The scarab transformation and magnetic analogies for three-dimensional curves in connection with a certain class of extremals

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The hodograph of the unit vector tangent to an arbitrary three-dimensional curve forms a curve on a sphere and can be "reprinted" on a plane by rolling the sphere. This "scarab transformation" is used to analyze several extremum problems of importance to the equilibrium of a plasma in a magnetic confinement system with a three-dimensional axis [B. A. Trubnikov and V. M. Glagolev, Sov. J. Plasma Phys. **10**, No. 2 (1984)], in the equilibrium of elastically deformed rods and fiber-optics cables, and in similar problems.

1. In this paper we introduce an operation, which we call the "scarab transformation," for arbitrary three-dimensional curves. Although this is a purely geometric operation, its analysis results in the formulation of several extremum problems which may find application in certain branches of physics, e.g., in calculating the equilibrium of deformed elastic rods and fiber-optics cables, in calculating the plasma equilibrium in a magnetic confinement system,[1] and in smaller problems.

The scarab transformation is defined by the following rules. For an arbitrary spatial curve $\mathbf{r}(s)$ we first write the Frenet equations

$$d\mathbf{r}/ds=\boldsymbol{\tau},\quad d\boldsymbol{\tau}/ds=K\mathbf{n},\quad d\mathbf{n}/ds=-K\boldsymbol{\tau}+\varkappa\mathbf{b}. \tag{1.1}$$

Here the coordinate s is the arc length along the curve, K is the curvature, $\varkappa$ is the torsion, and $\boldsymbol{\tau}$, $\mathbf{n}$, and $\mathbf{b}$ are three unit vectors. If s is replaced by the new quasitime variable $t=\int K ds$, the Frenet equations become[1)]

$$\dot{\boldsymbol{\tau}}=\mathbf{n},\quad \dot{\mathbf{n}}=\ddot{\boldsymbol{\tau}}=-\boldsymbol{\tau}-\omega[\dot{\boldsymbol{\tau}}\boldsymbol{\tau}], \tag{1.2}$$

where $\omega=\varkappa/K$ is the relative twisting, and the dot denotes a derivative with respect to t. This equation describes the motion of the end of the vector τ over a sphere of unit radius, $|\boldsymbol{\tau}|=1$, which we call the "hodograph sphere"; we call the $\boldsymbol{\tau}(t)$ trajectory itself the "spherical track" of the original curve. We then imprint this projection on a plane by the scarab method.

The scarab or dung beetle was held sacred in ancient Egypt since it constantly rolled with it a store of food in the form of a small ball, which was the symbol of the god Ra, worshipped by Egyptians. When a wet ball rolls over sand, the point of contact leaves tracks both on the ball itself (a spherical track consisting of the adhering grains of sand) and on the plane of the sand. Let us assume that there is no slippage or forcible rotation of the ball either as it sits in one place or as it rolls. If, for example, a ball of unit radius is rolled along a plane circle of R_1, no forcible additional rotations will occur if the ball rolls along with the cone circumscribed around it with vertex at the center of the plane circle. This circle is printed on the ball as a circle of radius

$$R_0=R_1/(1+R_1^2)^{1/2}. \tag{1.3}$$

For arbitrary trajectories this kinematic requirement must be satisfied for the instantaneous radii of curvature of the planar track (R_1) and the spherical track (R_0). It is this reprinting of hodograph (1.2) on the x, y plane which we call the "scarab transformation."[2)]

It can be seen from Eqs. (1.2) that the quasitime t is equal to the arc length of the hodograph, so that the normal ($\mathbf{n}$) to the original curve is also the unit tangent $\boldsymbol{\tau}_0$ for the trajectory of the hodograph. We denote by $\mathbf{n}_0$ the normal to this trajectory, and we rewrite the second equation in (1.2) in a form analogous to that of the second equation in (1.1):

$$\dot{\mathbf{n}}=\dot{\boldsymbol{\tau}}_0=K_0\mathbf{n}_0, \tag{1.4}$$

where K_0 is the curvature of the hodograph arc. Comparison of Eqs. (1.4) and (1.2) reveals

$$K_0^2=(\boldsymbol{\tau}-\omega\mathbf{b})^2=1+\omega^2=R_0^{-2},\quad \mathbf{n}_0=(\omega\mathbf{b}-\boldsymbol{\tau})R_0. \tag{1.5}$$

