Expanding self-similar discontinuities and shock waves in dispersive hydrodynamics

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Nonlinear flows in nondissipative dispersive hydrodynamics are examined. It is shown that the description of such flows requires the introduction of a new concept, namely, a discontinuity of a special kind, called self-similar. This is a nondissipative shock wave and replaces the strong discontinuity of ordinary hydrodynamics. The self-similar discontinuity expands linearly in time. It is shown that it can be inscribed into the solution of the Euler equations. Conditions formulated for the boundaries of the self-similar discontinuity permit the closure of the Euler equations of dispersive hydrodynamics, i.e., they replace the shock adiabat of ordinary dissipative hydrodynamics. A classification is carried out, and a complete solution is given, for the decay or arbitrary initial discontinuities in the hydrodynamics of highly nonisothermal plasmas.

The development of a finite-amplitude perturbation in the hydrodynamics of perfect fluids described by the Euler equations usually leads to the appearance of a singularity (Ref. 1, §94). A strong discontinuity appears behind the singular point, and is the origin of a shock wave. It is important to note that this shock front is followed by an expanding region in which a change of entropy has taken place, i.e., a change in the equation of state of the gas. The dissipation responsible for this change occurs in the region of the discontinuity. The nondissipative equations of a perfect fluid must therefore be augmented on the discontinuity by a special condition in the form of a shock adiabat that matches the thermodynamic quantities on the two sides of the discontinuity surface. The shock adiabat does not depend on the rate of dissipation: the latter determines only the width of the region of discontinuity.

The picture changes radically in the absence of dissipation, when all the higher-order corrections to the Euler equations are purely dispersive in character. Such conditions occur, for example, in collisionless plasma, in waves on water, in electroacoustics, and in other media.[2,3] Undamped oscillations develop in dispersive hydrodynamics. In the region behind the singularity, where their amplitude is at a maximum, they form a nondissipative shock wave.[4] The basic properties of this type of wave when its amplitude is small were determined by Pitaevskiĭ and one of the present authors,[5] who obtained rigorous asymptotic solutions for the case of a weak nonlinearity for which the equations of dispersive hydrodynamics degenerate to the Korteweg–de Vries (KdV) equation. It turns out that the initial discontinuity in the system described by the KdV equations spreads into the oscillation region that expands rapidly in time. The expansion process is determined by oscillations that have the following structure. They begin with a nonstationary chain of solitons whose separation grows logarithmically with time. The solitons gradually degenerate into a quasistationary wave. The amplitude of oscillations in this wave decreases with increasing distance from the soliton front, and their shape becomes nearly sinusoidal. The average (over the oscillations) densities and velocities on the leading wavefront have a weak rather than a strong discontinuity with infinite derivative, named singular discontinuity in Ref. 5. The singular discontinuity moves with ultrasonic velocity. This particular form of discontinuity appears only in dispersive hydrodynamics. Ultrasonic weak discontinuities are impossible in ordinary dissipative hydrodynamics.[1]

One can now understand the fundamental difficulties that arise when an attempt is made to solve nonstationary problems in dispersive hydrodynamics when the nonlinearity is not small. In ordinary hydrodynamics, the situation reduces to the solution of only the Euler equations augmented by the shock-adiabat condition, whereas in dispersive hydrodynamics there is a region of nonlinear oscillations in which the Euler equations are not valid. In its general form, the asymptotic set of equations describing the oscillatory part was formulated by Whitham[6] on the assumption that the wave picture was quasistationary as a whole, but it actually turned out to be too complex and has not been used in practice to solve nonlinear problems.[1)] Dubrovin and Novikov[7] have put forward a general method that can be used to write down the average equations in Hamiltonian form. As for direct numerical calculations, it is important to note that the presence of a small dispersive parameter governing the scale of the resulting oscillations leads to an essential limitation on the temporal and spatial dimensions of the region to be computed, and this complicates the determination of the asymptotic properties of the solutions.

The aim of this work is to investigate essentially nonlinear nonstationary flows in dispersive hydrodynamics, making simultaneous use of analytic and numerical methods. We shall demonstrate the necessity for introducing a new concept, namely, that of a special type of singularity, which we refer to as self-similar and which replaces the strong discontinuity of ordinary hydrodynamics. Nonlinear problems of dispersive hydrodynamics are then found to become in principle much simpler and reduce to the solution of the Euler equations augmented by special conditions on the boundaries of the self-similar discontinuity. As a specific example, we shall consider the decay of arbitrary initial discontinuities in the hydrodynamics of nonisothermal plasmas.

