Superradiance and dissipative instability in an inverted two-level system

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The semiclassical approximation is used to show that Dicke superradiance is a consequence of the dissipative instability of waves in an inverted two-level medium. This instability is related to the existence of negative-energy polarization waves and appears as a result of losses by emission of electromagnetic radiation through the boundaries of the specimen. Superabsorbance, which is the analog of superradiance and results from the development of dissipative instability due to volume ohmic losses, is also examined. The analysis is performed for an unbounded homogeneous medium, a one-dimensional model of a bounded medium, and a three-dimensional specimen in the form of a sphere.

§1. INTRODUCTION

Superradiance is produced in macroscopic specimens of an inverted medium when the concentration N of active molecules is high enough.[1–3] Decay of the excited state then occurs collectively in a time τ that is much shorter than the characteristic time T_1 for the spontaneous decay of an isolated molecule. The internal energy of the molecules stored in the specimen is radiated in the form of a short electromagnetic pulse whose power Q exceeds by several orders of magnitude the power $Q_{\rm spont}$ of noncoherent spontaneous emission by the same number of isolated molecules (see Fig. 1). The delay time t_0 of the superradiant pulse relative to the end of the pump pulse is usually greater than the superradiant pulse length τ by a factor of 10–20. The superradiant (phased) state of the set of molecules is formed during this time. In the initial stage, the phasing of the molecules is quantum-mechanical in character. However, when a sufficiently large number of photons appears, the electromagnetic field and the polarization of the medium become classical and the development of the superradiant process may be examined in the semiclassical approximation over most of the interval to (Refs. 2–5).

In this approximation, the high-frequency electromagnetic field is described by the Maxwell equations

$$\mathrm{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\qquad \mathrm{rot}\,\mathbf{B}=\frac{1}{c}\frac{\partial(\mathbf{E}+4\pi\mathbf{P})}{\partial t}+\frac{4\pi\sigma}{c}\mathbf{E}. \tag{1.1}$$

On the surface of a free specimen, the electric (**E**) and magnetic (**B**) fields satisfy boundary conditions that demand the continuity of tangential components. The possible presence of ohmic dissipation due to the electrical conductivity σ of the "background" medium is taken into account in (1.1). The active medium is an isotropic two-level molecular gas with transition frequency ω_0 and homogeneous line broadening $1/T_2$. It is described by the quantum-mechanical equations for the mean polarization **P** and population difference Δn per unit volume:

$$\frac{\partial^2\mathbf{P}}{\partial t^2}+\frac{2}{T_2}\frac{\partial\mathbf{P}}{\partial t}+\left(\omega_0^2+\frac{1}{T_2^2}\right)\mathbf{P}=\frac{\omega_c^2}{4\pi}\mathbf{E}, \tag{1.2}$$

$$\frac{\partial\Delta N}{\partial t}=-\frac{\Delta N-\Delta N_0}{T_1}+\frac{2}{\hbar\omega_0}E\frac{\partial\mathbf{P}}{\partial t}. \tag{1.3}$$

The coefficient relating the polarization and the field in (1.2) is determined by the square of the so-called plasma (cooperative) gas frequency ω_c:

$$\omega_c^2=-8\pi d^2\Delta N\omega_0/3\hbar, \tag{1.4}$$

where d is the high-frequency dipole moment of the molecular transition. In the inverted gas, $\Delta N>0$ and $\omega_c^2<0$.

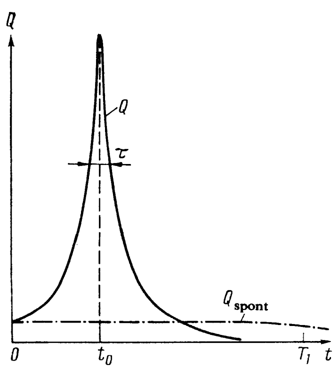

FIG. 1. Emission-pulse profiles $Q(t)$ for superradiant decay and $Q_{\rm spon}(t)$ for incoherent spontaneous decay of a specimen of an inverted two-level medium.

§2. DISSIPATIVE INSTABILITY OF TRANSVERSE WAVES IN A HOMOGENEOUS INVERTED MEDIUM

We shall consider plane waves $\mathbf{E}=\frac{1}{2}[\mathbf{E}_0\exp(-i\omega t+i\mathbf{kr})+\text{c.c.}]$ with a real wave vector **k** in an unbounded homogeneous medium with fixed inversion ΔN. In accordance with (1.2), the electromagnetic properties of the active medium will be described by the permittivity

$$\varepsilon(\omega)=1+i\frac{4\pi\sigma}{\omega}-\frac{\omega_c^2}{(\omega+i/T_2)^2-\omega_0^2}. \tag{2.1}$$

According to (1.1), the transverse-wave dispersion is determined by

$$c^2k^2=\omega^2\varepsilon(\omega), \tag{2.2}$$

which is a quartic in the complex frequency $\omega=\omega'+i\omega''$. Let us rewrite it in the following form:

$$\left[\omega\left(1+i\frac{4\pi\sigma}{\omega}\right)^{1/2}+ck\right]\left[\omega\left(1+i\frac{4\pi\sigma}{\omega}\right)^{1/2}-ck\right]\times\left(\omega+\frac{i}{T_2}+\omega_0\right)\left(\omega+\frac{i}{T_2}-\omega_0\right)=\omega^2\omega_c^2. \tag{2.3}$$

Assuming that $1/T_2\omega_0$, $2\pi\sigma/\omega_0$, $|\omega_c|/\omega_0\ll1$, and using the

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resonance approximation $|\omega_0 - ck| \ll \omega_0$ for the roots $\omega \approx \omega_0$ (i.e., for waves propagating in a single direction), we find that the first and second factors in (2.3) can be replaced with 2ω. Since $(1 + i4\pi\sigma/\omega)^{1/2} \approx 1 + i\cdot 2\pi\sigma/\omega)$, the dispersion relation defines two modes, namely, the electromagnetic wave and a polarization wave (Fig. 2)[1)]

$$\omega_{e,p}=\omega_0-\frac{i}{T_2}+\frac{1}{2}\left[ck-\omega_0+i\left(\frac{1}{T_2}-2\pi\sigma\right)\right]\times\left\{1\pm\left[1+\frac{\omega_c^2}{[ck-\omega_0+i(1/T_2-2\pi\sigma)]^2}\right]^{1/2}\right\}. \quad (2.4)$$

When the directions of propagation of the two waves are reversed, the frequencies are still given (2.4) except that the sign of the real part is reversed.

For a given field amplitude E_0, the amplitude of the polarization of the active medium in the polarization wave is greater (usually much greater) than in the electromagnetic wave.[2)] This is readily verified by substituting the eigenfrequencies $\omega = \omega_{e,p}$ of the normal wave (2.4) into the expression for the complex polarization amplitude

$$\mathbf{P}_0=\chi\mathbf{E}_0, \quad \chi=-\frac{\omega_c^2}{4\pi[(\omega+i/T_2)^2-\omega_0^2]}. \quad (2.5)$$

It is clear from (2.4) that only one of the two waves can be stable for given wave number k, i.e., either the electromagnetic wave or the polarization wave is stable. This is also valid outside the resonance region ($ck \not\approx \omega_0$), which follows from the Vieta relation for the sum of two parts of complex-conjugate roots of the exact dispersion relation (2.2): $\omega_e'' + \omega_p'' \equiv -1/T_2 - 2\pi\sigma < 0$.

The maximum wave growth (decay) rates are reached at the line center for $ck = (\omega_0^2 + 1/T_2^2)^{1/2} = |\omega_{e,p}|$ and are given by

$$\omega_{e,p}=-\frac{1}{T_2}+\frac{1}{2}\left(\frac{1}{T_2}-2\pi\sigma\right)\left\{1\pm\left[1-\frac{\omega_c^2}{(1/T_2-2\pi\sigma)^2}\right]^{1/2}\right\}. \quad (2.6)$$

Their magnitude determines the evolution of the waves and depends on the difference $1/T_2 - 2\pi\sigma$.

