torque exerted by the liquid on the vessel when the temperature of the liquid is varied. A torque will also arise when the pressure or the magnetic field is varied. The estimation of the latter effect requires the introduction into the theory of spin degrees of freedom. In this paper we consider the spin-locked regime, in which the spin vector $\mathbf{d}$ remains all the time parallel to $\mathbf{l}$.

In conclusion I wish to express my sincere gratitude to G. E. Volovik and V. P. Mineev for numerous fruitful discussions.

APPENDIX A

The modified Gross-Pitaevskii theory

We consider a slightly nonideal gas of point bosons whose internal structure has the same symmetry as the Cooper pairs in ^{3}He-A. Since fairly large temporal and spatial scales are considered in hydrodynamics, we shall assume the fulfillment of the conditions for a dipole-locked regime, when the spin vector $\mathbf{d}$ is not an independent variable, being parallel to the orbital vector $\mathbf{l} = \boldsymbol{\Delta}_1\times\boldsymbol{\Delta}_2$, where $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are two unit vectors orthogonal to each other. Then the boson state is determined by a complex wave function $\boldsymbol{\psi} = \boldsymbol{\psi}_1 + i\boldsymbol{\psi}_2$, where $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are two real vectors, orthogonal to each other and equal in magnitude; in this case $\boldsymbol{\Delta}_1 = \boldsymbol{\psi}_1/|\boldsymbol{\psi}_1|$ and $\boldsymbol{\Delta}_2 = \boldsymbol{\psi}_2/|\boldsymbol{\psi}_2|$.

In the Gross-Pitaevskiĭ theory the boson creation and annihilation operators in the Hamiltonian for the slightly nonideal Bose gas are replaced by c numbers. As a result, the Hamiltonian becomes a nonlinear functional of the classical field of the complex vector $\boldsymbol{\psi}$ (in Gross's[23] and Pitaevskiĭ's[24] papers the wave function is a scalar). Let us separate out the gradient energy in this functional, and represent the latter in the following form:

$$H=\frac{\hbar^2}{2M}\{K_1\nabla_i\psi_j^*\nabla_i\psi_j+K_2\nabla_i\psi_i^*\nabla_j\psi_j\}+H_0(\psi^*,\psi), \qquad (\mathrm{A}.1)$$

where the last term is a nonlinear functional of ψ^* and ψ (but not of their gradients), whose specific form will not be needed by us below. The invariant $\nabla_i\psi_j^*\nabla_j\psi_i$, which is not included in (A.1), reduces, after integration by parts, to the second invariant. Therefore, the expression (A.1) represents the gradient energy in a fairly general form.

The equations of motion are the Hamilton equations for the canonically conjugate fields ψ^* and ψ:

$$i\hbar\frac{\partial\psi}{\partial t}-\frac{\delta H}{\delta\psi^*}=0, \quad i\hbar\frac{\partial\psi^*}{\partial t}+\frac{\delta H}{\partial\psi}=0 \qquad (\mathrm{A}.2)$$

or for the pair of real fields ψ_1 and ψ_2:

$$\hbar\frac{\partial\psi_1}{\partial t}-\frac{\delta H}{\delta\psi_2}=0, \quad \hbar\frac{\partial\psi_2}{\partial t}+\frac{\delta H}{\partial\psi_1}=0, \qquad (\mathrm{A}.3)$$

where the functional derivatives are defined in the usual manner, i.e.,

$$\frac{\delta H}{\delta\psi}=\frac{\partial H}{\partial\psi}-\nabla_j\left(\frac{\partial H}{\partial\nabla_j\psi}\right).$$

The equations of motion can also be obtained in the Lagrangian formalism from the Lagrangian with density

$$\mathscr{L}=\frac{\hbar}{2}\left(\psi_2\frac{\partial\psi_1}{\partial t}-\psi_1\frac{\partial\psi_2}{\partial t}\right)-H(\psi_1,\psi_2). \qquad (\mathrm{A}.4)$$

Using the Noether theorem, we can obtain the laws of conservation of mass, energy, momentum, and angular momentum, which follow from the invariance of theory under gauge transformation, translations in time and in space, and three-dimensional rotations.