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Section 1 investigates the general structure of nonlinear flows in the hydrodynamics of a nonisothermal plasma. Analysis of oscillating components in the region of the rarefaction wave, in the region of weak discontinuities, and in the region of the plateau has shown that they all decay in time in a power-law manner (in proportion to $t^{-\alpha}$, where $\alpha = 1/3$, $1/2$, and $2/3$) because of dispersive effects. The only exception is the singular region where there is eventually established a quasistationary wave whose parameters, i.e., amplitude, wavelength, and so on, depend only on $\tau = x/tc_s$, where c_s is the velocity of sound. The asymptotic form of the solution as $t \to \infty$ in terms of the self-similar variables τ is thus identical with the solution obtained in Euler hydrodynamics, except for the region of the quasistationary wave.

The quasistationary-wave region $\tau_- \leqslant \tau \leqslant \tau_+$ is discussed in §2. It is shown that the structure of the quasistationary wave is completely analogous to that investigated in Ref. 5. In particular, the leading wavefront $\tau \approx \tau_+$ always has soliton properties, and the distance between the solitons increases logarithmically with time. The average density $\bar{n}(\tau)$ and average velocity $\bar{v}(\tau)$ undergo a large change in the region of the quasistationary wave. This change is particularly large on the leading wavefront where these two quantities have a singular discontinuity: $\bar{n},\bar{v} \propto (\ln|\tau_+ - \tau|)^{-1}$. Since the self-similar solutions of the Euler equations contain only the rarefaction wave or the plateau (n and v are constants), the transition from the plateau to undisturbed gas in the absence of the rarefaction wave can occur only in the region of the quasistationary wave. The quasistationary wave thus replaces the shock wave of ordinary hydrodynamics. It can naturally be referred to as the nondissipative shock wave (NSW).

Thus, the Euler equations are asymptotically valid in dispersive hydrodynamics of nonisothermal plasmas everywhere except for the self-similar expansion region $\tau_- < \tau < \tau_+$. We shall refer to this region as the self-similar discontinuity. It is shown in §3 that a transition across the self-similar discontinuity leads to a fully defined change in the gas velocity v, which depends only on the density discontinuity:

$$\Delta v = v_+ - v_- = c_s[F(n_+) - F(n_-)],$$

$$v_+ = v(\tau_+), \quad v_- = v(\tau_-), \quad n_+ = n(\tau_+), \quad n_- = n(\tau_-). \quad (1)$$

In the hydrodynamics of nonisothermal plasmas, $F(n) = \ln n$. The relation (1) assures the closure of the Euler equations. In dispersive hydrodynamics, it plays the same role as the shock-adiabat condition on a strong discontinuity in ordinary hydrodynamics. The important point is, however, that there is no change in entropy in the region behind the self-similar discontinuity, i.e., behind the NSW. This is the fundamental distinction between the self-similar discontinuity and the strong discontinuity of ordinary hydrodynamics.

Thus, when asymptotic solutions are obtained in dispersive hydrodynamics, we can proceed as in ordinary hydrodynamics by confining our attention to the Euler equations augmented by the conditions (1) on the self-similar discontinuity. This enables us to perform a classification, and to construct a complete solution, for the decay of an arbitrary initial dicontinuity (§4)

§1. STRUCTURE OF ONE-DIMENSIONAL FLOWS

Consider the structure of nonstationary one-dimensional flows in dispersive hydrodynamics of a highly nonisothermal plasma described by the equations[8]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{e}{M}\frac{\partial \varphi}{\partial x} = 0; \quad (2)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -4\pi e\left(n - n_0 \exp\frac{e\varphi}{T_e}\right), \quad (3)$$

where n is the ion density, v is the hydrodynamic velocity of the ions, M is their mass, φ is the electric potential, e and T_e are the charge and temperature of the electrons, and n_0 is the density of undisturbed plasma. The initial conditions corresponding to the decay of an arbitrary initial discontinuity at $t = 0$ are of the form

$$n_{x<0} = n_0, \quad n_{x>0} = n_1; \quad v_{x<0} = 0, \quad v_{x>0} = v_1. \quad (4)$$

Numerical integration of (2) and (3) subject to the initial conditions (4) yields the results shown in Fig. 1. It is clear that nonlinear flow is accompanied by efficient excitation of oscillations. The scale of these oscillations is determined by a parameter with the dimensions of length, which is contained in (3) viz., the Debye length:

$$D(n) = (T_e/4\pi e^2 n)^{1/2}. \quad (5)$$

Letting $D \to 0$, i.e., neglecting the term containing the principal derivative in (3), we arrive at the quasineutral limit in which (2) and (3) assume the form of the Euler equations of isothermal hydrodynamics:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c_s^2}{n}\frac{\partial n}{\partial x} = 0, \quad c_s^2 = \frac{T_e}{M}. \quad (6)$$

