Rigorous theory of cooperative spontaneous emission of radiation from a lumped system of two-level atoms: Bethe ansatz method

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(Submitted 27 June 1984)
Zh. Eksp. Teor. Fiz. **87**, 1617–1630 (November 1984)

It is shown that a lumped system of two-level atoms interacting with a quantum electromagnetic field can be described by a one-dimensional fully integrable model in quantum field theory. The Bethe ansatz method is used to obtain rigorously many-particle eigenstates of the "atoms + field" system and these states can be used to solve the problem of spontaneous decay of an excited state in an atomic subsystem (Dicke superradiance problem).

1. INTRODUCTION

Much theoretical and experimental work has been done (for reviews see Refs. 2 and 3) on the cooperative spontaneous emission of radiation predicted by Dicke.[1]

In the original formulation of the problem it is assumed that all the atoms are concentrated in a volume of radius $R \ll \omega_{12}^{-1}$, where ω_{12} is the frequency of a transition in a two-level atom (Dicke model[1–3]). Subsequent experiments carried out on an extended medium with a small transverse size and of length $l \gg \omega_{12}^{-1}$ led MacGillivray and Feld[4] to suggest a spatially one-dimensional model for the description of superradiance. The one-dimensional approach is justified by the characteristic feature of the observed cooperative effect which is a strong directionality along the axis of a sample. Therefore, in developing a theory of superradiance for an extended system of atoms it is sufficient to allow for the interaction of atoms with those photons whose wave vector is directed along the axis of a sample.

Rupasov[5] used the method of the quantum inverse scattering problem (for reviews see Ref. 6) to show that the one-dimensional model of MacGillivray and Feld is fully integrable. The present authors[7] used the Bethe ansatz method[8] (for reviews see also Refs. 6 and 9) to obtain explicit expressions for the eigenstates in the MacGillivray-Feld model with discrete atoms which, firstly, made it possible to avoid certain mathematical inaccuracies associated with the quantization of a continuous spin field[1)] and, secondly, to obtain general expressions describing the dynamics of spontaneous decay of excitations of the atomic subsystem. An analysis of these general expressions meets with a number of mathematical difficulties, which are to a great extent common to all problems encountered in the theory of fully integrable systems, which makes it difficult to provide a complete quantitative description of cooperative emission of radiation from an extended system.

The one-dimensional nature of the model is a necessary condition for the application of the methods of the theory of fully integrable systems. Therefore, one might assume that the Dicke model (representing a lumped system of atoms) in which it is necessary to allow for the interaction of atoms with a three-dimensional electromagnetic field differs qualitatively from the initially one-dimensional model of MacGillivray and Feld (extended system of atoms). However, we shall show that the Dicke model is equivalent to a certain one-dimensional fully integrable model in field theory. We shall use the Bethe method to diagonalize the Dicke Hamiltonian and to determine the state of the radiation field resulting from spontaneous decay of an initial excitation of the atomic subsystem.

The present treatment therefore achieves a certain unification of the description of the cooperative radiation-emission effects: the Dicke and MacGillivray-Feld models are described by similar fully integrable models of field theory. Since the Dicke model is characterized by a greater mathematical "simplicity" due to the absence of the field propagation effects in a medium, it can be analyzed until a complete description is obtained of any observable physical system.

The Bethe ansatz method has been used so far only in studies of equilibrium (thermodynamic) properties of fully integrable systems.[6,8,9] The present treatment is an example of the use of the Bethe method in solving the dynamic problem of the time evolution of the initial state of a many-particle system.

2. DICKE MODEL

The Dicke model is considered subject to the following physical conditions[1–3]:

1) it is assumed that all the atoms are concentrated in a volume of radius $R \ll \omega_{12}^{-1}$;

2) an atom is regarded as a two-level system with the ground state corresponding to the momentum $J_1 = 0$ and the excited state to the momentum $J_2 = 1$ and to a specific projection of the momentum M_2 on the quantization axis of an atom (we shall consider the specific case when $M_2 = 0$);

3) an allowance is made for the interaction of atoms only with photon states of frequency ω lying near a resonance frequency ω_{12}, i.e., it is assumed that $|\omega - \omega_{12}| \ll \omega_{12}$.

It should be noted that selection of a specific projection of the momentum M_2 of an excited state of an atom in radiation emission problems is clearly a necessary condition for the validity of the two-level model of an atom.

For our purpose it is convenient to retain the framework of the usual dipole (condition 1) resonance (condition 3)

 0038-5646/84/110927-08$04.00

approximation and expand the radiation field operators in terms of spherical harmonics.[10] In this expansion the electric-type photons are described by the operators $c(\omega, j, m)$ which depend on three quantum numbers: the frequency ω (or the modulus of the wave vector $k=\omega$), the momentum j, and the projection of the momentum m. The condition 1 allows us to limit the treatment in the $\omega_{12}R \ll 1$ case to an allowance for the interaction of the atomic subsystem solely with the $j=1$ harmonic of the field. The condition 2 fixes the second quantum number as $m=0$. The Dicke model is therefore essentially one-dimensional.

The Hamiltonian of the model is

$$H=\omega_{12}N+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Big\{(\omega-\omega_{12})c^{+}(\omega)c(\omega) +\varkappa^{1/2}\sum_{a=1}^{M}[s_a^{+}c(\omega)+s_a^{-}c^{+}(\omega)]\Big\}, \tag{1a}$$

where $\varkappa=2\omega_{12}^3 d^2/3\pi$; d is the dipole moment of the transition; M is the number of atoms. The excitation number operator

$$N=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}c^{+}(\omega)c(\omega)+\sum_{a=1}^{M}(s_a^{z}+{}^{1}/_{2}) \tag{2a}$$

commutes with the Hamitonian so that we can subsequently drop the term $\omega_{12}N$ from Eq. (1a) and measure all the frequencies from the resonance value ω_{12}. In view of the resonance approximation, we can take the interaction constant $\varkappa$ out of the integrand when the frequency is ω_{12} and to extend the lower limit of integration to $-\infty$. The operator $c(\omega)\equiv c(\omega, j=1, m=0)$ and the spin operator $s_a^i (i=x, y, z; s_a=1/2)$ describe respectively the photons and two-level atoms and satisfy the commutation relationship