Usual treatments of maser (induced) field instability[3,4,6] are concerned with an electromagnetic wave $\omega_e(k)$ under the conditions of strong relaxation of polarization and weak-field dissipation: $1/T_2 > |\omega_c|/2 > 2\pi\sigma$. According to (2.6), the maser growth rate is

$$\omega_e''=-\frac{\omega_c^2 T_2}{4}-2\pi\sigma=\frac{2\pi\omega_0 T_2 d^2\Delta N}{3\hbar}-2\pi\sigma \quad (2.7)$$

(for $1/T_2 \gg |\omega_c|/2$). The polarization wave $\omega_p(k)$ then describes the highly attenuated polarization oscillations with decay rate $1/T_2$, and is of little interest.

However, let us now turn to the case of weak relaxation (and strong inversion) when the inequality $1/T_2 < |\omega_c|/2$ that characterizes superradiance[2,3] is satisfied. To get our bearings, we note that, in the usual collisional relaxation in a gas, for which $1/T_2 \sim r_m^2 v_T N$, this case can occur when the degree of inversion is not too low: $\Delta N/N \gtrsim 5\cdot 10^{-10} N/\omega_0$ (the cross section of the molecules is taken to be $r_m^2 \approx 10^{-14}$ cm^2, their thermal velocity $v_T \sim 10^5$ cm/s, and the transition dipole moment is $d \sim 1$ D).

When field dissipation is small ($2\pi\sigma < 1/T_2$), the maximum growth rate (2.6) exhibits anomalous saturation:

$$\omega''=\frac{|\omega_c|}{2}-\frac{1}{2T_2}-\pi\sigma\approx\left(\frac{2\pi\omega_0 d^2\Delta N}{3\hbar}\right)^{1/2} \quad (2.8)$$

(for $1/T_2 \ll |\omega_c|/2$). In contrast to the maser growth rate (2.7), which is proportional to the inversion ΔN, the anomalous growth rate (2.8) increases only as the square root of inversion.[7] The significance of this saturation is clear from the energy balance condition

$$2\omega''|E_0|^2/8\pi=\hbar\omega_0\Delta N\rho-\sigma|E_0|^2/2, \quad (2.9)$$

in which the rate of growth of field energy is determined by the power carried by radiation emitted as a result of induced decay of two-level molecules and ohmic losses. The probability ρ of an induced transition from the upper to the lower level of a molecule per unit time is given by (1.3):

$$\rho=\mathrm{Im}(-\omega E_0^* P_0)/2\hbar\omega_0\Delta N.$$

Using (2.5) with $|\omega_c|/2 \gg 1/T_2, 2\pi\sigma$ we find that

$$\rho=-\omega_c^2|E_0|^2/16\pi\omega''\hbar\omega_0\Delta N.$$

This means tht the power emitted as a result of induced decay of the molecules is determined by the radiation spectral density, which is inversely proportional to the growth rate. Hence, we have from (2.9)

$$2\omega''\frac{|E_0|^2}{8\pi}=-\frac{\omega_c^2}{4}\frac{|E_0|^2}{4\pi\omega''}.$$

This equation leads directly to the anomalous growth rate (2.8) and corresponds to the well-known Einstein relation[4,6] for radiation whose spectral width $\Delta\omega \sim \omega'' \approx |\omega_c|/2$ exceeds the relaxation width $1/T_2$ of the transition and the field decay rate $2\pi\sigma$.

The situation undergoes a qualitative change when field dissipation predominates over the relaxation of polarization, i.e., $2\pi\sigma > 1/T_2$. Instead of the electromagnetic-wave instability, we then have the polarization-wave instability (see Fig. 2). So long as $2\pi\sigma \ll |\omega_c|/2$, the instability in the interior of the line $|ck - \omega_0| \lesssim |\omega_c|$ has the anomalous character described by (2.8). For strong dissipation, when $2\pi\sigma \gtrsim |\omega_c|/2$, the background gradually smooths out the spectrum $\omega_p(k)$ (Fig. 2b) and reduces the polarization wave instability: at the line center,

$$\omega_p''=-\omega_c^2/8\pi\sigma-1/T_2 \quad (2.10)$$

[see (2.6) for $2\pi\sigma \gg |\omega_c|/2$]. When $2\pi\sigma \gg 1/T_2$ it follows from (2.4) that the wave-number band in which the polarization-

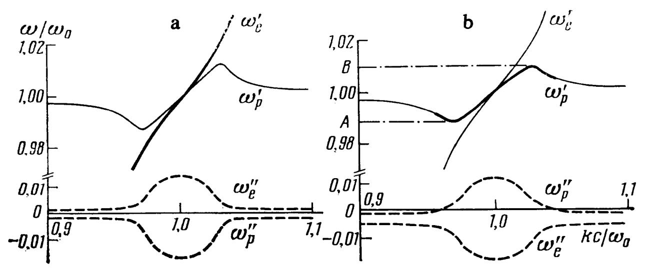

FIG. 2. Dispersion branches of normal waves in an inverted two-level medium at $1/T_2\omega_0 = 10^{-3}$, $-\omega_c^2/\omega_0^2 = 10^{-3}$, when $1/T_2 \ll |\omega_c|$: a) $\sigma = 0$, b) $\sigma = 10^{-2}\omega_0/4\pi \sim |\omega_c|/4\pi$. The thick line represents the section corresponding to instability.

wave instability develops with a growth rate of the order of the maximum value given by (2.6) is determined by

$$\Delta k\sim\frac{1}{c}[(2\pi\sigma)^2-\omega_c^2]^{1/2}\left[\frac{-\omega_c^2T_2/4-2\pi\sigma}{-\omega_c^2T_2/4+2\pi\sigma}\right]^{1/2}. \qquad (2.11)$$

We shall show that the polarization-wave instability is a dissipative instability of a negative-energy wave. (An instability of this kind is known, for example, for charged-particle beams in plasma physics and electronics.) We shall use the following equations for the rate of change of electromagnetic field energy

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}\right)=-\mathbf{E}\frac{\partial\mathbf{P}}{\partial t}-\sigma\mathbf{E}^2-\frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{B}]$$

and the polarization energy of two-level molecules

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial\mathbf{P}}{\partial t}\right)^2+\frac{1}{2}\left(\omega_0^2+\frac{1}{T_2^2}\right)\mathbf{P}^2\right]$$
$$=-\frac{2}{T_2}\left(\frac{\partial\mathbf{P}}{\partial t}\right)^2+\frac{\omega_c^2}{4\pi}\mathbf{E}\frac{\partial\mathbf{P}}{\partial t},$$

which follow from (1.1) and (1.2). Eliminating $\mathbf{E}\partial\mathbf{P}/\partial t$ from these equations, we obtain

$$\frac{\partial}{\partial t}\left\{\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}+\frac{2\pi}{\omega_c^2}\left[\left(\frac{\partial\mathbf{P}}{\partial t}\right)^2+\left(\omega_0^2+\frac{1}{T_2^2}\right)\mathbf{P}^2\right]\right\}$$
$$=-\frac{8\pi}{\omega_c^2T_2}\left(\frac{\partial\mathbf{P}}{\partial t}\right)^2-\sigma\mathbf{E}^2-\frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{B}]. \qquad (2.12)$$

The rate of change of the total field and polarization energy is determined in this expression by the relaxation of molecular polarization, ohmic dissipation, and the inhomogeneity of the energy flux. Averaging over the high-frequency period for homogeneous normal waves in the active medium (2.5) then yields the wave energy density w and the power loss Q per unit volume:

$$w_{e,p}=\frac{|E_0|^2}{16\pi}\left[1+\frac{c^2k^2}{|\omega_{e,p}^2|}+\frac{\omega_c^2(|\omega_{e,p}^2|+\omega_0^2+1/T_2^2)}{|(\omega_{e,p}+i/T_2)^2-\omega_0^2|^2}\right]$$
$$\times\exp(2\omega_{e,p}t). \qquad (2.13)$$

$$Q_{e,p}=\frac{|E_0|^2}{2}\left[\sigma+\frac{\omega_c^2|\omega_{e,p}^2|}{2\pi T_2|(\omega_{e,p}+i/T_2)^2-\omega_0^2|^2}\right]\exp(2\omega_{e,p}''t). \qquad (2.14)$$