If we describe the plane track by the vector $\mathbf{r}_1=\mathbf{e}_x x+\mathbf{e}_y y$, which also has arc length t, then the unit tangent vector to it will be $\boldsymbol{\tau}_1=\mathbf{e}_x\dot{x}+\mathbf{e}_y\dot{y}$; again for this plane vector there must be an equation like (1.4):

$$\dot{\boldsymbol{\tau}}_1=K_1\mathbf{n}_1, \tag{1.6}$$

where $\mathbf{n}_1$ is the normal to the plane track with curvature $K_1=1/R_1$ and radius of curvature R_1, which is related to $R_0=(1+\omega^2)^{-1/2}$ by (1.3). We can thus find the plane curvature $K_1=\omega$. Introducing $\mathbf{e}_z=\boldsymbol{\tau}_1\times\mathbf{n}_1$, which is the binormal to the plane curve, and assuming $\mathbf{n}_1=\mathbf{e}_z\times\boldsymbol{\tau}_1$, we can rewrite (1.6) as

$$\ddot{\mathbf{r}}_1=-\omega[\dot{\mathbf{r}}_1\times\mathbf{e}_z],\quad |\dot{\mathbf{r}}_1|=1. \tag{1.7}$$

It is useful to note that this equation for the plane track of an arbitrary three-dimensional curve which is found by the scarab method is formally the same as the equation

$$\ddot{\mathbf{r}}=\dot{\mathbf{v}}=-\frac{|e|}{mc}[\mathbf{v}\times\mathbf{B}]=-\omega_B[\dot{\mathbf{r}}\times\mathbf{e}_z],\quad \omega_B=\frac{|e|B}{mc}, \tag{1.8}$$

which describes the motion of an electron in a magnetic field

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$\mathbf{B} = B\mathbf{e}_z$ with straight lines of force along the z axis, so that the relative torsion $\omega = \varkappa/K$ serves as the cyclotron frequency ω_B. Equation (1.2), the equation of a spherical track, would formally be the same as the equation of motion of the electron in the field of an ion if this ion had, in addition to its electric charge, a magnetic charge of the nature of a Dirac monopole. These magnetic analogies help us obtain a graphic representation of the plane and spherical tracks of the curve as the drift trajectories of an electron in a magnetic field.

2. Since the scarab transformation can be incorporated in a natural way in the general equations of three-dimensional curves, we can use it to formulate several problems involving the extremal properties of curves. One such problem, and apparently the most "fundamental" in this sense, is the following scarab problem: A scarab keeps his ball at a certain point O (his antrum or house), but from time to time he must dry out the ball by turning it upside down, but he can do this only by rolling the ball without slippage. What is the shortest closed curve on a plane (Fig. 1) over which the scarab must roll his ball (of unit radius) to reposition the ball in the antrum precisely upside down; i.e., the point which is initially the low point on the ball must end up as its upper pole.

It is not difficult to see that a purely circular planar trajectory does not solve the problem. The simplest solution would be a equilateral triangle of side π and total length 3π. There are other possiblilities. For example, we could take a trajectory consisting of three circular arcs of length $T_{0,1,0}$ such that the first and third arcs are identical and join smoothly with the central arc T_1. Here we must take care to ensure that the spherical track has its beginning and end at opposite points—the poles of the sphere. This condition imposes a certain relationship between the parameters of T_0 and T_1. If we then vary one of them, we can attempt to minimize the total length $2T_0 + T_1$. A numerical calculation shows that this minimum length is 2.43839π. However, the shortest plane curve which we found in Ref. 1 is the line described by the following parametric equation on the interval $-\pi/2 < \xi < \pi/2$:

$$x = x_{\max}\cos\xi, \qquad y = \frac{1}{k}x_{\max}Z, \qquad Z = E(k,\xi) - \frac{1}{2}F(k,\xi), \tag{2.1}$$

where F and E are the incomplete elliptic integrals of the first and second kinds which depend on the parameter $k = 0.90891$ which is the solution of the equation $K(k) = 2E(k)$. The length of this shortest "scarab's loop" is

$$T_{\min} = \frac{2}{\mu}K(k) = 2.4350\pi,$$

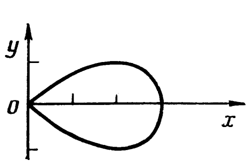

FIG. 1. The minimal scarab's loop.

where the parameter $\mu = 0.606817$ determines the functional dependence of the quasitime $t = F/\mu$ on the angle ξ. Here it is convenient to assume that the time $t = 0$ corresponds to the middle of the loop. We also note that we have $x_{\max} \approx 2k/\mu = 2.99566$.