$$[c(\omega), c^{+}(\nu)]=2\pi\delta(\omega-\nu), \tag{3a}$$

where

$$[s_a^i, s_b^j]=ie^{ijk}s_a^k\delta_{ab}, \quad a, b=1, \ldots, M.$$

Introducing the notation

$$\varepsilon(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}c(\omega)e^{i\omega x}, \qquad S^i=\sum_{a=1}^{M}s_a^i$$

and rewriting Eqs. (1a)–(3a) in terms of the operators $\varepsilon(x)$ and S^i, we find that

$$H=\int_{-\infty}^{\infty}dx\{-i\varepsilon^{+}(x)\partial_x\varepsilon(x)+\varkappa^{1/2}\delta(x)[S^{+}\varepsilon(x)+S^{-}\varepsilon^{+}(x)]\}, \tag{1b}$$

$$N=\int_{-\infty}^{\infty}dx\varepsilon^{+}(x)\varepsilon(x)+\left(S^{z}+\frac{M}{2}\right), \tag{2b}$$

$$[\varepsilon(x), \varepsilon^{+}(y)]=\delta(x-y), \quad [S^i, S^j]=ie^{ijk}S^k. \tag{3b}$$

The vacuum state $|0\rangle$ of the model described by Eqs. (1)–(3) will be assumed to be a direct product $|0\rangle=|0\rangle_F\otimes|0\rangle_A$ of the vacuum field state $|0\rangle_F[\varepsilon(x)|0\rangle_F=0]$ and of the ground state of the atomic subsystem $|0\rangle_A(S^{-}|0\rangle_A=0, S^{z}|0\rangle_A=(-M/2)|0\rangle_A)$ in which all the atoms are at the lowest ground level. The state $|0\rangle$ is clearly the eigenstate of the model: $N|0\rangle=H|0\rangle=0$.

We can therefore see that the Dicke model is equivalent to the model of a one-dimensional Bose field with an unbounded linear spectrum $\omega=k$ $(-\infty<k<\infty)$ interacting at the point $x=0$ with an impurity characterized by a spin $S=M/2$ that is governed by the number of atoms forming the atomic subsystem.

It should be noted that the reduction of the Dicke model to the one-dimensional form performed above differs from the corresponding procedure in the Kondo problem[9] by the fact that in the lowest order with respect to the parameter $\omega_{12}R \ll 1$ it is necessary to include the field harmonic with the momentum $j=1$.

Following Ref. 1, we shall consider the states of the atomic subsystem:

$$|D_m\rangle=(S^{+})^m|0\rangle_A, \quad m=0, 1, \ldots, M. \tag{4}$$

Since the Hamiltonian of the model contains only the total spin operators $S^{\pm}=\Sigma s_a^{\pm}$, it follows that the vectors $|D_m\rangle$ define a subspace of states of the atomic subsystem of dimensions $M+1$ (Dicke subspace) close to the action of the Hamiltonian. Consequently, if at the initial moment $t=0$ the atomic subsystem is in one of the states of Eq. (4), then at any subsequent moment in time its state belongs to the Dicke subspace.

It should be stressed that this is true only in the case when all the M atoms have the same transition frequency ω_{12}. However, when the transition frequencies of different atoms are different (i.e., when an allowance is made for an inhomogeneous broadening of atomic transition lines), then for any initial condition the evolution of the atomic subsystem in time occurs in the complete space of states of dimensions 2^M. The Bethe ansatz method makes it possible to solve the problem also in its general form when the broadening is inhomogeneous. We shall confine ourselves to an analysis of the Dicke problem in its standard formulation[1–3] and we shall assume that there is no inhomogeneous broadening and that initially the atomic subsystem is in one of the Dicke states. The inhomogeneous broadening case will be discussed in Sec. 6.

3. DIAGONALIZATION OF THE DICKE HAMILTONIAN

The existence of an integral of motion N allows us to classify the eigenstates of the "atoms + field" system in respect of the number of quasiparticles (excitations), i.e., in respect of the eigenvalue of the operator N. It is convenient to consider first the model of Eqs. (1)–(3) specified in a finite interval $-L/2\leqslant x\leqslant L/2$ and then to substitute $L\to\infty$ only in the final expressions.

We shall seek the one-particle state $|\lambda\rangle$ $(N|\lambda\rangle=1|\lambda\rangle)$ in the form

$$|\lambda\rangle=\left[\int dx e^{i\lambda x}f(\lambda, x)\varepsilon^{+}(x)+\sum_{a=1}\xi_a s_a^{+}\right]|0\rangle. \tag{5}$$

Then, the Schrödinger equation $H|\lambda\rangle=E|\lambda\rangle$ reduces to the following system of equations:

$$i\partial_x f(\lambda, x)=\varkappa^{1/2}\left(\sum_a \xi_a\right)\delta(x), \quad \lambda\xi_a=\varkappa^{1/2}\int dx\delta(x)f(\lambda, x). \tag{6}$$

The singularity of this system is typical of problems with a linear unrenormalized spectrum and a point-like interaction, and it can be removed by replacing the δ function with a regular function $u(x)$ followed by going to the limit $u(x)\to\delta(x)$. From solutions of the system (6) we can separate a class which obeys the conditions $f\equiv 0$, $\lambda = 0$, and $\Sigma\xi_a = 0$. These solutions describe stationary (and nonradiative) excited states of an ensemble of atoms, since the matrix element of the dipole moment operator $d = d(S^+ + S^-)$ between any of these states and the ground state is zero:

$$\langle 0|\hat{d}\sum \xi_a s_a^+|0\rangle=\sum \xi_a=0.$$

From now on we shall be interested only in radiative states of the "field + atoms" system. The relevant solution of the system (6) is

$$f(\lambda, x)=\theta(x<0)+\frac{\lambda-i\varkappa M/2}{\lambda+i\varkappa M/2}\theta(x>0),$$
$$f(\lambda, 0)=1/2[f(\lambda, 0+)+f(\lambda, 0-)], \tag{7}$$
$$\xi_a=\frac{\varkappa^{1/2}}{\lambda+i\varkappa M/2}=\frac{\varkappa^{1/2}}{\lambda}f(\lambda, 0).$$