The mass conservation law:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}_M=0, \quad \rho=\frac{M}{2}(\psi_1{}^2+\psi_2{}^2),$$

$$(\mathbf{j}_M)_i=\frac{M}{\hbar}\left(\psi_1\frac{\partial H}{\partial\nabla_i\psi_2}-\psi_2\frac{\partial H}{\partial\nabla_i\psi_1}\right). \qquad (\mathrm{A}.5)$$

The energy conservation law:

$$\frac{\partial E}{\partial t}+\operatorname{div}\mathbf{Q}=0; \quad Q_i=-\frac{\partial\psi_1}{\partial t}\frac{\partial H}{\partial\nabla_i\psi_1}-\frac{\partial\psi_2}{\partial t}\frac{\partial H}{\partial\nabla_i\psi_2}. \qquad (\mathrm{A}.6)$$

The momentum conservation law:

$$\frac{\partial(j_P)_i}{\partial t}+\nabla_j\Pi_{ij}=0, \quad (j_P)_i=\frac{\hbar}{2}(\psi_1\nabla_i\psi_2-\psi_2\nabla_i\psi_1), \qquad (\mathrm{A}.7)$$

$$\Pi_{ij}=\delta_{ij}\mathscr{L}+\nabla_i\psi_1\frac{\partial H}{\partial\nabla_j\psi_1}+\nabla_i\psi_2\frac{\partial H}{\partial\nabla_j\psi_2}.$$

The conservation law for the total angular momentum, which consists of the intrinsic angular momentum $\mathbf{L}=\hbar\psi_1\times\psi_2$ and the extrinsic orbital angular momentum $\mathbf{r}\cdot\mathbf{j}_p$:

$$\frac{\partial}{\partial t}(L_i+[\mathbf{r}\mathbf{j}_P]_i)+\nabla_j J_{ij}=0, \qquad (\mathrm{A}.8)$$

$$J_{ij}=\left[\frac{\partial H}{\partial\nabla_j\psi_1},\psi_1\right]_i+\left[\frac{\partial H}{\partial\nabla_j\psi_2},\psi_2\right]+\varepsilon_{int}r_n\Pi_{tj}.$$

We see that in the general case the mass flux $\mathbf{j}_M$ does not coincide with the momentum density $\mathbf{j}_p$. They coincide if $K_2=0$ and $K_1=1$ in the Hamiltonian (A.1). It is only in this case that the theory is Galilean invariant; the momentum flux tensor Π_{ij} (see (A.7)) is then symmetric, and the intrinsic and extrinsic angular momenta are conserved separately.

Let us introduce in place of the vector fields ψ_1 and ψ_2 the density ρ (see (A.5)), the vector $\mathbf{l}$, and the variation of the phase Φ:

$$\mathbf{l}=2\frac{[\psi_1\psi_2]}{|\psi_1|^2+|\psi_2|^2}, \quad \delta\Phi=2\frac{\psi_1\,\delta\psi_2-\psi_2\,\delta\psi_1}{|\psi_1|^2+|\psi_2|^2}. \qquad (\mathrm{A}.9)$$

Then for Φ and $\mathbf{l}$ we obtain the equations

$$\frac{\partial\Phi}{\partial t}+\frac{\delta H}{\delta\rho}=0, \quad \frac{\hbar}{M}\rho\frac{\partial\mathbf{l}}{\partial t}+\left(\frac{\partial H}{\partial\mathbf{v}_s}\nabla\right)\mathbf{l}=-\left[\mathbf{l},\frac{\delta H}{\delta\mathbf{l}}\right]. \qquad (\mathrm{A}.10)$$

If the conditions ($K_2=0$ and $K_1=1$) for Galilean invariance are fulfilled, then these equations, together with the continuity equation (A.5), constitute a closed system of equations that is equivalent to the system, discussed in Sec. 4, of hydrodynamic equations for $T=0$. Here it should be borne in mind that the functional derivative $\delta H/\delta\rho$ is defined at fixed values of $\mathbf{v}_s$ and $\mathbf{l}$ and for the case when the current $\mathbf{j}_0=\mathbf{j}-\rho\mathbf{v}_s=0$, and therefore differs from the true chemical potential μ, which is defined for fixed values of $\mathbf{v}_s$, $\mathbf{l}$, L, and the current j (see Sec. 4).