We can now use these two relations to verify that the polarization wave in the inverted medium in which $\omega_c^2<0$ does, in fact, have negative energy (the electromagnetic wave has positive energy). This means that, when $Q_p>0$, the loss of energy due to field dissipation produces an increase in the amplitude of the polarization wave at the rate[3)]

$$\omega_p''=-Q_p/2w_p. \qquad (2.15)$$

According to (2.6) and (2.15), the dissipative instability described above occurs in a finite range of conductivities:

$$1/T_2<2\pi\sigma<-\omega_c^2T_2/4. \qquad (2.16)$$

When $-\omega_c^2T_2/4<\omega_0$, the instability of the polarization wave in the region described by (2.16) and the instability of the electromagnetic wave in the region $2\pi\sigma<1/T_2$ are convective and concentrated in the wave number interval

$$k_{1,2}=\frac{1}{c}\left\{\omega_0^2+\frac{1}{T_2^2}+\left(\frac{1}{2}+\frac{1}{4\pi\sigma T_2}\right)\left[-\omega_c^2-\frac{8\pi\sigma}{T_2}\mp\left(-\omega_c^2-\frac{8\pi\sigma}{T_2}\right)^{1/2}(-\omega_c^2+8\pi\sigma T_2\omega_0^2)^{1/2}\right]\right\}^{1/2} \qquad (2.17)$$

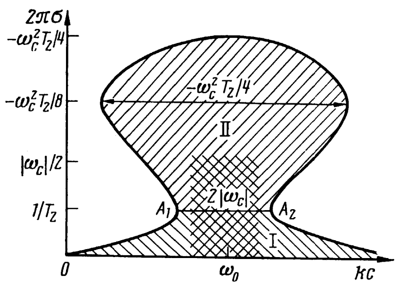

FIG. 3. Region of unstable wave numbers (2.17) vs conductivity in the case $4\sqrt{2}/T_2<|\omega_c|$ (shown schematically). The electromagnetic wave is unstable in region I and the polarization wave is unstable in region II (line A_1A_2 is the boundary between these regions). The region of anomalous instability of the waves is doubly hatched.

(Fig. 3). We note that this interval coincides with the region of negative values of the Hurwitz determinant

$$\begin{vmatrix} a_1 & 1 & 0\\ a_3 & a_2 & a_1\\ 0 & a_4 & a_3\end{vmatrix}<0 \qquad (2.18)$$

for the dispersion relation (2.2), written in the form

$$(-i\omega)^4+a_1(-i\omega)^3+a_2(-i\omega)^2+a_3(-i\omega)+a_4=0;$$
$$a_1=2(1/T_2+2\pi\sigma),\quad a_2=c^2k^2+\omega_0^2+\omega_c^2+1/T_2^2+8\pi\sigma/T_2,$$
$$a_3=2c^2k^2/T_2+4\pi\sigma(\omega_0^2+1/T_2^2),\quad a_4=c^2k^2(\omega_0^2+1/T_2^2).$$

Since the coefficients a_j written out above are positive, we can use the Lienard-Chipart theorem,[9] according to which the inequality (2.18) is the necessary and sufficient condition for stability $\omega''>0$.

It is clear from Fig. 3 that, when $4\sqrt{2}/T_2<|\omega_c|$, an increase in conductivity σ in the region $2\pi\sigma>1/T_2$ is accompanied by an expansion in the size of the region of instability of polarization waves (2.17), which is in contrast to the maser instability of electromagnetic waves for $2\pi\sigma<1/T_2$. This expansion occurs because of the appearance of dissipative instability on the line wings $|ck-\omega_0|\gtrsim|\omega_c|$. According to (2.4) and Fig. 2b, up to the value $2\pi\sigma=-\omega_c^2T_2/8$, the range of unstable frequencies coincides with the dispersion band AB occupied by the polarization-wave branch.

We note that the above polarization waves do not play an appreciable role in existing lasers. The point is that these lasers employ high-Q cavities for which the equivalent loss is $2\pi\sigma\equiv\omega_0/2q<1/T_2$. Moreover, generation occurs in accordance with (2.7) with a relatively low level of inversion for which there are no weakly-damped polarization waves because the condition $1/T_2<|\omega_c|/2$ is not satisfied. On the contrary, this inequality is satisfied when superradiance can take place.

§3. SELF-EXCITATION OF MODES IN A BOUNDED ACTIVE MEDIUM (ONE-DIMENSIONAL MODEL). SUPERRADIANCE AND SUPERABSORBANCE EFFECTS

In the one-dimensional model of superradiance by a bounded active medium, described by (1.1)–(1.3), it is usual[2,3] to consider plane waves propagating in the direction of

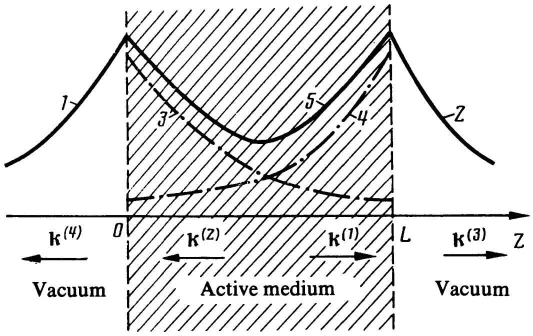

FIG. 4. Spatial structure of dissipatively unstable modes in the one-dimensional superradiance model: 1—$|E_0^{(4)}e^{ik(4)z}|$, 2—$|E_0^{(3)}e^{ik(3)z}|$, 3—$|E_0^{(2)}e^{ik(2)z}|$, 4—$|E_0^{(1)}\times e^{ik(1)z}|$, 1—5—2—$|E_0(z)|$.

the z axis that is perpendicular to the open plane layer of the medium of thickness $L\gg\lambda\equiv 2\pi c/\omega_0$ (see Fig. 4). By linearizing (1.1)–(1.3), i.e., by assuming $\Delta N=$ const, we obtain the solutions of the homogeneous equations (1.1) and (1.2) in the form

$$\mathbf{E}=(\mathbf{x}^0/2)\,[E_0(z)e^{-i\omega t}+\text{c.c.}],$$

where the unit vector $\mathbf{x}^0$ is perpendicular to the z axis. It is assumed that the magnetic field $\mathbf{B}$ and polarization $\mathbf{P}$ are given by similar expressions with amplitudes $B_0(z)$ and $P_0(z)$. We now write the field as the superposition of two waves propagating in opposite directions in the layer:

$$E_0(z)=E_0^{(1)}e^{ik^{(1)}z}+E_0^{(2)}e^{ik^{(2)}z},\quad 0\leqslant z\leqslant L, \tag{3.1}$$

and in the form of waves leaving the layer for the vacuum region:

$$E_0(z)=E_0^{(3)}e^{ik^{(3)}z},\quad z\geqslant L,$$
$$E_0(z)=E_0^{(4)}e^{ik^{(4)}z},\quad z\leqslant 0. \tag{3.2}$$

According to the Maxwell equations (1.1) and the equations for the polarization of the active medium (1.2), the spatial structures of the magnetic field and the polarization have similar form, and

$$k^{(j)}E_0^{(j)}=\frac{\omega}{c}B_0^{(j)},\quad P_0^{(j)}=\chi(\omega)E_0^{(j)},$$
$$k^{(1)}=-k^{(2)}=\frac{\omega}{c}\varepsilon^{1/2}(\omega),\quad k^{(3)}=-k^{(4)}=\frac{\omega}{c} \tag{3.3}$$