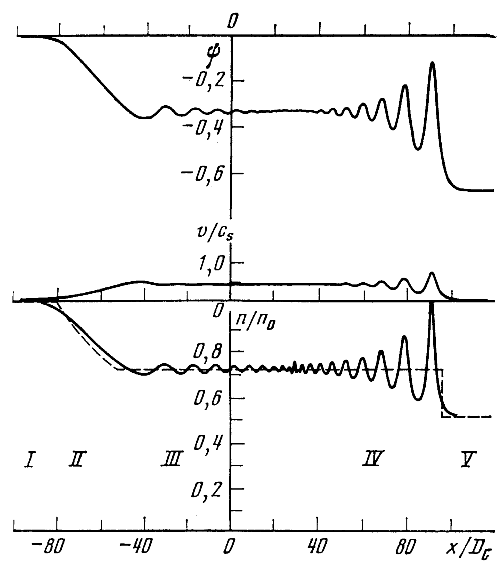

FIG. 1. Distribution of the potential $\psi = e\varphi/T_e$, velocity v, and density n as plasma flows into plasma ($n_1 = 0.5n_0$, $v_1 = v_0 = 0$) at time $t = 80\,\Omega^{-1}$, $\Omega = (4\pi e^2 n_0/M)^{1/2}$. Dashed line—solution of the equations of dissipative isothermal hydrodynamics (6).

Thus, in the Euler limit, the equations of dispersive hydrodynamics become identical with the equations of ordinary dissipative hydrodynamics, as should indeed be the case. The solution of (6) with initial conditions (4), obtained by the standard method in ordinary hydrodynamics (Ref. 1, §85), is shown in Fig. 1 by the dashed line.

When we compare the solution in Fig. 1 with the solution obtained in ordinary hydrodynamics, we cannot but note the first considerable qualitative similarity between the structures. This is particularly evident in the self-similar variables $\tau = x/tc_s$. There are well-defined regions on all these curves: the region of unperturbed dense plasma (I), the self-similar expansion (II), the plateau (III), the shock wave (IV), and the unperturbed rarefied plasma (V). An analogous correspondence arises when we compare the other solutions of (2), (3), and (6) under the same initial conditions. It follows hence that the overall structure of nonlinear one-dimensional flows in dispersive hydrodynamics of nonisothermal plasmas is determined by the Euler equations, just as it is in ordinary hydrodynamics.

We emphasize that the rigorous procedure of eliminating the small parameter with dimension of length from the equations of dispersive hydrodynamics does not lead at all to the Euler equations but to the very complex chain of nonlinear equations obtained by Whitham.[6] Despite this, we see that the exact (numerical) solutions of the equations of dispersive hydrodynamics remain very similar in their structure to the solutions of the Euler equations.

It is important to note that it is only within the framework of the Euler equations without parameters having the dimension of length that we can distinguish unequivocally between the singularities that arise in nonlinear flows, namely, the strong and weak discontinuities that are the most important qualitative characteristics of the process. The singularities are smoothed out in equations in which the small dimensional parameters are retained. The following singularities are well known to arise in ordinary Euler hydrodynamics: weak discontinuities, tangential discontinuities, and strong discontinuities. They can be seen in the example illustrated in Fig. 1. The character of these discontinuities is substantially transformed when we pass to dispersive hydrodynamics. Let us therefore perform a detailed investigation of the structure of the solution of (2) and (3), using analytic approximations.

Weak discontinuity on the boundary of the unperturbed region. The equations (6) of isothermal hydrodynamics have the following self-similar solution:

$$n=n_0,\quad v=0,\quad \psi=e\varphi/T_e=\text{const}=0,\quad \tau=x/tc_s<-1, \tag{7a}$$

$$n=n_0e^{-\tau-1},\quad v=c_s(\tau+1),\quad \psi=-\tau-1,\quad \tau>-1. \tag{7b}$$

A weak discontinuity appears on the boundary of the region of unperturbed plasma (7a) and rarefaction wave (7b). The departure of the exact numerical solution of (2) and (3) from (7) is illustrated in Fig. 2. It is clear that it falls rapidly with time. To estimate the rate of this decrease, we recognize that the dimensionless potential Ψ changes little in the region of the weak discontinuity: $\psi \ll 1$. We can therefore confine our