Using these expressions, we can rewrite Eq. (5) in a more compact form

$$|\lambda\rangle=\int_{-L/2}^{L/2} dx e^{i\lambda x}f(\lambda, x)r^+(\lambda, x)|0\rangle, \tag{8}$$

where $r^+(\lambda, x) = \varepsilon^+(x) + (\varkappa^{1/2}/\lambda)\delta(x)S^+$. It follows from the expression for the wave function of a photon

$$\psi(x)\equiv\langle 0|\varepsilon(x)|\lambda\rangle=e^{i\lambda x}\theta(x<0)+\frac{\lambda-i\varkappa M/2}{\lambda+i\varkappa M/2}e^{i\lambda x}\theta(x>0)$$

that the scattering of a photon by an impurity simply causes an abrupt change in the phase. The "rapidity" parameter λ satisfies the equation

$$e^{i\lambda L}\frac{\lambda-i\varkappa M/2}{\lambda+i\varkappa M/2}=1, \tag{9}$$

which appears when $\psi(x)$ is subjected to a periodic boundary condition $\psi(x = -L/2) = \psi(x = L/2)$. Taking logarithms of Eq. (9), we obtain the relationship

$$\lambda L-2\,\mathrm{arctg}\,(\varkappa M/2\lambda)=2\pi I \tag{10}$$

between the rapidity λ and the quantum number of the system $I = 0, \pm 1, \ldots$. The states of Eq. (8) form an orthogonal basis

$$\langle\mu|\lambda\rangle=\|\lambda\|\delta_{\lambda\mu} \tag{11}$$

in the one-particle sector of the space of states in the Dicke model and the norm of the sector

$$\|\lambda\|=\langle\lambda|\lambda\rangle=L+\frac{\varkappa M}{\lambda^2+(\varkappa M/2)^2} \tag{12}$$

is identical with the formal derivative $\partial I/\partial\lambda$. Therefore, in the limit $L\to\infty$ we can transform summation in respect of the quantum number I to integration in respect of the rapidity λ:

$$\sum_I\left(\frac{1}{\|\lambda\|}\ldots\right)\to\int_{-\infty}^{\infty}d\lambda\frac{\partial I}{\partial\lambda}\left(\frac{1}{\|\lambda\|}\ldots\right)=\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}(\ldots). \tag{13}$$

Simple expressions in Eqs. (9)–(13) given here can account for similar but more cumbersome formulas in the many-particle case.

In the case of a many-particle eigenstate in the Dicke model

$$H|\lambda_1\ldots\lambda_m\rangle=E|\lambda_1\ldots\lambda_m\rangle,\ N|\lambda_1\ldots\lambda_m\rangle=m|\lambda_1\ldots\lambda_m\rangle \tag{14}$$

which is parametrized by a set of rapidities $\{\lambda_j, j = 1, \ldots, m; \lambda_1\neq\lambda_j \text{ for } i\neq j\}$, we can use the Bethe ansatz method[8]

$$|\lambda_1\ldots\lambda_m\rangle=\int_{-L/2}^{L/2}dx_1\ldots dx_m\prod_{i<j}\left[1+\frac{i\varkappa}{\lambda_i-\lambda_j}\mathrm{sign}\,(x_i-x_j)\right]$$
$$\times\prod_{j=1}^{m}e^{i\lambda_j x_j}f(\lambda_j, x_j)r^+(\lambda_j, x_j)|0\rangle, \tag{15}$$

where

$$\mathrm{sign}\,x=\{1, x>0;\ 0, x=0;-1, x<0\}.$$

In Eq. (15) the Bethe factor

$$\prod_{i<j}\left[1+\frac{i\varkappa}{\lambda_i-\lambda_j}\mathrm{sign}\,(x_i-x_j)\right]$$

reflects the appearance of the photon-photon correlations in the scattering of photons in the atomic subsystem. These correlations are responsible for the cooperative effects in the emission of photons from an excited system of atoms. It should be stressed here that correlations in the photon subsystem appear also when photons are scattered by one atom ($M = 1$) and are responsible for the cooperative effect observed in the resonance fluorescence of an atom in a strong field.

The rapidities $\{\lambda_j\}$ satisfy the following system of transcendental Bethe equations:

$$e^{i\lambda_j L}\frac{\lambda_j-i\varkappa M/2}{\lambda_j+i\varkappa M/2}=\prod_{\substack{i=1\\ i\neq j}}^{m}\frac{\lambda_i-\lambda_j+i\varkappa}{\lambda_i-\lambda_j-i\varkappa}, \tag{16}$$

which appears when the wave function of photons is subjected to periodic boundary conditions

$$\psi(x_1, \ldots, x_j=-L/2, \ldots, x_m)=\psi(x_1, \ldots, x_j=L/2, \ldots, x_m),$$

and the energy of the state described by Eq. (15) is given by the simple expression

$$E=\sum_{j=1}^{m}\lambda_j.$$

Taking logarithms of the Bethe equations, we obtain the relationship

$$\lambda_j L-2\mathrm{arctg}\frac{\varkappa M}{2\lambda_j}+2\sum_{\substack{i=1\\ i\neq j}}^{m}\mathrm{arctg}\frac{\varkappa}{\lambda_i-\lambda_j}=2\pi I_j \tag{17}$$

between the rapidity $\{\lambda_j\}$ and the quantum numbers of the system $I_j = 0, \pm 1, \ldots$. Following Ref. 5, we can show that the commutation relationships for the matrix elements of the monodromy of the Dicke model are identical with those for the quantum nonlinear Schrödinger equation model.[6] Consequently, we can calculate the norm of the Bethe vectors using the results obtained by Korepin.[11] According to Korepin, the norm of the Bethe vector with the set of rapidities $\{\lambda_j\}$ is given by the expression

$$\|\lambda_1 \ldots \lambda_m\| = (2\pi)^m \prod_{i<j} \left[1 + \frac{\varkappa^2}{(\lambda_i - \lambda_j)^2}\right] \frac{D(I_1 \ldots I_m)}{D(\lambda_1 \ldots \lambda_m)}, \quad (18)$$

where $D(I_1 \ldots I_m)/D(\lambda_1 \ldots \lambda_m)$ is the Jacobian of the transformations of Eq. (17).