[see (2.1) and (2.5)]. Equation (3.3) and the boundary conditions demanding the continuity of the tangential components of the electric and magnetic fields across the surface of the layer lead to a homogeneous set of algebraic equations for the relative amplitudes of the plane waves $E_0^{(j)}$ ($j=1, 2, 3, 4$). Its solution gives

$$E_0^{(1)}=\frac{\varepsilon^{1/2}-1}{2\varepsilon^{1/2}}E_0,\quad E_0^{(2)}=\frac{\varepsilon^{1/2}+1}{2\varepsilon^{1/2}}E_0,$$
$$E_0^{(3)}=\frac{E_0}{2\varepsilon^{1/2}}[(\varepsilon^{1/2}-1)e^{i\omega L(\varepsilon^{1/2}-1)/c}+(\varepsilon^{1/2}+1)e^{-i\omega L(\varepsilon^{1/2}+1)/c}],$$
$$E_0^{(4)}=E_0 \tag{3.4}$$

and the characteristic equation

$$\text{ctg}\,[\omega\varepsilon^{1/2}(\omega)L/c]=i[1+\varepsilon(\omega)]/2\varepsilon^{1/2}(\omega).$$

The latter can also be written in a form analogous to (2.2):

$$\omega_m\varepsilon^{1/2}(\omega_m)=\frac{\pi mc}{L}-i\delta_m,\quad \delta_m=\frac{c}{L}\,\text{arth}\left[\frac{2\varepsilon^{1/2}(\omega_m)}{1+\varepsilon(\omega_m)}\right], \tag{3.5}$$

where m is an integer. This equation determines the discrete frequency spectrum ω_m for the modes of the one-dimensional model of the open layer of active medium (Fig. 4). In the case of superradiance by an inverted specimen, the electromagnetic field and polarization in the interior of the layer can be written as superpositions of these modes with amplitudes depending on the initial conditions. The field amplitudes of modes in the specimen, $A_m(t)=E_0\exp(\omega_m''t)$, are functions of time because their eigenfrequencies are complex ($\omega_m''=\text{Im}\,\omega_m$). Fields with continuous spectra are discussed in Refs. 2 and 3.

As in the preceding section, we shall seek the solution of the characteristic equation in the resonance approximation

$$\omega_m\approx\omega_0\gg|\omega_0-\pi mc/L|,\quad |\varepsilon(\omega_m)-1|\ll 1,$$

assuming that the parameters $1/T_2\omega_0, 2\pi\sigma/\omega_0, |\omega_c|/\omega_0, c/L\omega_0\ll 1$ are small. Let $k_m=\pi m/L$ and recall that the quantity

$$\delta_m\equiv\delta_m'+i\delta_m''\approx\frac{c}{L}\ln\left[\frac{2}{[\varepsilon^{1/2}(\omega_m)]-1}\right] \tag{3.6}$$

is small in comparison with ω_0. We then have from (3.5)

$$\omega_m+\frac{i}{T_2}=\left[\omega_0^2+\frac{\omega_c^2\omega_m}{2[\omega_m+i(\delta_m+2\pi\sigma)-ck_m]}\right]^{1/2}$$
$$\approx\omega_0+\frac{\omega_c^2\omega_m/4\omega_0}{\omega_m+i(\delta_m+2\pi\sigma)-ck_m}. \tag{3.7}$$

Solving the quadratic (3.7) for the explicitly appearing frequency ω_m, and neglecting squares of the above small parameters, we find that

$$(\omega_m)_{e,p}=\omega_0-\frac{i}{T_2}+\frac{1}{2}\left[ck_m-\omega_0+i\left(\frac{1}{T_2}-2\pi\sigma-\delta_m\right)\right]$$
$$\times\left\{1\pm\left[1+\frac{\omega_c^2}{[ck_m-\omega_0+i(1/T_2-2\pi\sigma-\delta_m)]^2}\right]^{1/2}\right\}. \tag{3.8}$$

Substitution of (3.8) in (2.4) will show that there are two types of mode in the bounded layer: electromagnetic (m,e) and polariton (m,p). Their properties are analogous to those of electromagnetic waves and polarization waves in an unbounded medium (see §2). In particular, only one of the two modes can be unstable. The principal difference between the modes (3.8) and those in (2.4) is that their spectrum is discrete and the correction $\delta_m\propto 1/L$ to the characteristic equation [see (3.5), (3.8), (2.2), and (2.4)] appears in them. The latter leads to an additional decay (growth) rate and to the inhomogeneity of the spatial structure of the modes (3.1) and (3.2) ($\text{Im}\,k^{(j)}\neq 0$). In the limit of an infinitely long layer ($L\to\infty$), the frequency spectrum (3.8) becomes denser, $\delta_m\to 0$, and the modes of the bounded specimen take the form of waves in unbounded space [see (2.4) for $m\to\infty$, $k_m=\pi m/L\to k$]. For a specimen of finite length L, and using the above approximations, we have

$$\delta_m\approx\delta_m'\equiv 2\pi\sigma_{\text{rad}}(m)=\frac{c}{L}\ln\left|\frac{2}{\varepsilon^{1/2}(\omega_m)-1}\right|, \tag{3.9}$$

where $\ln|2/(\varepsilon^{1/2}-1)|\gg 1$. The effective quantity $\sigma_{\text{rad}}(m)=\delta_m'/2\pi$ is introduced because it appears in the expression for the complex frequencies (3.8) as part of a sum involving the electrical conductivity σ, and characterizes the rate of dissipation of the field by radiation through the boundaries of the specimen.

The latter statement becomes clear if we turn to the expression for the rate of change of energy (2.12) and integrate it over the thickness L of the layer and unit cross-section area S, i.e., over the volume $V=LS$. After averaging over the high-frequency period, we obtain the following expressions for the energy and power loss in the mth mode:

$$H_m=\exp(2\omega_m''t)\int_V\Big[\frac{|\mathbf{E}_0|^2+|\mathbf{B}_0|^2}{16\pi}+\frac{\pi}{\omega_c^2}\Big(|\omega_m|^2+\omega_0^2+\frac{1}{T_2^2}\Big)|\mathbf{P}_0|^2\Big]dV, \quad (3.10)$$

$$Q_m=\exp(2\omega_m''t)\Big\{\int_V\Big[\sigma\frac{|\mathbf{E}_0|^2}{2}+\frac{4\pi|\omega_m|^2}{\omega_c^2T_2}|\mathbf{P}_0|^2\Big]dV+\frac{c}{8\pi}\int_S\mathrm{Re}[\mathbf{E}_0\times\mathbf{B}_0^*]d\mathbf{S}\Big\}. \quad (3.11)$$

Hence, using (3.1)–(3.3) together with the equation $\mathrm{Im}\,(\omega_m\varepsilon^{1/2}(\omega_m))=-\delta_m'$, we find that

$$H_m=\frac{|A_m|^2V}{16\pi\ln|2/(\varepsilon^{1/2}(\omega_m)-1)|}\Big[1+|\varepsilon(\omega_m)|+\frac{\omega_c^2(|\omega_m^2|+\omega_0^2+1/T_2^2)}{|(\omega_m+i/T_2)^2-\omega_0^2|^2}\Big], \quad (3.12)$$

$$Q_m=\frac{|A_m|^2V}{2\ln|2/(\varepsilon^{1/2}(\omega_m)-1)|}\Big[\sigma_{\text{изл}}(m)+\sigma+\frac{\omega_c^2|\omega_m^2|}{2\pi T_2|(\omega_m+i/T_2)^2-\omega_0^2|^2}\Big], \quad (3.13)$$

$$\omega_m''=-Q_m/2H_m,\quad A_m=E_0\exp(\omega_m''t). \quad (3.14)$$

The term containing $\sigma_{\rm rad}$ in the expression for the power loss (3.13) is the flux of the Poynting vector $c|A_m^2|S/4\pi$ through the boundaries of the specimen, and constitutes an additional channel for energy dissipation that was not present in the unbounded medium [see (2.14)]. The other consequence of the bounded nature of the open specimen of the active medium is the exponential inhomogeneity of the mode structure, noted above. This ensures that the energy characteristics of the modes, given by (3.12) and (3.13), are actually determined not by the entire volume of the medium, but only by the boundary layer, whose thickness[4)] is of the order of $L/\ln|2/(\varepsilon^{1/2}-1)|$ [see (2.13) and (2.14)].