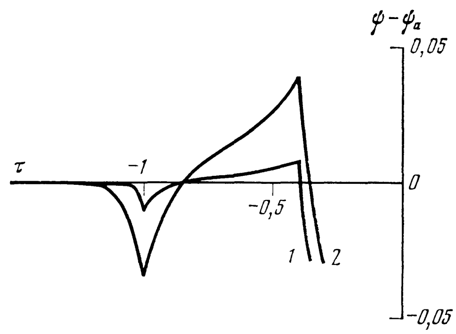

FIG. 2. Difference between the exact numerical solution and the approximation (7) in the region of the rarefaction wave $1 - t = 700/\Omega$; $2 - t = 100/\Omega$.

attention to the weak-nonlinearity approximation, i.e., we can use the KdV equation to which (2) and (3) reduce at $\psi \ll 1$ (Ref. 2). The structure of the weak discontinuities in the KdV approximation is examined in Ref. 9, where it is shown that a weak discontinuity is described the function

$$\psi=t^{-2/3}f(p),\quad p=x^*/t^{1/3},\quad x^*=x-c_0t, \tag{8}$$

where $c_0 = c_s + v$ is the velocity of the weak discontinuity. On the whole, this formula agrees with the numerical calculation illustrated in Figs. 1 and 2. However, it is found that the solution does not reach the asymptotic form (8) even for the longest computation time $t = 1000/\Omega$, $\Omega = (4\pi e^2 n_0/M)^{1/2}$.

Rarefaction wave. It is clear from Figs. 1 and 2 that, in the region of the rarefaction wave, the numerical solution rapidly approaches the self-similar limit (7). Deviations from it decrease with time in proportion to $1/t$.

Weak discontinuity on the boundary of the rarefaction wave and the region of the plateau. It is clear from Fig. 1 that this discontinuity is accompanied by oscillations in the region of the plateau. The general structure of the oscillations in the region of the discontinuity takes the form of the Airy function; their amplitude decreases with time in proportion to $1/t^{1/3}$ and the wavelength increases at $t^{1/3}$. The properties of this weak discontinuity are not consistent with the discontinuities established in Ref. 9 on the basis of the KdV equation. The reason for this discrepancy is that the analysis given in Ref. 9 does not take into account oscillations excited in the region of the plateau which, as will be shown later, have a dominating influence on the structure of the discontinuity.

Plateau region. For sufficiently long t, the average (i.e., the average over the oscillations) numerical values of $\bar{n}$, $\bar{v}$ and $\bar{\psi}$ turn out to be constant throughout the plateau region between $\tau = \tau_2 = -1 + \bar{v}/c_s$ and $\tau = \tau_-$ to within the maximum precision of the numerical calculation. For example, in the case shown in Fig. 1, ($n_1 = 0.5n_0$, $v_1 = v_0 = 0$), the average values $\bar{n}, \bar{v}, \bar{\psi}$ are related to the values n_p, v_p, ψ_p on the plateau as follows:

$$\bar{n}=n_p=0.707n_0,\quad \bar{v}=v_p=0.346c_s,\quad \bar{\psi}=\psi_p=-0.346.$$

Oscillations on the plateau. It is clear from Fig. 1 that oscillations on the plateau have a distinctive structure: the amplitude and wavelength vanish at the point $x_p \approx 0.346c_st$.

This property persists for all t. Consequently, τ_p is a constant:

$$\tau_p=x_p/tc_s=v_p/c_s, \tag{9}$$

with τ_p equal to the average velocity $\bar{v}/c_s$ of the gas on the plateau. The point x_p must therefore coincide with the point of degenerate tangential discontinuity of ordinary hydrodynamics (Ref. 1, §81). This is a singular point relative to which the medium is immobile and through which there is no flow of material. The region to the left of it contains the gas that occupied the left half-space prior to the start of decay, and the region to the right contains the gas that occupied the right half-space.

We shall use Eqs. (2) and (3) linearized in terms of n_p, v_p, ψ_p to analyze the behavior of the oscillations on the plateau. We then have

$$n_\sim(x,t)=\int_{-\infty}^{+\infty} n_0(k)\, e^{i(kx-\omega t)}\, dk, \quad n_\sim=n-n_p, \tag{10}$$

where $n_0(k)$ is the Fourier transform of the initial perturbation. Similar expressions can be obtained for $v\sim$, $\psi\sim$. The frequency $\omega=\omega(k)$ is defined by the dispersion relation

$$\omega(k)=kv_p+kc_s/[1+(kD_p)^2]^{1/2}, \tag{11}$$

where D_p is the Debye length (5) for $n=n_p$. It is clear from (2) and (3) that the initial perturbation $n\sim(x,0)$ is produced near $x\approx 0$. Formula (10) describes its spreading over the plateau region. Using the method of steepest descent to evaluate the integral (10), we find that

$$n_\sim(x,t)=\left(2\pi/t\left|\frac{d^2\omega}{dk^2}(k^*)\right|\right)^{1/2} n_0(k^*)$$
$$\exp[i(k^*x-\omega(k^*)t)+i\pi/4], \tag{12}$$

where $k^*(x,t)$ is determined by the equation

$$d\omega(k^*)/dk=x/t.$$

Using this together with (11), we find that

$$k^*=D_p^{-1}[(\tau^*)^{-2/3}-1]^{1/2}, \quad \tau^*=\tau-\tau_p,$$
$$\lambda=2\pi/k^*=2\pi D_p(\tau^*)^{1/3}[1-(\tau^*)^{2/3}]^{-1/2}. \tag{13}$$