Curiously, the form of the extremal in (2.1) is precisely the same as that of a circular rod which has been bent to the point that its ends are very close together, resembling an archer's bow bent by a bowstring of zero length. As we know from mechanics, when we bend an elastic rod a torque $M = \varepsilon K_1$ will arise in its cross section; this torque is proportional to the curvature $K_1 = y''(1+y'^2)^{-3/2}$ of the plane curve $y(x)$—the axis of the rod. The constant factor is $\varepsilon = (1/4)\pi E a^4$, where E is the Young's modulus, and a is the radius of the rod. For the geometry in Fig. 1 the equilibrium equation for a rod of length l can be written $M = xf$, where $f = 4\varepsilon l^{-2}K^2(k)$ is the tension in the bowstring. Setting $l = T_{\min}$, we find that in our case the curvature of the plane track is $K_1 = \omega(x) = x\mu^2$, so that the equation for the shortest track (1.7) is

$$\ddot{\mathbf{r}}_1 = -\omega[\dot{\mathbf{r}}_1\mathbf{e}_z], \qquad \omega = x\mu^2, \qquad |\dot{\mathbf{r}}_1| = 1 \tag{2.2}$$

and corresponds in terms of our magnetic analogy, (1.8), to the drift of an electron in a linearly nonuniform magnetic field, $B_z(x) = xB_0'$. This nonuniformity can be assumed the simplest.

3. The most remarkable property of the scarab transformation is that general equation (1.7) of a plane track is a linear equation, and if the function $\omega(t)$ is given this equation can easily be solved in quadratures:

$$x = \int \sin\alpha\, dt, \qquad y = \int \cos\alpha\, dt, \qquad \alpha = \int \omega\, dt. \tag{3.1}$$

In contrast, this is not possible for the equation of a spherical track, Eq. (1.2), unless we resort to numerical methods. At the same time, it is clear from the construction of the scarab transformation that the two tracks are in a one-to-one correspondence, and if one of the tracks is known the other can always be constructed by rolling the sphere "scarab-fashion." This situation is similar in part to a situation in the theory of solitons, where nonlinear equations can be solved by the method of the inverse scattering problem only when they can be reduced to Lax pairs of linear equations; the condition for the compatibility of these equations is the original equation. Our equations, however, contain no partial derivatives.

The meaning of general solution (3.1) for the plane track becomes clearer if we construct around the initial curve $\mathbf{r}(s)$ an orthogonal quasicylindrical Mercier coordinate system[3] ρ, Ω, s with the metric

$$(d\mathbf{r})^2 = (d\rho)^2 + (\rho d\Omega)^2 + (1-Kx)^2(ds)^2. \tag{3.2}$$

Here ρ and Ω are the polar radius and the polar angle in the $s = \text{const}$ plane, which is perpendicular to our curve; here $x = \rho\cos\vartheta$, where the angle ϑ is reckoned from the normal to the center of curvature and is given by

$$\vartheta=\Omega-\alpha(s), \quad \alpha=\int\varkappa\, ds=\int\omega\, dt.$$

We see that if point 1 lies in the $s=$ const plane, and point 2 in the plane $s+ds=$ const, then the length $|d\mathbf{r}|$ will be minimized if $d\rho=\rho_1-\rho_2=0$ and $d\Omega=\Omega_1-\Omega_2=0$. In other words, the line $\rho=$ const, $\Omega=$ const takes the shortest path from one $s=$ const plane to another, acquiring step by step a length

$$L=\int|d\mathbf{r}|=\int_0^s[1-K\rho\cos(\Omega-\alpha)]ds=S+\delta L, \tag{3.3}$$

$$\delta L=-\rho\cos\Omega\int\cos\alpha\, dt-\rho\sin\Omega\int\sin\alpha\, dt,$$

in which we see our plane tracks (3.1).