Substitution of (3.8) in (3.12) shows that, in the inverted specimen, the polariton (m,p)-modes have negative energy and the electromagnetic (m,e)-modes have positive energy. As in the unbounded medium, the polariton modes are therefore unstable during energy dissipation $Q_{m,p}>0$ (this is the dissipative instability), whereas electromagnetic modes are unstable during negative dissipation $Q_{m,e}<0$ (maser instability). The maximum growth (decay) rates of the modes are given by

$$(\omega_m'')_{e,p}=-\frac{1}{T_2}+\frac{1}{2}\Big[\frac{1}{T_2}-2\pi(\sigma+\sigma_{\rm rad})\Big]\times\Big\{1\pm\Big[1-\frac{\omega_c^2}{[1/T_2-2\pi(\sigma+\sigma_{\rm rad})]^2}\Big]^{1/2}\Big\}, \quad (3.15)$$

which differs from (2.6) by the replacement $\sigma\to\sigma+\sigma_{\rm rad}$. These rates are achieved for modes at the line center for $ck_m+\delta_m''\approx\omega_0$, where, according to (3.6), we have $\delta_m''\lesssim c/L\ll\omega_0$.

When all the above factors due to the finite length L of the specimen are taken into account, analysis of the conditions for the appearance of mode instability and different limiting cases is analogous to that given in §2 for an unbounded medium. In particular, formulas (2.7), (2.8), and (2.10) are extended to the case of a bounded active medium by making the replacement $\sigma\to\sigma+\sigma_{\rm rad}$. Dissipative instabilty of polariton modes arises for sufficiently large inversion $|\omega_c|>2/T_2$ for which

$$1/T_2<2\pi(\sigma+\sigma_{\rm rad})<-\omega_c^2T_2/4 \quad (3.16)$$

[see (2.16)]. The wave-number band for which the instability growth rates of polariton modes are of the order of the maximum rate in (3.15) is given by the following expression which is analogous to (2.11):

$$\Delta k_m\sim\frac{1}{c}[4\pi^2(\sigma+\sigma_{\rm rad})^2-\omega_c^2]^{1/2}\Big[\frac{-\omega_c^2T_2/4-2\pi(\sigma+\sigma_{\rm rad})}{-\omega_c^2T_2/4+2\pi(\sigma+\sigma_{\rm rad})}\Big]^{1/2}. \quad (3.17)$$

Let us now consider the special case corresponding to the well-known conditions for Dicke superradiance:[2,3] ohmic dissipation is absent ($\sigma=0$), inversion is large enough ($|\omega_c|\gg 2/T_2$), and losses by radiation are not too high ($2\pi\sigma_{\rm rad}\ll-\omega_c^2T_2/4$). The last of these means that the length of the specimen is much greater than the minimum length for which superradiance first becomes possible.[10]

$$L\gg L_{min}=4c\ln|2/(\varepsilon^{1/2}-1)|/(-\omega_c^2T_2).$$

For simplicity, we shall also assume that

$$2\pi\sigma_{\rm rad}\gg|\omega_c|\ln|2/(\varepsilon^{1/2}-1)|,$$

i.e., the length of the specimen is small in comparison with the so-called Arecchi-Courtens length:[11] $L\ll L_c\equiv c/|\omega_c|$ [see (3.9)]. These conditions are sufficient for inequalities (3.16), which lead to dissipative instability of polariton modes, to be satisfied. Only one of them, namely, the (m,p)-mode that is the closest to the line center and has the maximum growth rate, plays a dominant role:

$$\omega_p''\approx-\omega_c^2/8\pi\sigma_{\rm rad}. \quad (3.18)$$

The other modes with indices $m\pm l$ ($l=1,2,\ldots$) do not fall into the strong instability band (3.17) since $|k_{m\pm l}-k_m|\equiv\pi l/L\gtrsim\Delta k_m$.

At the beginning of the superradiance process, when $\Delta N=$ const, the square of the amplitude $|A_p(t)|^2$ of the above polariton mode obeys the law

$$d|A_p|^2/dt=2\omega_p''|A_p|^2. \quad (3.19)$$

Since the growth rate is $\omega_p''>0$, the polarization of the medium increases, i.e., the high-frequency dipole moments of the molecules increase and become phased. As the mode amplitude grows further, the inversion ΔN begins to vary. In the adiabatic approximation, the time dependence of $|A_p|^2$ can again be described by (3.19) in which the growth rate $\omega_p''(t)$ in

(3.18) is determined by the instantaneous difference $\Delta N(t)$ between the populations [see (1.4)]. The structure of the electromagnetic field in the interior of the layer of active medium is then set by the mode structure (3.1) with the instantaneous value of the amplitude $A_p(t)$, whereas, outside the layer, it is set by the structure of departing waves (3.2), whose amplitude at each point z is determined by the mode amplitude $A_p(t')$ at the retarded time $t'=t+z/c$ for $z<0$, and $t'=t-(z-L)/c$ for $z>L$. In other words, in the nonstationary problem that we are considering, the field outside the specimen is formed by radiation from the surface of the specimen.[12–14]

The population difference $\overline{\Delta N}(t)$ averaged over the length L of the specimen and over the high-frequency period $2\pi/\omega_p$ is the solution of the equation

$$\frac{d\overline{\Delta N}}{dt}=-\frac{2}{\hbar\omega_0}\frac{c|A_p|^2}{4\pi L}, \tag{3.20}$$

which follows from (1.3) in the absence of noncoherent relaxation ($T_1=\infty$). For simplicity, we shall also neglect the slow time dependence of $\sigma_{\rm rad}$ and substitute $\omega_p''\propto\overline{\Delta N}$ in accordance with (3.18). It is important to note that the use of the average inversion $\overline{\Delta N}$ in the growth rate (3.18) is a very crude approximation, similar to that adopted in the mean field model.[2,15] In this approximation, (3.19) and (3.20) can be integrated and the result of this is referred to as the conservation of the length of the Bloch vector:[2,3]

$$|A_p(t)|^2+\frac{2\pi\hbar\omega_0 L\omega_p''(0)}{c\overline{\Delta N}(0)}[\overline{\Delta N}(t)]^2=\text{const}. \tag{3.21}$$

Equation (3.19)–(3.21) lead to the following well-known expression for the superradiant pulse shape:[2,3,15]

$$\overline{\Delta N}(t)=-\overline{\Delta N}(0)\,\text{th}\left(\frac{t-t_0}{2\tau}\right),$$

$$Q(t)=\frac{\hbar\omega_0\overline{\Delta N}(0)}{4\tau}\text{ch}^{-2}\left(\frac{t-t_0}{2\tau}\right), \tag{3.22}$$

where $Q=-(\hbar\omega_0/2)d\,\overline{\Delta N}/dt=c|A_p|^2/4\pi L$ is the power radiated per unit volume of the specimen,

$$\tau=3c\hbar\ln|2/(\varepsilon^{1/2}(\omega_m)-1)|/4\pi\omega_0 d^2\overline{\Delta N}(0)L$$

is the length of the superradiant pulse, and $t_0=\tau\ln[Q_{\max}/Q(0)]$ is the delay time (see Fig. 1). It is assumed in (3.22) that, at the initial time $t=0$, the radiated power has the fluctuation value

$$Q(0)\ll Q_{max}=\hbar\omega_0\overline{\Delta N}(0)/4\tau,$$

so that $t_0\gg\tau$.