The numerical calculations are compared with (13) in Fig. 3. As can be seen, there is complete agreement.

We emphasize that the initial perturbation of the potential and of the electron density, determined in accordance with (2) and (3), does not resemble a sharp discontinuity such as (4), but is spread out with a scale $\sim D$ over the region near $x\approx 0$. The oscillations on the plateau are the result of the decay of this initial perturbation. They vanish in the limit as $t\to\infty$, and, as is clear from (12), their amplitude a tends to zero: $a\propto t^{-1/2}$ (Fig. 4). This result is then an obvious consequence of energy conservation in nondissipative media: the energy density of the oscillations is proportional to a^2 and the region of the plateau occupied by them expands in time in proportion to t.

We empashize that the vanishing of the oscillations on the plateau signifies simultaneously the vanishing of the tangential discontinuity. Consequently, in the asymptotic limit as $t\to\infty$, there is no degenerate tangential discontinuity in the hydrodynamics of nonisothermal plasmas.

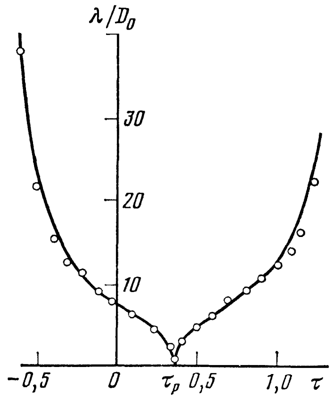

FIG. 3. Wavelength λ of potential oscillations on the plateau as a function of τ. The curve is a plot of (13), points—numerical solution, $D_0=(T_e/4\pi e^2 n_0)^{1/2}$.

It is clear from Fig. 1 that the oscillation amplitude increases sharply near the boundary between the plateau region and the rarefaction wave. This should indeed by the case since this boundary is defined by

$$\tau=-1+v_p/c_s,$$

i.e., $|\tau^*|=1$, and, as is clear from (12) and (13), the wavelength λ and the oscillation amplitude a increase sharply as $|\tau^*|\to 1$. This is the region of the caustic for the oscillations on the plateau. It follows from (13) that their structure near the caustic is described by the equation

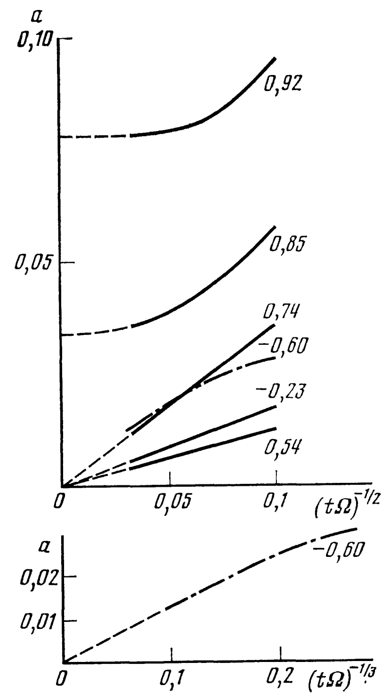

FIG. 4. Time dependence of the potential oscillation amplitude a on the plateau for different values of τ.

$$\frac{d^2n_\sim}{d\xi^2}+\xi n_\sim=0, \quad \xi=\left(\frac{2}{3D_p{}^2c_s}\right)^{1/3}\frac{x-x_2}{t^{1/3}}, \tag{14}$$

where $x_2 = t(c_s - v_p)$ is a point of weak discontinuity on the boundary of the plateau region. It follows from (14) that oscillations near the discontinuity are described by the Airy function, their wavelength increases with time in proportion to $t^{1/3}$, and their amplitude decreases as $\propto t^{-1/2}$. This is in agreement with numerical calculations (Fig. 3, and dot-dash curve in Fig. 4).