As a graphic mechanical analog we could imagine an elastic rod—a long pencil—with a central "lead" which cannot be stretched and with an elastic cladding in the form of a (wooden) cylinder. In the undeformed state the wood fibers are straight, run parallel to the lead, and have the same length S. Upon twisting, the wood fibers become longer, but elastic forces tend to shorten them. If the rod is bent into the three-dimensional curve $\mathbf{r}(s)$, but if no twisting torques are applied along the axis, then the lateral fibers, tending to shorten, themselves turn in such a way as to conform to a family of "locally shortest" extremals ρ, $\Omega=$ const. The same property is exhibited by the magnetic lines of force around a solenoid curved in three dimensions.[1]

We find a particularly interesting case when the plane trace is closed, and the integral increment in length in (3.3) is zero, so that the lengths L of all the lateral fibers are equal to the length of the central axis, S. As a result, the entire sheaf of elastic fibers is in an equilibrium state of such a nature that no bending moments must be applied to its ends. In contrast with an elastic rod, here we are allowing a free slippage of the fibers along each other.

4. We turn now to the next "scarab problem." What should be the shape of a closed plane track if, for a total track length T, we need to maximize the angle $\gamma(<\pi)$, which is the angle between the initial vector $\tau(0)$ and the final vector $\tau(T)$ on the hodograph?

For rolling along an equilateral triangle with a perimeter T, for example, we find $\gamma=2\arcsin[\sin^3(T/6)]$. Surprisingly, however, here again the optimal curve is the same scarab's loop as in (2.1), but with the new parameter $x_{\max}=kT/K(k)$ corresponding to the given length T. The equations of the spherical track described by the angles θ and φ are thus

$$\cos\theta=\cos\theta_{min}\cos\xi, \quad \omega=\omega_0\cos\xi,$$
$$\dot\xi=\mu\Delta=\mu(1-k^2\sin^2\xi)^{1/2}, \tag{4.1}$$

where

$$\mu=2K(k)/T, \quad \omega_0=2k\mu, \quad K(k)=2E(k).$$

The minimum angle $\theta_{\min}$ is determined by

$$\operatorname{tg}\theta_{min}=\nu^{-1/2}=(2k)^{-1}[2K(k)/T-T/2K(k)]$$
$$=1/x_{max}-x_{max}/4k^2, \tag{4.2}$$

and for $x_{\max}<2k=1.81782$ for the center of the loop ($t=0$) we should assume $\dot\varphi\sin\theta=-1$, while for $x_{\max}>2k$, we have $\dot\varphi\sin\theta=1$ for $t=0$. For $x_{\max}<2k$, the angle φ on the hodograph is determined by the integral

$$\varphi(\xi)=k\left(1+\frac{1}{\nu}\right)^{1/2}\int_0^\xi\left(1-\frac{1+\nu^{1/2}/k\mu}{1+\nu\sin^2\xi}\right)\frac{d\xi}{\Delta}, \tag{4.3}$$

and the maximum angle γ which we want is

$$\gamma=2k(1+1/\nu)^{1/2}[K(k)-(1+\nu^{1/2}/k\mu)\Pi(k,\nu)], \tag{4.4}$$

where Π is the complete elliptic integral of the third kind. Numerical values of $\gamma(T)$ for the entire interval $0<\gamma<\pi$ are listed in Table I. These values given the optimum solution of the scarab problem of the maximum turning of a sphere of unit radius when the sphere is rolled along a closed plane trajectory without slippage. The ρ, $\Omega=$ const curve drawn near the main three-dimensional axis $\mathbf{r}(s)$ forms with the axis a so-called surface curvature band,[4] so that the scarab problem solved above gives us the shortest path for a sheaf of such bands of identical length. Such problems must be solved, for example, in the optimization of the "Drakon" closed magnetic confinement system for plasmas which was proposed in Refs. 1 and 5 and in which the lines of ρ, $\Omega=$ const correspond to the lines of force of the magnetic field if it is uniform at the axis.

5. In conclusion we examine the relationship between our scarab problem and the so-called Delaunay variational problem which was solved by Weierstrass[6] back in 1884 and which can be formulated as follows: We are to connect two spatial points $\mathbf{r}_1$ and $\mathbf{r}_2$ by a curve of constant curvature $K_0=1$ in such a manner that the curve has unit vectors $\boldsymbol{\tau}_{1,2}$ of specified direction at the points $\mathbf{r}_{1,2}$ and in such a manner that the arc length of this curve is minimized.