It is clear from the solution (3.22) that the superradiant pulse length τ is $1/2\omega_p''$, i.e., it is equal to the reciprocal of the growth rate of dissipatively unstable polariton mode closest to the line center. This is valid even for extended specimens of length exceeding the Arecchi-Courtens length (in the so-called oscillator superradiance regime[2,3,10]). In this case, $2\pi\sigma_{\rm rad}\lesssim|\omega_c|$ and the growth rate $\omega_{p''}$ assumes the maximum possible value $|\omega_c|/2$ [see (2.8)], which determines the maximum rate of the superradiant process. Correspondingly, the minimum superradiant pulse length is equal to the Arecchi-Courtens cooperative time, well known in superradiance theory:[11] $\tau_c=1/|\omega_c|$. As far as the conditions necessary for dissipative instability (3.16) are concerned, they are the same in both cases as the previously established conditions for superradiance in the one-dimensional model.[3,10]

The above account shows that the superradiance effect is a consequence of the dissipative instability of polariton modes with negative energy, which arises as a result of energy loss by radiation through the boundaries of the specimen of inverted medium.[5)] When $2\pi\sigma_{\rm rad}<1/T_2$, i.e., the length of the specimen is very large so that $L>cT_2\ln|2/(\varepsilon^{1/2}-1)|$, it is clear from (3.15) that, instead of the dissipative instability of polariton modes, we have the usual maser instability of electromagnetic modes. This corresponds to a transition to maser amplification of spontaneous emission which is referred to as "superluminescence."[3] This transition is due to the fact that energy dissipation by radiation through the boundaries becomes ineffective and the total power loss (3.13) is negative [see the discussion of (3.12)–(3.14) and footnote 3].

Next, let us examine the case opposite to superradiance, in which the main factor in field-energy dissipation is volume ohmic loss [see (3.13)], and radiation losses through the boundaries are of minor significance: $\sigma\gg\sigma_{\rm rad}$, i.e.,

$$L\gg(c/2\pi\sigma)\,[\ln|2/(\varepsilon^{1/2}-1)|].$$

For sufficiently strong inversion $|\omega_c|\gg 2/T_2$ and, according to (3.15) and (3.16), we again have dissipative instability. In particular, when $2\pi\sigma\gg|\omega_c|$, the nonlinear dynamics of the development of this instability is described by (3.22). The latter follows from formulas analogous to (3.18)–(3.21) if we replace $\sigma_{\rm rad}$ with σ and define Q as the ohmic power released in the form of heat rather than radiation. This process of collective relaxation of the population difference in the inverted medium, which gives rise to a pulsed heat release, may be referred to as superabsorbance.

In the general case of an arbitrary ratio between ohmic power dissipation and power lost by radiation through the boundaries of the specimen, the existence of dissipative instability of polariton modes under the conditions defined by (3.16) leads to a composite superradiance + superabsorbance effect. The result is collective relaxation of the inverted molecules to the lower energy level and, after a delay $t_0\gg 1/2\omega_p''$ following the end of the pump pulse, the emission of a powerful pulse of electromagnetic radiation from the specimen, accompanied by its ohmic heating for the time[6)] $\tau=1/2\omega_p''$ (see Fig. 1).

§4. SUPERRADIANCE FROM A THREE-DIMENSIONAL SPECIMEN

Let us now consider superradiance from an open three-dimensional specimen of an inverted medium in the form of a sphere (this problem was posed in a number of papers; see Ref. 17 and the references cited therein). For a homogeneous sphere of arbitrary radius R, the solution of the linearized equations (1.1) and (1.2) ($\Delta N=$ const) for the complex amplitudes of electric and magnetic fields can be expressed in spherical polar coordinates in terms of the Debye potentials[14] u and v:

$$E_{0r}=\left(\frac{\partial^2}{\partial r^2}+k^2\right)(ru),\quad B_{0r}=\left(\frac{\partial^2}{\partial r^2}+k^2\right)(rv),$$

$$E_{0\theta}=\frac{1}{r}\frac{\partial^2(ru)}{\partial r\,\partial\theta}+\frac{i\omega}{c\sin\theta}\frac{\partial v}{\partial\varphi},$$

$$B_{0\theta}=-\frac{ik\varepsilon^{1/2}}{\sin\theta}\frac{\partial u}{\partial\varphi}+\frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\theta},\tag{4.1}$$

$$E_{0\varphi}=\frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\,\partial\varphi}-i\frac{\omega}{c}\frac{\partial v}{\partial\theta},$$

$$B_{0\varphi}=ik\varepsilon^{1/2}\frac{\partial u}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\varphi}.$$

The complex amplitude of the polarization is given by (2.5); $k=(\omega/c)\varepsilon^{1/2}(\omega)$ for $r<R$, and $k=\omega/c$ for $r>R$. The potentials u and v satisfy the same wave equation, namely,

$$\Delta f+k^2f=0\quad(f=u,v).\tag{4.2}$$

The solution of (4.2) can be written in the form of an expansion in terms of the spherical harmonics $Y_n^{(j)}(\theta,\varphi)$ and spherical Bessel and Hankel functions

$$j_n(kr)=(\pi/2kr)^{1/2}J_{n+1/2}(kr),$$

$$h_n^{(1)}(\omega r/c)=(\pi c/2\omega r)^{1/2}H_{n+1/2}^{(1)}(\omega r/c)$$

in the form

$$f^{(i)}=\sum_{n=1}^{\infty}\sum_{j=-n}^{n}f_{nj}^{(i)}j_n(kr)Y_n^{(j)}(\theta,\varphi)\tag{4.3}$$

in the interior of the sphere ($r\leqslant R$), and

$$f^{(e)}=\sum_{n=1}^{\infty}\sum_{j=-n}^{n}f_{nj}^{(e)}h_n^{(1)}(\omega r/c)Y_n^{(j)}(\theta,\varphi)\tag{4.4}$$

outside the sphere ($r\geqslant R$). The latter expansion is a superposition of spherical waves leaving the sphere. On the surface of the sphere ($r=R$), the Debye potentials must satisfy the boundary conditions

$$\varepsilon u^{(i)}=u^{(e)},\quad\frac{\partial}{\partial r}(ru^{(i)})=\frac{\partial}{\partial r}(ru^{(e)}),\tag{4.5}$$

$$v^{(i)}=v^{(e)},\quad\frac{\partial}{\partial r}(rv^{(i)})=\frac{\partial}{\partial r}(rv^{(e)}),\tag{4.6}$$

which represent the continuity of the tangential field components $E_{0\theta},E_{0\varphi},B_{0\theta},B_{0\varphi}$. Substituting the expansions (4.3) and (4.4) into them, and recalling the orthogonality of the spherical harmonics $Y_n^{(j)}$, we find that the conditions for a nontrivial solution of (4.5) and (4.6) leads to the following characteristic equations, respectively:

$$\frac{j_{n-1}(Z)}{\varepsilon^{1/2}(\omega)j_n(Z)}=\frac{h_{n-1}^{(1)}(Z')}{h_n^{(1)}(Z')}-\frac{cn(1-\varepsilon(\omega))}{\omega R\varepsilon(\omega)}\tag{4.7}$$

$$\frac{\varepsilon^{1/2}(\omega)j_{n-1}(Z)}{j_n(Z)}=\frac{h_{n-1}^{(1)}(Z')}{h_n^{(1)}(Z')},$$

$$Z=\omega\varepsilon^{1/2}(\omega)R/c,\quad Z'=\omega R/c.\tag{4.8}$$

The first of these determines the discrete spectrum ω_m of electric modes with radial index n and angular index $j=-n,\dots,0,\dots,n$, in which the field structure is described by the potential u. For these modes, $B_r\equiv0$. The second equation refers to the magnetic modes with $E_r\equiv0$, described by the potential v. As in the one-dimensional model [see the discussion after (3.5) and (3.19)], the field in the specimen in the case of superradiance by a sphere can be written as a superposition of the above nonstationary modes.[7)]