APPENDIX B

Force acting on the surface bounding an anisotropic superfluid

Let us consider the momentum balance for an element of volume $\Delta S\Delta z$ including a section of the bounding surface (this can be the boundary with a diaphragm pressure gauge) with area ΔS. The z axis is oriented along the outward normal to the surface of the liquid. The balance equation has the form

$$\int\left(\frac{\partial g_i}{\partial t}+\nabla_j\sigma_{ij}\right)dx\,dy\,dz=0, \qquad (\mathrm{B}.1)$$

where the integration is performed over the volume $\Delta S\Delta z$. Further, we consider the limit $\Delta S\to0$, $\Delta z\to0$, where Δz becomes small compared to the hydrodynamic scales, but remains fairly large compared to the thickness of the layer in which the surface currents connected with the existence of the intrinsic angular momentum occur. We obtain

$$f_i=\sigma_{iz}|_--\left\{\frac{\partial g_i^{(S)}}{\partial t}+\nabla_j^{(S)}\sigma_{ij}^{(S)}\right\}. \qquad (\mathrm{B}.2)$$

Here $\sigma_{iz}|_-$ is the momentum flux from the liquid to the boundary surface, while $f_i=\sigma_{iz}|_+$ is the momentum flux from the wall, and is equal to the density f_i of the force exerted by the wall. Standing in the curly brackets in (B.2) is the contribution from the surface-current-related terms, which are singular at the surface. Here

$$g_i^{(S)}=\int_{-\Delta z/2}^{\Delta z/2}g_i\,dz, \quad \sigma_{ij}^{(S)}=\int_{-\Delta z/2}^{\Delta z/2}\sigma_{ij}\,dz$$

are the surface-current density and the surface-momentum flux tensor, and the gradient vector ∇^S has components only in the plane of the boundary.

We assume that the intrinsic angular momentum exists only in the liquid, and that, in the material of the wall, the current g_i and the momentum flux σ_{ij}, which include the contributions from the intrinsic angular momentum, coincide with the current j_i and the momentum flux Π_{ij}, which do not include these contributions. Therefore, we can repeat the arguments that led us to (B.2), considering the balance equation for the momentum with density **j**, and not **g**. As a result, instead of (B.2), we obtain

$$f_i=\Pi_{iz}|_{-}-\left\{\frac{\partial j_i^{(S)}}{\partial t}+\nabla_j^{(S)}\Pi_{ij}^{(S)}\right\}, \tag{B.3}$$

where the surface current $\mathbf{j}_S$ and the surface momentum flux Π_{ij}^{s} are obtained from **j** and Π_{ij}, using the same procedure as before. We can obtain Eq. (B.3) from (B.2) if we take into account the fact that the intrinsic angular momentum **L** has its own balance equation (32), according to which the contributions from **L** to g_i^S and Π_{ij}^S cancel each other out.

The transition from the current **g** to the current **j** does not, generally speaking, eliminate the singular surface terms, which can arise from the terms with z derivatives in the expressions for **j** and Π_{ij}. They give the quantity standing in the curly brackets in (B.3). In linear hydrodynamics only the last term in the expression for Π_{ij} can become singular (see (30) and (37)). But it is antisymmetric, and does not make any contribution ot the force f_z, which can be measured during the determination of the pressure. The curl **l**-related surface singularities in the current **j** can be eliminated if the boundary condition, adopted in the hydrodynamics of the A phase, that stipulates the perpendicularity of the vector **l** to the surface of the wall is fulfilled. A surface current could arise from the terms of the type $\nabla\rho\times\mathbf{l}$ and $\nabla S\times\mathbf{l}$ in the current **j**. But everywhere in this paper we assume that such terms appear only as part of the vortex current $\frac{1}{2}$ curl **L**, and therefore they occur only in the current **g**, and not in **j**. In this case the quantity standing in the curly brackets in (B.3) vanishes, and we can further verify that, in the linear theory, the magnitude f_z of the force density normal to the surface coincides with the pressure P.

[1)] Such a derivation has been carried out by G. E. Volovik, but has not been published by him.

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