In the absence of atoms ($M = 0$), the operators

$$N_0 = \int dx \varepsilon^+(x)\varepsilon(x), \qquad H_0 = -i\int dx \varepsilon^+(x)\partial_x \varepsilon(x) \quad (19)$$

describe a free radiation field and the Bethe states of Eq. (15) reduce to

$$|\lambda_1 \ldots \lambda_m\rangle_0 = \int_{-L/2}^{L/2} dx_1 \ldots dx_m \prod_{i<j} \left[1 + \frac{i\varkappa}{\lambda_i - \lambda_j} \operatorname{sign}(x_i - x_j)\right] \times \prod_{j=1}^{m} e^{i\lambda_j x_j} \varepsilon^+(x_j)|0\rangle. \quad (20)$$

A Bethe state described by Eq. (20) is an eigenstate for the free-field operators

$$N_0|\lambda_1 \ldots \lambda_m\rangle_0 = m|\lambda_1 \ldots \lambda_m\rangle_0,$$
$$H_0|\lambda_1 \ldots \lambda_m\rangle_0 = \left(\sum_{j=1}^{m} \lambda_j\right)|\lambda_1 \ldots \lambda_m\rangle_0, \quad (21)$$

which can easily be shown by recalling that, for example, the operators N_0 and H_0 are identical with the particle number and momentum operators in the nonlinear Schrödinger equation model. We in fact obtain

$$e^{i\lambda_j L} = \prod_{\substack{i=1 \\ i\neq j}}^{m} \frac{\lambda_i - \lambda_j + i\varkappa}{\lambda_i - \lambda_j - i\varkappa} \quad (22)$$

and also relationships analogous to the expressions (17) and (18).

This somewhat unusual method of describing a free field will be used, in addition to the conventional method of an expansion of the states of the field in terms of free photons, to solve the Dicke problem.

We shall now classify the m-particle eigenstates (15) of the Dicke model in the limit $L \to \infty$. Bethe[8] demonstrated that in general the solutions of a system of transcendental equations of the (16) type lie in a complex plane and are grouped in "strings"

$$\lambda_j^l = \Lambda_l + \frac{i\varkappa}{2}(2j - m_l - 1), \quad j = 1, \ldots, m_l, \quad (23)$$

where m_l ($1 \leqslant m_l \leqslant m$) is the number of rapidities λ_j^l forming a string with the number l, and Λ_l is the total real part. We shall call m_l the length and Λ_l the carrier rapidity of a string

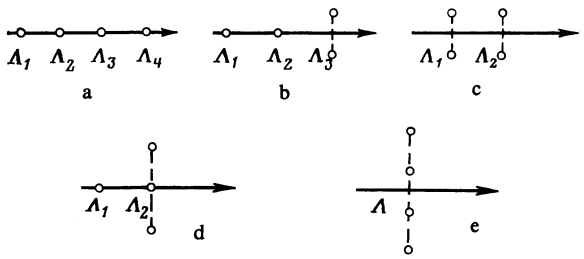

FIG. 1.

l. The total number of strings of an m-particle Bethe vector n ($l = 1, \ldots, n$) can vary from 1 to m and we obviously have $\Sigma m_l = m$. By way of illustration, Figs. 1a–1e show all possible sets of strings (configurations) of a four-particle Bethe vector.

A system of m Bethe equations can be rewritten in such a way that it contains only n equations for the carrier rapidities $\{\Lambda_l\}$. Then, as is known from Refs. 6, 9, and 12–14, the remaining $(m-n)$ equations are satisfied with an accuracy to exponentially small terms of the order of $\exp(-\varkappa L)$. We shall not give here the equations for $\{\Lambda_l\}$, because their exact form is important only in studies of thermodynamic properties of the model. In our case, the only important fact is that, in the limit $L \to \infty$, the carrier rapidities $\{\Lambda_l\}$ assume independent values throughout the real axis and the norm of the Bethe vector $|\Lambda_1 \ldots \Lambda_n\rangle$ is still described in terms of the Jacobian of transformations from the quantum numbers $\{I_l\}$ to the carrier rapidities $\{\Lambda_l\}$:

$$\|\Lambda_1 \ldots \Lambda_n\| = (2\pi)^n \left(\prod_{l=1}^{n} \frac{m_l}{\varkappa^{(m_l-1)}}\right) \prod_{l<k} \prod_{i=1}^{m_l} \prod_{j=1}^{m_k} \left[1 + \frac{\varkappa^2}{(\lambda_i^l - \lambda_j^k)^2}\right] \times \frac{D(I_1 \ldots I_n)}{D(\Lambda_1 \ldots \Lambda_n)}. \quad (24)$$

Classification of the Bethe states of a free radiation field (20) does not require separate consideration because it is fully identical with the above classification of the Bethe states in the Dicke model. It must be stressed that in the limit $L \to \infty$ any solution of the system of equations (16) with an arbitrary set of strings characterized by carrier rapidities $\{\Lambda_l\}$ corresponds to a solution of the system (22) with the same set of strings and carrier rapidities.