§2. QUASISTATIONARY WAVE

Consider the region $\tau > \tau_p$. It is clear from Fig. 1 that the oscillations in this region grow with increasing τ. Moreover, their asymptotic behavior changes radically near the point τ_-. Whereas at $\tau < \tau_-$, their amplitude decreases with time as $t^{-1/2}$, it tends to a stationary, t-independent level (Fig. 4), at $\tau > \tau_-$. The quasistationary wave will thus begin at $\tau > \tau_-$. It fills the self-similarly expanding region from the trailing edge $\tau = \tau_-$, on which the generation of oscillations takes place, to the leading front $\tau = \tau_+$ on which these oscillations assume the form of individual solitons. In the case shown in Fig. 1, for example, $\tau_- = x_-/c_s t \approx 0.84$ and $\tau_+ = x_+/c_s t \approx 1.24$.

The quasistationary wave is an almost stationary nonlinear wave with slowly (in comparison with wavelength and the oscillation frequency) varying parameters, namely, the amplitude, velocity, and wavelength. The stationary wave itself is readily found as the stationary oscillating solution of (2) and (3). It is described by

$$\xi=\frac{1}{[2(W_{max}-W_{min})]^{1/2}}\int d\psi\left[s^2+\frac{W_{min}-W(\psi)}{W_{max}-W_{min}}\right]^{-1/2}, \tag{15}$$
$$W(\psi)=e^\psi+u^2[1-(1-2\psi/u^2)^{1/2}]-C, \quad \xi=x-utc_s,$$

where $W_{\min}$ and $W_{\max}$ are, respectively, the maximum and minimum values of the function $W(\psi)$, uc_s is the phase velocity of the wave, and s^2 is a parameter characterizing the oscillations (it varies from zero to 1). When $s^2 \to 0$, the oscillations have a small amplitude and sinusoidal shape. When $s^2 = 1$, they take the form of individual solitons whose velocity is known[4] to be uniquely related to the amplitude ψ_m:

$$u_s{}^2=(e^{\psi_m}-1)^2/2(e^{\psi_m}-\psi_m-1). \tag{16}$$

As $s^2 \to 1$, the solution (15) takes the form of a chain of solitons whose separation λ increases logarithmically with decreasing $1 - s^2$:

$$\lambda_{s^2\to1}\approx\frac{u_s}{2(u_s{}^2-1)^{1/2}}[\ln(1-s^2)^{-1}+C_1]. \tag{17}$$

The other parameters, for example the wavelength, remain finite as $s^2 \to 1$:

$$u=u_s-C_2(1-s^2). \tag{18}$$

The quantities C_1 and C_2 are determined by the parameters of the stationary wave.

Let us now examine in greater detail the structure of the quasistationary wave in the neighborhood of its leading $\tau \approx \tau_+$ and trailing $\tau \approx \tau_-$ edges.

Trailing edge. In the neighborhood of the trailing edge $\tau \approx \tau_-$, the amplitude a of the oscillations is small, so that it is natural to examine the solution by expanding it in powers of a:

$$n=n_p(1+n_1a+n_2a^2+\ldots), \quad v=v_p+c_s(v_1a+v_2a^2+\ldots),$$
$$\psi=\psi_p+\psi_1a+\psi_2a^2+\ldots, \tag{19}$$

where n_p, v_p, ψ_p are the constant density, velocity, and potential on the plateau. Substituting this expansion in (2) and (3) we find that, in the linear approximation (10),

$$n_1=1+(kD_p)^2, \quad v_1=[1+(kD_p)^2]^{1/2}, \quad \psi_1=1. \tag{20}$$

Here a is the amplitude of the oscillations of the field ψ. The wave frequency will, of course, satisfy the dispersion relation (11). We recognize also that, under quasistationary conditions, the number of waves is always conserved:[6]

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x}(Vk)=0,$$

where $V = \omega/k$ is the phase velocity of the wave. In terms of the self-similar variables $\tau = x/tc_s$, this equation assumes the form

$$(u-\tau)\frac{dk}{d\tau}+k\frac{du}{d\tau}=0, \quad u=\frac{\omega}{kc_s}. \tag{21}$$

In the linear approximation, we obtain from (21) and (11), the relations (13) that define the wave vector $k(\tau^*)$ and hence the coefficients n_1 and v_1 in (20) as well:

$$n_1=(\tau_-{}^*)^{-2/3}, \quad v_1=(\tau_-{}^*)^{-1/3}, \quad \tau_-^*=\tau_--v_p. \tag{22}$$