Let us now consider a sphere of large radius $R\gg\lambda$. Using the asymptotic expansion of the Bessel and Hankel functions[19] with $Z\to\infty$ for modes with a fixed radial index n

$$J_n(Z)\approx\left(\frac{2}{\pi Z}\right)^{1/2}\cos\left(Z-\frac{n\pi}{2}-\frac{\pi}{4}\right),$$

$$H_n^{(1)}(Z)\approx\left(\frac{2}{\pi Z}\right)^{1/2}\exp\left[i\left(Z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]$$

and again assuming that the parameters $1/T_2\omega_0,2\pi\sigma/\omega_0,|\omega_c|/\omega_0,|\varepsilon(\omega)-1|,|\omega-\omega_0|/\omega_0$ are small in comparison with unity, we find from (4.7) and (4.8) that $\mathrm{ctg}[Z-n\pi/2]=i/D$ where $D=1/\varepsilon^{1/2}(\omega)$ for electric modes and $D=\varepsilon^{1/2}(\omega)$ for magnetic modes. Let us write this characteristic equation in the form

$$\omega_m\varepsilon^{1/2}(\omega_m)=\pi c(n+2m)/2R-i\delta_m,\tag{4.9}$$

where the factor δ_m, which governs the inhomogeneity of the mode field, is given by

$$\delta_m=\frac{c}{R}\,\mathrm{arth}\,D\approx\frac{c}{2R}\ln\left(\frac{2}{1-D}\right).\tag{4.10}$$

Equations (4.9) and (4.10) coincide with the corresponding equations (3.5) and (3.6) in the one-dimensional model if we introduce the replacement $m\to(n+2m)$, $L\to2R$ in the latter. Consequently, all our conclusions about the properties of modes in a bounded specimen (§3) remain valid for the electric and magnetic modes of the three-dimensional open sphere with low radial indices n [see (3.7)–(3.11) and (3.14)–(3.17)]. In particular, for each n in (3.16), there is a dissipative instability of electric and magnetic polariton-type (m,p)-modes with negative energy and positive losses. The contribution of radiative energy loss through the surface of the sphere to the growth rate of $\omega_{m,p}^{\parallel}$ is given by

$$2\pi\sigma_{\mathrm{rad}}(m)=\frac{c}{2R}\ln\left|\frac{2}{\varepsilon^{1/2}(\omega_m)-1}\right|\tag{4.11}$$

[see (3.9) and (3.15)]. Hence, in an isotropic inverted medium in the form of a sphere, the superradiance and superabsorbance effects are determined by the dissipative instability of negative-energy modes.

Let us estimate the number M of disipatively unstable modes with growth rate of the order of the maximum value (3.15) under the following conditions that are typical for superradiance:

$$L_{min}\equiv4c\ln|2/(\varepsilon^{1/2}-1)|/(-\omega_c^2T_2)\ll2R\leqslant L_c\equiv c/|\omega_c|,$$

$$|\omega_c|\gg2/T_2,\quad\sigma=0.$$

We have already shown that dissipatively unstable polariton modes exist under these conditions for sufficiently low radial indices n (or, more precisely, for[8)] $n\ll\omega_0R/c$). For large values of n, we use the asymptotic expansions of the Bessel and Hankel functions of the form[19)] $H_n^{(1)}=J_n+iY_n$ for $n\to\infty$ and fixed Z:

$$J_n(Z) \approx \frac{1}{(2\pi n)^{1/2}} \left(\frac{eZ}{2n}\right)^n,$$

$$Y_n(Z) \approx -\left(\frac{2}{\pi n}\right)^{1/2} \left(\frac{eZ}{2n}\right)^{-n}.$$

Substituting these into the characteristic equation (4.7) for the electric modes, we obtain an equation of the form $\varepsilon(\omega) \approx -1 - i\nu$ where ν gives the contribution of radiative losses to the dissipative instability growth rate. For large n, this quantity turns out to be small: $\nu \propto (\omega_0 R/cn)^{2n+1}$. It follows from the solution of this equation, namely,

$$\omega = \omega_0 + \omega_c^2/4\omega_0 - i/T_2 - i\omega_c^2\nu/8\omega_0$$

and the fact that the polarization relaxation time is finite ($T_2 \neq \infty$), that all the electric modes are damped out $n \gg \omega_0 R/c$ for ($\omega'' < 0$). The characteristic equation (4.8) for the magnetic modes then reduces to $\omega R/c = 2n$, which has no resonance roots $\omega \simeq \omega_0$. We may therefore conclude from this that the number $\bar{n}$ of different radial structures generating dissipatively unstable modes is of the order of[8] $\omega_0 R/c$. Using (3.17) and recalling the $(2n+1)$-fold degeneracy of the modes, which is due to the fact that the characteristic equations (4.7) and (4.8) are independent of the angular index $j = -n, \ldots, 0, \ldots, n$ of the spherical harmonics $Y_n^{(j)}$, we obtain the required estimate:

$$M \sim \bar{n}(2\bar{n}+1) \sim (\omega_0 R/c)^2 \gg 1. \tag{4.12}$$

The presence of a large number of simultaneously unstable modes is the fundamental difference between superradiance in a sphere and superradiance in the one-dimensional model. As we know,[2,3] the latter is closest to a cylinder of length L and cross section $S \sim \lambda L$, since the Fresnel number is then $F = S/\lambda L \sim 1$ and the inhomogeneity of the field in the lateral direction is at a maximum. Most papers on superradiance[2,3,10] are devoted to the study of superradiance by this kind of "single-mode" cylinder within the framework of the one-dimensional model. There is some point, therefore, in comparing superradiance from a sphere and from a cylinder of length $L \sim 2R \gg \lambda$ and lateral cross section $S \sim 2R\lambda$ for the same inversion density ΔN. In accordance with the foregoing, the length of the superradiant pulse is the same for the sphere and the cylinder: $\tau = 1/2\omega_p''$ [see (3.18)]. The peak superradiant power Q_{max} from the sphere is greater by the factor $2R/\lambda$ as compared with the cylinder. This factor is determined by the ratio of the energies stored in these two specimens:

$$(4\pi/3)R^3\hbar\omega_0\Delta N/RS\hbar\omega_0\Delta N \sim 2R/\lambda.$$

For the same level of fluctuations (thermal and spontaneous), the initial amplitudes of unstable modes are roughly equal, so that the total initial power $Q(0)$ in the sphere is greater by the factor M as compared with the cylinder. Since $t_0 = \tau \ln[Q_{max}/Q(0)]$ and $M \sim (2R/\lambda)^2$ [see (3.22) and (4.12)], it follows that the superradiance pulse delay time for the sphere is shorter than for the cylinder by an amount of the order of $\tau \ln(2R/\lambda)$. Next, we must allow for the fact that superradiance from a sphere consists of fields due to a large number of electric and magnetic modes with multipole radiation patterns [see (4.1) and (4.3)] and random initial phases. The superradiance from the sphere is therefore almost isotropic and its intensity per unit solid angle is smaller by the factor $2R/\lambda$ than the superradiance intensity from a single-mode cylinder radiating into the narrow solid angle $\sim(\lambda/2R)^2$ along its axis.

Finally, we shall show that, in the limit as $R/\lambda \ll 1$, the exact characteristic equations (4.7) and (4.8) lead to the well-known results obtained in the granular model of superradiance (Dicke model).[1-3] We now use the following expansion for the spherical Bessel functions:[19]

$$j_n(Z) = \frac{Z^n}{1\cdot 3\ldots(2n+1)}\left(1 - \frac{Z^2/2}{2n+3} + \ldots\right),$$

$$y_n(Z) = -\frac{1\cdot 3\ldots(2n-1)}{Z^{n+1}}\left(1 + \frac{Z^2/2}{2n-1} + \ldots\right);$$

and recall that the first-order spherical Hankel function is $h_n^{(1)} = j_n + iy_n$, we obtain from (4.7) the following approximate expression for the electric dipole ($n = 1$) mode:[9]

$$\varepsilon(\omega) \approx -2\left[1 + \frac{6}{5}\left(\frac{\omega}{c}R\right)^2\right] - 2i\left(\frac{\omega}{c}R\right)^3. \tag{4.13}$$

If we use the permittivity (2.1), the solution of this equation gives $\omega' = \omega_0$ and

$$\omega'' + \frac{1}{T_2} = -\frac{\omega_c^2}{9\omega_0^2}\left(2\pi\sigma + \frac{\omega_0^4 R^3}{c^3}\right). \tag{4.14}$$

In the inverted medium in which $\omega_c^2 < 0$, the growth rate becomes positive for sufficiently small values of $1/T_2$.