Strictly speaking, the Bethe vectors in the Dicke model do not form a complete set. This can easily be shown by noting that the state (15) is projected only on the Dicke subspace of the complete space of states in the atomic subsystem. Therefore, in the absence of an inhomogeneous broadening the Bethe vectors form a complete set only in the subspace of states of the "atoms + field" system, which is a direct product of the Dicke subspace and the complete space of the states of the radiation field. This circumstance together with an allowance for the closed nature of the Dicke subspace relative to the Hamiltonian of the model is sufficient for investigating the time evolution of the system which is

initially (at $t = 0$) in one of the Dicke states. We can show (see Conclusions) that the problem of time evolution of an arbitrary initial state of the atomic subsystem reduces formally to the problem of evolution in the Dicke subspace considered below.

4. QUANTUM CAUCHY PROBLEM. FINAL STATE OF THE RADIATION FIELD

We shall assume that initially (at $t = 0$) the atomic subsystem is in one of the Dicke states and the field in the vacuum state, i.e., the state of the system, is

$$|in\rangle=\left[\frac{(M-m)!}{m!M!}\right]^{1/2}(S^+)^m|0\rangle, \quad \langle in|in\rangle=1. \tag{25}$$

The state (25) is an eigenstate of the integral of motion N ($N|in\rangle = m|in\rangle$) so that it can be represented by a superposition of m-particle Bethe spectra

$$|in\rangle=\sum_{\{I_j\}}A_L(\{\lambda_j\})\frac{|\lambda_1\ldots\lambda_m\rangle}{(m!\,\|\lambda_1\ldots\lambda_m\|)^{1/2}}, \tag{26a}$$

$$A_L(\{\lambda_j\})=\frac{\langle\lambda_1\ldots\lambda_m|in\rangle}{(m!\,\|\lambda_1\ldots\lambda_m\|)^{1/2}}$$

$$=\left[\frac{\varkappa^m M!}{(M-m)!\,\|\lambda_1\ldots\lambda_m\|}\right]^{1/2}\prod_{j=1}^{m}\frac{1}{\lambda_j-i\varkappa M/2}, \tag{26b}$$

where the summation is carried out over all possible solutions of the Bethe equations (16), i.e., over all sets of the quantum numbers $\{I_j\}$.

The time evolution of the initial state $\Psi_L(t) = \exp(-iHt)|in\rangle$ subject to Eq. (14) can be represented by

$$\Psi_L(t)=\sum_{\{I_j\}}A_L(\{\lambda_j\})\exp[-iE(\{\lambda_j\})t]\frac{|\lambda_1\ldots\lambda_m\rangle}{(m!\,\|\lambda_1\ldots\lambda_m\|)^{1/2}}. \tag{27}$$

In the limit $L\to\infty$, the state (27) becomes

$$\Psi(t)=\sum_{(\text{конф})}\int\frac{d\Lambda_1\ldots d\Lambda_n}{(2\pi)^n}A(\{\Lambda_l\})e^{-iEt}|\Lambda_1\ldots\Lambda_n\rangle, \tag{28a}$$

$$A(\{\Lambda_l\})=\left[\frac{\varkappa^m M!}{m!(M-m)!}\right]^{1/2}\left(\prod_{l=1}^{n}\frac{\varkappa^{(m_l-1)}}{m_l}\right)$$

$$\times\left(\prod_{l<k}\prod_{i=1}^{m_l}\prod_{j=1}^{m_k}\frac{(\lambda_i^l-\lambda_j^k)^2}{(\lambda_i^l-\lambda_j^k)^2+\varkappa^2}\right)$$

$$\times\prod_{l=1}^{n}\prod_{j=1}^{m_l}\frac{1}{\Lambda_l-{}^1/_2 i\varkappa(M+m_l+1-2j)} \tag{28b}$$

where the summation is carried out over all the possible configurations of m-particle Bethe vectors; $E = \Sigma m_l\Lambda_l$.

In the derivation of Eqs. (28a) and (28b) we have gone over from summation over quantum numbers to integration of rapidities and summation over configurations. We have used Eq. (24) and have reduced the Jacobian of the transformation [see Eq. (13)].

The expressions (28a) and (28b) determine completely the state of the "atoms + field" system at any moment in time $t\geqslant 0$ and, consequently, they give a solution of the quantum Cauchy problem for a given initial state (25). The physical formulation of the problem of decay involves determination of the characteristics of the radiation field, i.e., it reduces to a study of the asymptotic behavior of the expression (28a) in the limit $t\to\infty$, when the excitation of the atomic subsystem is converted completely to the excitation of the field.

We can easily see that in the limit $t\to\infty$ the contribution of the atomic variables to the state $\Psi(t)$ decays exponentially and the state reduces to

$$\Psi(t)\to\Phi(t)=e^{-iH_0t}|\text{out}\rangle, \tag{29a}$$

$$|\text{out}\rangle=\sum_{(\text{conf})}\int\frac{d\Lambda_1\ldots d\Lambda_n}{(2\pi)^n}A(\{\Lambda_l\})\prod_{j=1}^{m}f(\lambda_j,0+)\,|\Lambda_1\ldots\Lambda_n\rangle_0. \tag{29b}$$

The expressions (29a) and (29b) describe the final state of the free radiation field formed as a result of spontaneous decay of the initial excited state of the atomic subsystem and, consequently, they solve the Dicke problem in the basis of the Bethe functions of the free radiation field.

The final state of the radiation field considered in the basis of the Bethe vectors is a sum over all possible configurations and, therefore, the radiation corresponding to a certain set of strings should be regarded as decay of the excitation of an ensemble of atoms in a certain channel.