Although the Delaunay problem is not the same as our scarab problem, the two can be combined by posing an additional question: For which value of the curvature $K_0=$ const does the Delaunay-Weierstrass curve have the closed plane track which could be obtained from a hodograph with a unit tangent vector $\boldsymbol{\tau}$ by rolling in the scarab method?

Using vector notation, we will first write the Weierstrass solution (which uses a complicated coordinate version of the equations).

We describe the curve which we are seeking by $\mathbf{r}(p)$, where p is some arbitrary parameter (e.g., the length along a straight line between the points 1 and 2). The arc length is then

$$S=\int ds=\int s'(p)\, dp,$$

TABLE I. T and γ as functions of $\theta_{\min}$.

θ_{min}, deg	T	γ, deg	θ_{min}, deg	T	γ, deg
80	0,4461	0,0675	20	3,3538	27,378
70	0,8949	0,6009	10	3,9573	41,845
60	1,3497	2,2993	0	4,6420	61,741
50	1,8151	4,9871	−10	5,4452	89,095
40	2,2977	9,8279	−20	6,4251	127,13
30	2,8064	16,997	−30	7,6783	181,28

where

$$s'=\frac{ds}{dp}=|\mathbf{r}'|,\quad \mathbf{r}'=\boldsymbol{\tau}s',\quad \mathbf{r}''=\boldsymbol{\tau}s''+\mathbf{n}K_0s'^2. \tag{5.1}$$

We introduce the vector $\mathbf{w}=\mathbf{r}'\times\mathbf{r}''=\mathbf{b}K_0s'^3$, where $\mathbf{b}=\boldsymbol{\tau}\times\mathbf{n}$; we can then write $\mathbf{w}^2=K_0^2s'^6$. Upon a variation $\mathbf{r}\to\mathbf{r}+\delta\mathbf{r}$, the condition of constant curvature means that we have

$$F=\mathbf{w}\delta\mathbf{w}-3K_0^2s'^5\delta s'=0, \tag{5.2}$$

where $\delta\mathbf{w}=(\delta\mathbf{r})'\times\mathbf{r}''+(\mathbf{r}'\times\delta\mathbf{r})''$, so we can write

$$F=\boldsymbol{\alpha}(\delta\mathbf{r})'+\boldsymbol{\beta}(\delta\mathbf{r})''-3V\delta s'=0. \tag{5.3}$$

Here we have introduced the temporary notation $V=K_0^2s'^5$ and also

$$\boldsymbol{\alpha}=V[\boldsymbol{\tau}-\mathbf{n}(s''/K_0s'^2)],\quad \boldsymbol{\beta}=\mathbf{n}V/K_0s'. \tag{5.4}$$

We should also note that we have $\delta s'=\boldsymbol{\tau}(\delta\mathbf{r})'$, and the minimization of the arc, $\delta S=\int\delta s'dp=0$, under the additional equation $F=0$ implies the requirement

$$\int_1^2(\delta s'-\chi F)\,dp=0, \tag{5.5}$$

where χ is an arbitrary function to be determined. The term with the second derivative, $(\delta\mathbf{r})''$, can be integrated by parts if we rewrite (5.5) as

$$\int_1^2\mathbf{Q}(\delta\mathbf{r})'\,dp=0,\quad \mathbf{Q}=(1+3\chi V)\boldsymbol{\tau}-\boldsymbol{\alpha}\chi+(\boldsymbol{\beta}\chi)'. \tag{5.6}$$

since $(\delta\mathbf{r})'$ is arbitrary, the solution of the Delaunay problem is the equation $\mathbf{Q}=$ const, which can be rewritten as

$$\mathbf{Q}=(1+2\psi)\boldsymbol{\tau}+\frac{d}{dt}(\dot\psi\boldsymbol{\tau})=(1+\psi)\boldsymbol{\tau}+\dot\psi\mathbf{n}+\omega\psi\mathbf{b}=\mathbf{c}_1=\text{const}. \tag{5.7}$$

Here $\omega=\varkappa/K_0$, and the combination $\psi=\chi V$ is treated as a function of the variable $t=sK_0$ which we introduced earlier. Taking the vector product of (5.7) and $\boldsymbol{\tau}$, we find

$$\left[\boldsymbol{\tau}\times\frac{d}{dt}(\psi\dot{\boldsymbol{\tau}})\right]=\frac{d}{dt}(\psi\mathbf{b})=K_0\frac{d}{dt}[\mathbf{r}\times\mathbf{c}_1]. \tag{5.8}$$