In the second approximation, we consider only the corrections $\bar{n}_2$, $\bar{v}_2$, $\bar{\psi}_2$ averaged over the oscillations. After some simple algebra, we find from (2), (3), (19), and (22) that

$$\bar{n}_2=-\frac{1+(\tau_-{}^*)^{-2/3}}{4[1-(\tau_-{}^*)^2]}, \quad \bar{v}_2=-\frac{2-(\tau_-{}^*)^2+(\tau_-{}^*)^{4/3}}{4\tau_-{}^*[1-(\tau_-{}^*)^2]},$$
$$\bar{\psi}_2=-\frac{2+(\tau_-{}^*)^{-2/3}-(\tau_-{}^*)^2}{4[1-(\tau_-{}^*)^2]}. \tag{23}$$

It is thus clear from (19) and (23) that all the quadratic corrections are negative, i.e., the mean density, the velocity, and the potential decrease in the region of the quasistationary wave in proportion to the square of the amplitude (a^2). The amplitude itself increases linearly with τ near the point τ_-:

$$a=C(\tau_-)(\tau-\tau_-). \tag{24}$$

This result is obtained from (21) by taking into account quadratic corrections proportional to a^2 in the expressions for u and k for stationary nonlinear waves. The function $C(\tau_-)$ is shown in Fig. 5. The derivation of (24) for small amplitude discontinuities corresponding to the lower branch of the curve $C(\tau_-)$ is given in the Appendix. It follows from (19), (22), and (24) that τ_- is the only free parameter characterizing the structure of the solutoin in the region of the trailing edge. Equations (19), (23), and (24) are in complete agreement with numerical calculations.

Leading (soliton) front. Our numerical calculations have shown that individual solitons whose separation increases slowly with time gradually appear on the leading wavefront.

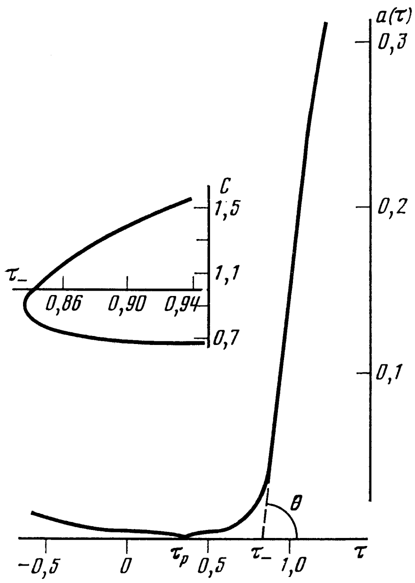

FIG. 5. Potential-oscillation amplitude as a function of τ near the point τ_- (for $n_1 = 0.5n_0$, $v_1 = v_0 = 0$) and the dependence of $C = \tan\theta$ on τ_- (24).

To analyze the asymptotic structure of the soliton front, we shall use Eq. (21) which expresses the conservation of the number of waves. Substituting in (21) Eqs. (18) and (17) for u and $k = 2\pi/\lambda$ we obtain

$$1-s^2=\tau_1^*/C_2 \ln\left[(\tau_1^*)^{-1}C_2 e\right], \quad \tau_1^*=\tau_+-\tau. \tag{25}$$

This expression describes the variation in the parameter s^2 near the leading wavefront. Substituting (29) in (17), we see that solitons appear as we approach the wavefront, and the separation between them increases logarithmically with time: $\lambda \sim \ln(\tau_1^*)^{-1}$. Accordingly, the mean density $\bar{n} - n_1$, potential $\bar{\psi} - \psi_1$, and wave velocity $\bar{v}$ vanish near the wavefront in accordance with the logarithmic law.

$$(\bar{n}-n_1)\sim(\bar{\psi}-\psi_1)\sim\bar{v}\sim[\ln(\tau_1^*)^{-1}]^{-1}. \tag{26}$$

We note that this asymptotic behavior can be discerned in numerical calculations only for very long times $t \gg 10^4/\Omega$. It follows from (26) that a "singular" discontinuity is always present on the leading front of the quasistationary wave.[5,8]

The velocity of the leading soliton is $u_s = \tau_+$. Accordingly, its amplitude is $\psi_m = \psi_m(\tau_+)$ (16). When $\psi_m - \psi_1 \gtrsim 1.3$, i.e., $\tau_+ \geqslant 1.6$, the soliton breaks,[4] so that the structure of the quasistationary wave for $\tau_+ > 1.6$ is no longer described by single-flow hydrodynamics.

§3. SELF-SIMILAR DISCONTINUITY

The above analysis has shown that as $t \to \infty$ the asymptotic solution of the equations of dispersive hydrodynamics (2) and (3) is identical, except for the region $\tau_- \leqslant \tau \leqslant \tau_+$, with the solution of the Euler equations (6). For $\tau_- \leqslant \tau \leqslant \tau_+$, a transition occurs from the plateau n_p to the unperturbed gas n_1; the quasistationary wave is excited in this region.