The probability of emission from n strings of lengths m_l ($l = 1, \ldots, n$)is given by the expression

$$W(\text{conf})=\int\frac{d\Lambda_1\ldots d\Lambda_n}{(2\pi)^n}|A(\{\Lambda_l\})|^2. \tag{30}$$

Thus, for example, the probability of decay accompanied by the emission of the longest (mth) string is

$$W(m-tu\ \text{string})=\frac{m}{2m-1}\frac{M!\,(2m-1)!}{m!\,(M+m-1)!}. \tag{31}$$

The number of possible decay channels rises very rapidly on increase in the number m. Nevertheless, the probability of decay of a completely excited atomic subsystem ($m = M$) in a channel representing the emission of the M th string is finite and dominant [$W(M\text{ th string})\gtrsim 1/2$] in the limit $M\to\infty$. On the other hand, the probability of emission of M single-particle strings is exponentially small:

$$W[(m-1)-tu\ \text{string}]$$

$$=\frac{M!\,(M-1)!}{(M-m)!\,(M+m-1)!}\underset{m=M\gg 1}{\propto}M^{1/2}2^{-M}. \tag{32}$$

In the opposite limiting case of a weakly excited subsystem ($m\ll M$) the emission of one-particle strings predominates. According to Eq. (32), the probability of decay in such a channel tends to unity and the probability of decay in the other channels consequently tends to zero when $M\to\infty$.

Therefore, the distribution of the probabilities of decay in various channels depends strongly on the degree of initial excitation of the atomic subsystem. These results can be given a natural physical explanation. The nonlinearity of the "atoms + field" system is due to the finite nature of the spectrum of excitations in the atomic subsystem ($-M/2\leqslant S^z\leqslant M/$

2). Therefore, the correlation effects appear most strongly for a completely inverted initial state. In the opposite case of a weak excitation the system "does not know" about the finite nature of its spectrum and its behavior hardly differs from the behavior of a system with an unbounded equidistant spectrum, i.e., from the behavior of a harmonic oscillator.

The wave function of photons in the Bethe state with rapidities collected in the mth string is

$$\psi(\{x_j\}) = \text{const} \exp\left\{ i\Lambda \sum_{j=1}^{m} x_j - \frac{\varkappa}{2} \sum_{i<j} |x_i - x_j| \right\}. \quad (33)$$

The function (33) describes the momentum of a photon with its center of gravity $X = (1/m)\Sigma x_j$ moving in accordance with the wave vector $K = m\Lambda$. The pulse duration $\tau_p \approx (\varkappa m)^{-1}$ is inversely proportional to the number of photons m.

A study of the "out" state shows that in the case of a lumped system under discussion, the decay in any channel occurs in a time $0 \leqslant t \lesssim (\varkappa m)^{-1}$, so that experimental identification of a specific channel requires determination of higher-order correlation functions. The physical quantities measured in conventional experiments are governed by the sum over all possible configurations of the Bethe vectors.

The problem of calculation of the physically observable quantities is very complex because of the enormous number of configurations if m is large. However, it is found that a superposition of the Bethe vectors which distinguishes the "out" states of the radiation field (29b) is characterized by a latent simplicity due to an interesting mathematical property, which we shall demonstrate by considering the decay of a doubly excited ($m = 2$) atomic subsystem. In this case, we have

$$\Phi(t) = \varkappa \left(\frac{M(M-1)}{2} \right)^{1/2} \times \left\{ \frac{\varkappa}{2} \int_{-\infty}^{\infty} \frac{d\Lambda}{2\pi} \frac{e^{-2i\Lambda t}}{[\Lambda + {}^1/_2 i\varkappa(M-1)][\Lambda + {}^1/_2 i\varkappa(M+1)]} |\Lambda\rangle \right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{d\Lambda_1 d\Lambda_2}{(2\pi)^2} \frac{(\Lambda_1 - \Lambda_2)^2}{(\Lambda_1 - \Lambda_2)^2 + \varkappa^2} \left(\prod_{j=1}^{2} \frac{e^{-i\Lambda_j t}}{\Lambda_j + i\varkappa M/2} \right) |\Lambda_1 \Lambda_2\rangle \right\}. \quad (34)$$

Here, the first integral corresponds to the contribution of the second string, whereas the second term represents the contributions of two one-particle strings. The first integral is governed by the contribution from two poles $\Lambda^{(1)} = -i\varkappa(M-1)/2$ and $\Lambda^{(2)} = -i\varkappa(M+1)/2$ (Fig. 2). Simple calculations show that the contribution of the pole $\Lambda^{(2)}$ cancels out exactly the second integral in Eq. (34) and, therefore, the state $\Phi(t)$ is given by the following simple expressions

$$\Phi(t) = \int_{-\infty}^{\infty} dx_1\, dx_2 \psi(x_1, x_2; t)\, \varepsilon^+(x_1)\, \varepsilon^+(x_2) |0\rangle, \quad (35a)$$

$$\psi(x_1, x_2; t) = \varkappa \left(\frac{M(M-1)}{2} \right)^{1/2} \theta(t-x_1)\, \theta(t-x_2) \times \exp\left\{ \frac{\varkappa}{2}(M-1)(x_1 + x_2 - 2t) - \frac{\varkappa}{2} |x_1 - x_2| \right\}. \quad (35b)$$

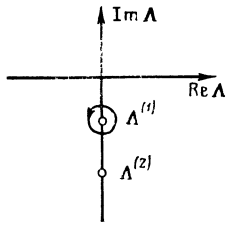

FIG. 2.

We can easily show that if $m = 3$, the state $\Phi(t)$ can be calculated similarly: it is still found by calculating the residue at the pole $\Lambda^{(1)} = -i\varkappa(M-2)/2$ of the configuration nearest to the real axis when all three velocities are collected in the third string.

We shall apply this rule to an arbitrary m, although unfortunately, we cannot provide a rigorous mathematical proof. If we consider in Eq. (28) the configuration with the mth string and if we calculate the integral with respect to the carrier rapidity including only the pole $\Lambda^{(1)} = -i\varkappa(M-m+1)/2$ nearest to the real axis, we obtain

$$\Phi(t) = \int_{-\infty}^{\infty} dx_1 \ldots dx_m \psi(\{x_j\}; t) \prod_{j=1}^{m} \varepsilon^+(x_j) |0\rangle, \quad (36a)$$

$$\psi(\{x_j\}; t) = \left[\frac{\varkappa^m M!}{m!(M-m)!} \right]^{1/2} \times \exp\left\{ \frac{\varkappa}{2}(M-m+1) \sum_{j=1}^{m} (x_j - t) - \frac{\varkappa}{2} \sum_{i<j} |x_i - x_j| \right\} \times \prod_{j=1}^{m} \theta(t - x_j). \quad (36b)$$

This state is automatically normalized $\langle \Phi(t) | \Phi(t) \rangle = \langle in | in \rangle = 1$, which provides an additional proof of the validity of the above procedure of summing over the Bethe vector configurations. One should point out that the application of this summation rule to the right-hand side of the expansion of the "in" state in terms of the Bethe vectors (26a) gives the correct initial expression (25). Therefore, the above property exhibited already by the initial superposition of the Bethe states is retained also during the time evolution of the system.