Integrating, we find

$$\psi\mathbf{b}=K_0[\mathbf{r}\times\mathbf{c}_1]+\mathbf{c}_2$$

and also $\mathbf{b}\cdot\mathbf{c}_1\psi=\mathbf{c}_1\cdot\mathbf{c}_2$, where $\mathbf{c}_2$ is a new constant vector. On the other hand, taking the scalar product of Eq. (5.7) and $\mathbf{b}$, we find $\mathbf{b}\cdot\mathbf{c}_1=\omega\psi$ and thus

$$\omega\psi^2=\mathbf{c}_1\mathbf{c}_2=\text{const}. \tag{5.9}$$

Squaring (5.7), we find

$$(1+\psi)^2+\dot\psi^2+\omega^2\psi^2=\mathbf{c}_1^2=\text{const}. \tag{5.10}$$

If we use (5.9) to express ω in terms of ψ and use the notation

$$l=|\mathbf{c}_1|^{-1},\quad h=\mathbf{c}_1\mathbf{c}_2l^2,\quad \lambda(t)=(1+\psi)l,$$

we find the basic equation

$$(\lambda-l)^2\dot\lambda^2=(1-\lambda^2)(\lambda-l)^2-h^2. \tag{5.11}$$

This equation was analyzed by Weierstrass, who solved it in elliptic functions. As we saw above, however, it is more graphic to replace the function λ by the function $\omega=\varkappa/K_0$, which serves as the magnetic field (more precisely, as the cyclotron frequency) in "magnetic equation" (1.7) for the plane track of the curve. We then find in place of (5.11) the equation

$$(\dot\omega/2\omega)^2=(1-l^2)h^{-1}\omega-1-\omega^2-2l(\omega/h)^{1/2}>0, \tag{5.12}$$

which has real solutions only if $l^2<1$. In particular, we could have a steady-state solution $\dot\omega=0$, with $\omega=$ const being the root of the right side. This solution corresponds to an initial curve which is a helix with a plane track (1.7) in the form of the "Larmor circle" of an electron in a uniform magnetic field. A slight nonuniformity of the magnetic field will obviously make plane track (1.7) similar to the drift displacement of a circle, and to solve our auxiliary problem with the requirement that the plane track be closed for the Delaunay-Weierstrass curve it is sufficient to take a single loop of the drift trajectory with one intersection point.

For a general solution of Eq. (5.12) we replace t by the new variable $\alpha=\int\omega dt$ and rewrite (5.12) as

$$d\omega/d\alpha=2(2\omega_0\omega-1-\omega^2-\sigma\omega^{1/2})^{1/2},$$

$$\omega_0=(1-l^2)/2h,\quad \sigma=2l/h^{1/2}. \tag{5.13}$$

We find a particularly simple case if $l = \theta = 0$; in this case, Eq. (5.13) can be integrated in terms of elementary functions, and it yields

$$\omega=[1+(1-4h^2)^{1/2}\cos 2\alpha]/2h, \tag{5.14}$$

from which we find the functional dependences

$$\omega(t)=ab/(a^2-\cos^2 t), \quad \alpha(t)=\mathrm{arctg}(ab^{-1}\,\mathrm{tg}\,t), \tag{5.15}$$

where

$$a=2^{-1/2}[1+1/(1-4h^2)^{1/2}]^{1/2}=\mathrm{const}, \qquad b=(a^2-1)^{1/2}.$$

We can thus find a closed plane track of a given Delaunay-Weierstrass curve for the interval $-\pi<t<\pi$,

$$x=a(\arccos(-1/a)-\arccos(a^{-1}\cos t)),$$

$$y=b\,\mathrm{Arsh}(b^{-1}\sin t), \tag{5.16}$$

but this is not the shortest scarab's loop.

It can be hoped that the scarab transformation introduced here will also prove useful in other problems because of the simplicity of the "magnetic equation" for a plane track, (1.7).

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[1)] This equation is somewhat similar to the Landau-Lifshitz equation for a magnetic material. In particular, with ω = const, Eq. (1.2) becomes the equation of simple spin waves.

[2)] Chaplygin[2] has analyzed the rolling of an inert sphere with an eccentric center of gravity over a plane. In the present paper we are dealing with a forced rolling without inertia.

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