The growth rate ω'' can also be deduced from energy considerations if we recall that we are dealing with the oscillations of a high-frequency point dipole $(4\pi/3)R^3\mathbf{P}$, and we neglect the growth rate $1/T_2$ for simplicity. The power loss per unit volume, $Q = Q_{rad} + Q_{ohm}$, consists of radiation into the space surrounding the grain[12] $Q_{rad} = (\omega_0^4|P_0|^2/3c^3)(4\pi R^3/3)$ and ohmic losses $Q_{ohm} = \sigma|E_0|^2/2$. The complex polarization and field amplitudes in these expressions are related by $P_0 = (\varepsilon - 1)E_0/4\pi \approx -3E_0/4\pi$ [see (4.13)]. The energy density in the uniformly polarized grain is given by the following well-known electrodynamic formula:[8,13]

$$w = \frac{|E_0|^2}{16\pi}\frac{d[\omega\varepsilon(\omega)]}{d\omega} \approx \frac{|E_0|^2}{8\pi}\frac{9\omega_0^2}{\omega_c^2} < 0. \tag{4.15}$$

We can then readily verify that the quantity $\omega'' = -Q/2w$, is in complete agreement with the growth rate (4.14). The electric dipole mode of the grain is therefore a dissipatively unstable polariton mode of negative energy.

In the adiabatic approximation, the dynamics of dissipative instability in a small specimen $R \ll \lambda$ is described by the equations

$$dQ/dt = 2\omega''Q, \quad d\Delta N/dt = -2Q/\hbar\omega_0 \tag{4.16}$$

[compare this with (3.19) and (3.20)]. When $T_2 = \infty, \sigma = 0$, which is the case usually examined in superradiance theory,[2,3] the solution of (4.16) is similar to that given in §3 and leads to the classical superradiant pulse shape (3.22) (see Fig. 1). In particular, the superradiant pulse length is $\tau = 1/2\omega'' = -9c^3/2\omega_c^2\omega_0^2R^3$ which agrees with the Dicke granular model.[1-3]

§5. CONCLUSION

We have shown that two types of modes, namely, electromagnetic and polariton, can be excited in an inverted two-level medium. Electromagnetic mode instability developes when the degree of inversion is small and there is considerable relaxation of polarization ($|\omega_c| < 2/T_2$). Polariton-mode instability is possible in a highly inverted medium ($|\omega_c| > 2T_2$). Mode energy calculations show that the polariton modes have negative energy whereas electromagnetic modes have positive energy. Polariton modes are therefore unstable for true (positive) energy dissipation whereas electromagnetic modes are unstable for negative dissipation. The latter situation corresponds to the maser mechanism of amplification of electromagnetic radiation. In contrast to this, the instability of polariton modes is the dissipative instability of negative-energy modes. It arises as a result of volume ohmic dissipation and also radiation through the boundaries of the specimen of active medium.

It follows from the account presented above that Dicke superradiance is the result of the development of the above dissipative instability of negative-energy polariton modes and is due to energy loss by radiation through the specimen boundaries. Superabsorbance is possible when ohmic losses predominate. It is analogous to superradiance in that it leads to rapid collective relaxation of inverted molecules to the lower energy state in a time much shorter than the time of noncoherent relaxation of the isolated molecule ($\tau \ll T_1$). However, superabsorbance converts the energy stored in the active medium into heat rather than electromagnetic radiation. Superabsorbance is gradually transformed into superradiance as ohmic losses are reduced, i.e., the electrical conductivity of the medium is reduced. In both cases, the length of the collective relaxation pulse is determined by the dissipative instability growth rate: $\tau = 1/2\omega_p''$. The growth rate ω_p'' and the polariton mode structure depend on the shape of the specimen. They were found above for the one-dimensional model (flat layer) and the three-dimensional model (sphere).

Our description of the connection between Dicke superradiance and dissipative instability of polariton modes can be exploited to provide a more complete macroscopic description of different features of this effect. This will require further analysis of the dynamics of collective excitations of three-dimensional specimens of active media under superradiant conditions. The nonlinear interaction between modes and the spatial distribution of inversion within the specimen will be of particular interest. In addition, the statistical properties of superradiance can be investigated by quantizing the polariton and electromagnetic mode oscillators.

1)This paper is confined to the problem with initial condition. When the boundary-value problem is formulated, the dispersion relation given by (2.2) must be looked upon as an equation in the complex wave number k. In the absence of spatial dispersion, it is a quadratic and its solution $k = \pm(\omega/c)\varepsilon^{1/2}(\omega)$ determines only one type of propagating forced oscillation.

2)It is precisely for this reason that the nonelectromagnetic branch is referred to as a polarization wave. We note that, when $|\omega_c^2|T_2/4 \ll \omega_0$, which is usually satisfied in gases, the susceptibility of the active medium is small ($|\chi| \ll 1$), and the magnetic field in both waves is roughly the same (Since $ck/|\omega_{e,p}| = |\varepsilon^{1/2}(\omega_{e,p})| \approx 1$; see (1.1)].

3)For the electromagnetic wave $w_e > 0$ and the wave amplitude increases ($\omega_e'' = -Q_e/2w_e > 0$) for negative loss $Q_e < 0$, which is possible only for $2\pi\sigma < 1/T_2$ [see (2.14)] and corresponds to the maser instability [see (2.7)]. In an uninverted medium[8] in which the energy and power loss are positive for both waves, there is no instability.

4)We note, for comparison, that, for an active medium in a closed resonator with perfectly reflecting walls, the spatial structure of these modes is homogeneous: $\mathrm{Im}[\omega\varepsilon^{1/2}(\omega)] = 0$. When the reflection coefficient R is small, we have $2\pi\sigma_{\rm rad} = (c/2L)\ln R^{-1}$.

5)An analogous superradiance instability is found in nonequilibrium plasma in magnetic traps.[16]

6)When $1/T_2 = 0, |\omega_c| \ll 2\pi\sigma$, $\sigma_{\rm rad} = 0$, the quantity $1/2\omega_p''$ is the pulse length noted in Ref. 5, which is described by the equations of the one-dimensional model of superradiance when periodic boundary conditions are imposed and radiative losses are taken into account by introducing effective volume losses.

7)The modes of an active spherical specimen in a nontransparent medium that prevents superradiance are discussed in Ref. 18.

8)To show this more rigorously, we have to use the uniform asymptotic expansions of the Bessel and Hankel functions of the form[19] $J_n(nZ)$, $H_n^{(1)}(nZ)$ for $n \to \infty$.

9)All the magnetic and higher-order multipole electric ($n \geqslant 2$) modes are damped in the Dicke granular model of superradiance. This follows from (4.7) and (4.8).

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Note added in proof (30 September 1984). The existence of modes with symmetric structure (Fig. 4) is confirmed by observations of synchronous superradiance in the form of identical pulses from the two ends of a $KCl{:}O_2$-crystal [Florian *et al.*, Phys. Rev. A **29**, 2709 (1984)]. This regime is not described by the unidirectional propagation model [Haake *et al.*, Phys. Rev. A **29**, 3208 (1984)]. It arises when the reflection coefficient R is mall but finite (10^{-2}–10^{-6}), and is due to the discontinuity in permittivity across the boundary of $L \lesssim L_c = c/|\omega_c|$, when $2\pi\sigma_{\rm rad} = (c/2L)\ln R^{-1} > |\omega_c|/2$ [see (3.9) and (3.18)].