Equations (36a) and (36b) describe completely the final state of the field in the usual basis of free photons and, therefore, they solve explicitly the problem of Dicke superradiance for a lumped system of two-level atoms.

The wave function of the emitted photons $\psi(\{x_j\}; t)$ describes a pulse of duration $\tau_p \approx (\varkappa m)^{-1}$, with the center of gravity $X = (1/m)\Sigma x_j$ distributed over an interval $\Delta \approx [\varkappa m(M - m + 1)]^{-1}$. In the case of a weakly excited initial state of the atomic subsystem ($m \ll M$), we find that the corresponding ratio is $\tau_p/\Delta \sim M \gg 1$ and, therefore, we can observe experimentally only the exponential tail of the fall of the radiation intensity, exactly as in the case of conventional spontaneous emission from a system of m independent atoms. In the opposite limiting case of a strongly excited atomic subsystem ($m \lesssim M$) the ratio is $\tau_p/\Delta \sim 1$ and a bell-

shaped time dependence of the radiation intensity can be detected.

5. CALCULATION OF PHYSICAL QUANTITIES

The expressions (36) allow us to calculate any physical characteristic of a radiation field. For example, the radiation intensity at a distance r from a system of atoms is described by the correlation function

$$I(r,\ t)\sim\langle\Phi(t)\,|\varepsilon^{+}(r)\varepsilon(r)\,|\Phi(t)\rangle. \tag{37}$$

The process of calculation on the basis of Eqs. (36) and (37) reduces to $(m-1)$-fold integration with respect to the coordinates of all the photons, except one. After integration, the expression for $I(r,t)$ can be written in the form of a finite sum of elementary functions. We shall not follow this procedure, because it gives results obtained earlier from the equation for the density matrix ρ_A of the atomic subsystem (see Ref. 3 and the literature cited there).

We recall that in the derivation of the closed equation for the density matrix ρ_A we need to use not only the physical conditions (1)–(3) governing the Dicke model, but also additional approximations. In the cited papers these approximations are the factorization of the total density matrix of the "atoms + field" system ρ_{A+F} (Born approximation)

$$\rho_{A+F}(t)\approx\rho_A(t)\otimes|0\rangle_{FF}\langle 0| \tag{38}$$

and the assumption that the atom-photon correlations disappear in a time considerably shorter than the characteristic time of a change in the state of the atomic subsystem (Markov approximation). It should also be stressed that the use of only the vacuum term of the density matrix of the photon subsystem in Eq. (38) implies neglect of the contribution of the induced radiation emission processes.

A rigorous solution of the problem shows that none of the approximations mentioned above applies to the exact form of the total matrix $\rho_{A+F}(t)$: it cannot be factorized and the time scales of changes are the same for all the correlation functions of the "atoms + field" system. Nevertheless, if we take the trace of the photon variables in the exact density matrix ρ_{A+F}, we obtain an expression for the density matrix of the atomic subsystem $\rho_A=\mathrm{Tr}_F\,\rho_{A+F}$, which satisfies the equation obtained using the Born-Markov approximation. Clearly, this can be explained by the extremely simple nature of the time evolution of the atomic subsystem, which can be described qualitatively as discontinuities in a ladder of levels.[1-3] This is supported by a calculation of the spectral density of the radiation

$$G(\omega)=\langle\mathrm{out}\,|c^{+}(\omega)c(\omega)\,|\mathrm{out}\rangle$$
$$=\sum_{k\leqslant l}^{m}\frac{1}{\Gamma_l}\prod_{j=k}^{l}\frac{\Gamma_j}{i\omega+(\Gamma_j+\Gamma_{j-1})/2}+\text{c.c.}, \tag{39}$$

which corresponds to a system of coupled transitions in the atomic system with the width of the j-th level described by $\Gamma_j=\varkappa_j(M-j+1)$, identical with the width calculated by Dicke[1] in the second order of perturbation theory.

6. INHOMOGENEOUS BROADENING

We shall consider now a more common physical situation, namely the case of an inhomogeneous broadening of a transition line of a system of atoms. The Hamiltonian of the "atoms + field" system differs from Eq. (1a) by an additional term

$$\Delta H=\sum_{a=1}^{M}\Delta_a(s_a^{z}+{}^1/_2), \tag{40}$$

where $\Delta_a=\omega_a-\omega_{12}$ is the detuning of the transition frequency from the average value

$$\omega_{12}=\frac{1}{M}\sum_{a=1}^{M}\omega_a.$$

In accordance with the usual physical conditions, we shall assume that $|\Delta_a|\ll\omega_{12}$, which ensures the validity of the resonance approximation.

The procedure for determining the many-particle eigenstates of the system is similar to that described above. The one-particle state $|\lambda\rangle$ is determined by the equations

$$i\partial_x f(\lambda,x)=\varkappa^{1/2}\left(\sum_{a=1}^{M}\xi_a\right)\delta(x),$$
$$(\lambda-\Delta_a)\xi_a=\varkappa^{1/2}\int dx\,\delta(x)f(\lambda,x). \tag{41}$$

In the case when all the values of Δ_a are different, the system (41) has no degenerate nonradiation solutions. The only solution is

$$f(\lambda,x)=\theta(x<0)+\frac{k(\lambda)-i\varkappa M/2}{k(\lambda)+i\varkappa M/2}\theta(x>0), \tag{42}$$
$$f(\lambda,0)={}^1/_2[f(\lambda,0+)+f(\lambda,0-)],\qquad \xi_a=\frac{\varkappa^{1/2}}{\lambda-\Delta_a}f(\lambda,0),$$

where $k^{-1}(\lambda)=(1/M)\Sigma[1/(\lambda-\Delta_a)]$. Using Eq. (42), we can represent the state $|\lambda\rangle$ in the form of Eq. (8), when now we have

$$r^{+}(\lambda,x)=\varepsilon^{+}(x)+\varkappa^{1/2}\delta(x)\sum_{a=1}^{M}\frac{1}{\lambda-\Delta_a}s_a^{+}. \tag{43}$$

The many-particle eigenstates of the system still have the form given by Eq. (15), where the quantities $f(\lambda,x)$ and $r^{+}(\lambda,x)$ are given by Eqs. (42) and (43). The classification of the Bethe states does not differ from that described above.

The basis of the Bethe states obtained here can be used to describe the time evolution of an arbitrary initial state of the atomic subsystem. We shall give here the results for a simple special case of a system of two atoms ($M=2$), which is of some intrinsic interest and which makes it possible to grasp the behavior of a polyatomic system if we use the results obtained above.

1) We shall assume that at $t=0$ a system of two atoms is singly excited:

$$|\mathrm{in}\rangle=(\alpha_1 s_1^{+}+\alpha_2 s_2^{+})\,|0\rangle,\qquad |\alpha_1|^2+|\alpha_2|^2=1. \tag{44}$$

In the final state ($t\to\infty$) the phonon wave function is

$$\psi(x,t)=\frac{i\varkappa^{1/2}}{(\Delta^2-\varkappa^2)^{1/2}}\left\{\left[\alpha_1\left(\eta_{+}+\frac{\Delta}{2}\right)+\alpha_2\left(\eta_{+}-\frac{\Delta}{2}\right)\right]\right.$$
$$\left.\times\ e^{i\eta_{+}(x-t)}+(\eta_{+}\to\eta_{-})\right\}, \tag{45}$$

where

$$\eta_{\pm}=[-i\varkappa\pm(\Delta^2-\varkappa^2)^{1/2}]/2,\ \ \mathrm{Im}\,(\Delta^2-\varkappa^2)^{1/2}\geqslant 0.$$

If the detuning is sufficiently large ($\Delta > \varkappa$), the intensity of the radiation generally exhibits oscillations of frequency Δ with a decrement $\varkappa$.

2) In the case when $t=0$, both atoms are excited ($|in\rangle = s_1^+ s_2^+ |0\rangle$) and the final state of the radiation field is described by the expression

$$\Phi(t)=-\frac{\varkappa\Delta^2}{2}\int dx_1\,dx_2\theta(x_1<x_2<t)\times\left[\frac{\exp\{i[\eta_-(x_1-t)+\eta_+(x_2-t)]\}}{\eta_+(\eta_+-\eta_-)}+(\eta_+\leftrightarrow\eta_-)\right]\times\varepsilon^+(x_1)\varepsilon^+(x_2)|0\rangle. \tag{46}$$

The time dependence of the radiation intensity is

$$I(t-x)=\theta(t-x)\left\{\frac{2\varkappa\Delta^2}{\Delta^2-\varkappa^2}e^{-\varkappa(t-x)}-\frac{8\varkappa^3}{\Delta^2}e^{-2\varkappa(t-x)}+\frac{\varkappa^2\Delta^4}{8i(\Delta^2-\varkappa^2)}\left[\frac{1}{\eta_+^3}e^{-2i\eta_-(t-x)}+\frac{1}{\eta_-^3}e^{-2i\eta_+(t-x)}\right]\right\}. \tag{47}$$

It is clear from this formula that in the case of a small inhomogeneous width ($\Delta < \varkappa$) the fall of the intensity is exponential, whereas for $\Delta > \varkappa$ the decay is accompanied by oscillations of frequency $(\Delta^2-\varkappa^2)^{1/2}$. We cannot exclude the possibility that the experimentally observed jagged form of the intensity of cooperative radiation[2,3] is associated with a fairly strong inhomogeneous broadening in the system.

7. CONCLUSIONS

We obtained an exact solution of the Dicke model in its standard formulation. The approach employed is essentially a generalization of the Wigner-Weisskopf theory[15,10] to the case of an arbitrary number of atoms. Naturally, we must bear in mind that the Dicke model allows only for a resonance interaction of photons with two-level atoms and it does not describe, for example, the dipole-dipole interaction due to virtual nonresonance photons. The problem of the influence of the dipole-dipole interaction on the cooperative emission has been discussed frequently in the literature.[2,3] An allowance for this interaction disturbs the complete integrability of the model; its influence can be investigated, in principle, by the methods of perturbation theory using the exact Bethe states of the Dicke Hamiltonian obtained by us.

We shall conclude by noting that the decay of the initial excited state (4) of the atomic subsystem, belonging to the Dicke subspace, is considered here. V. Ya. Chernyak drew our attention to the fact that the above solution of this special Cauchy quantum problem in fact provides a complete description of the evolution of an arbitrary initial excited state of the atomic subsystem. An arbitrary state of a system of two-level atoms can indeed be expanded in terms of irreducible representations of the $SU(2)$ group corresponding to specific values of the "spin." The evolution of a state corresponding to a specific spin S is identical with the evolution—described above—of a Dicke state with an effective number of atoms $M_{\text{eff}} = 2S$.

The authors are grateful to V. M. Agranovich, P. B. Wiegmann, V. E. Zhakharov, I. B. Levinson, É. I. Rashba, and V. A. Fateev for their interest and valuable advice.

Note added in proof (October 3, 1984)

One of the authors (V. I. Yu.) has now proved the validity of the summation rule used above (Sec. 4). It is a consequence of a representation of an arbitrary physical state as a superposition of generalized Bethe vectors of a given configuration of strings (without summation over the configurations!).

[1)]We take this opportunity to thank L. D. Faddeev and E. K. Sklyanin for pointing this out to us.

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Translated by A. Tybulewicz