Thus, instead of the strong discontinuity of ordinary hydrodynamics, a relatively rapid self-similar transition occurs in dispersive hydrodynamics: the discontinuity appears to spread out into the self-similar expanding oscillating region. The precise structure of the solution in this region is not of particular interest, and we may confine our attention to the Euler equations, except that we introduce into them a discontinuity occupying the finite region between τ_- and τ_+ in terms of the self-similar variables. We shall call this the self-similar discontinuity. In dispersive hydrodynamics it replaces the strong discontinuity of ordinary dissipative hydrodynamics. It is important to emphasize that, by calling this discontinuity self-similar we emphasize the fact that, in the problem defined by (4), it expands so that its boundaries are displaced linearly in time, i.e., they are immobile in terms of the self-similar variables $\tau = x/t$. This type of discontinuity may also appear in other formulations of the problem (se, for example, Ref. 5), but the law of its expansion, i.e., the law of motion of its boundaries, may be different.

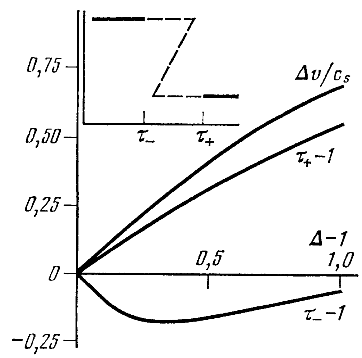

FIG. 6. Velocity jump $\Delta v/c_s = v_2/c_s$ and values of τ_+ and τ_- as functions of the relative jump Δ in density across the self-similar discontinuity. Inset shows the breaking of a simple wave; dashed line—multivalued solution.

Let us now formulate the conditions for matching the hydrodynamic variables on the two sides of the self-similar discontinuity. The first point to take into account is that there is no dissipation in the region of the self-similar discontinuity, i.e., the state of the gas does not change behind the discontinuity. It is also important that the self-similar discontinuity arises in the region of a simple wave in place of the multivalued solution that appears after the break (see inset in Fig. 6). The hydrodynamics variables v and n in a simple wave are known to be related by (Ref. 1, §94)

$$dv=\pm c_s(n)\,dn/n,$$

where $c_s(n)$ is the velocity of sound. This relation is valid everywhere outside the region of mutlivalued solution or, more precisely, outside the region of the quasistationary wave. Integrating it, we obtain the following difference across the boundaries of the self-similar discontinuity:

$$v(\tau_+)-v(\tau_-)=\pm\int_{n(\tau_-)}^{n(\tau_+)}\frac{c_s(n)}{n}\,dn. \tag{27}$$

In particular, in the case of isothermal hydrodynamics (6) considered here,

$$v(\tau_+)-v(\tau_-)=\pm c_s \ln\Delta, \quad \Delta=n(\tau_-)/n(\tau_+). \tag{28}$$

The last two expressions establish the relationship between the velocity difference and the density difference across the discontinuity. They constitute a closure of the Euler equations and play the same role in dispersive hydrodynamics as the shock adiabat in ordinary dissipative hydrodynamics. It is very important to note that (27) and (28) are determined exclusively by Euler hydrodynamics and do not depend on the nature of dispersive effects, or on the structure of the quasistationary wave. Equation (18) is illustrated in Fig. 6. It is in complete agreement with numerical calculations.

Figure 6 shows in addition the boundaries τ_+ and τ_-, of the self-similar discontinuity, which are also single-valued functions of the relative density jump Δ, given by (28). As for the structure of the solution in the region of the discontinuity i.e., the structure of the quasistationary wave, it also depends only on the jump Δ. More precisely, the functions $[\bar{n}(\tau) - n(\tau_+)]/n(\tau_+)$ and $[\bar{v}(\tau) - v(\tau_+)]/c_s$ averaged over the oscillations turn out to be the same for discontinuities with the same density jumps Δ. For given Δ, the oscillations themselves either expand or contract with similarity parameter $n^{-1/2}(\tau_+)$ because of the change in the Debye length $D(n)$ [Eq. (5)]. These conclusions follow from a general analysis of (2) and (3)[2)] and are in complete agreement with numerical calculations. This can be seen, for example, from Fig. 7, which gives the numerical solution of two problems with different initial conditions, and shows the same jump $\Delta = n_2/n_1 = 1.92$ for equal values of $n(\tau_+)$. It can be seen that the structure of the oscillations in the region of the quasistationary wave is the same for the two solutions.

We note that the self-similar discontinuity can be replaced by an "equivalent" strong discontinuity by writing down the usual continuity conditions for the flux and momentum across the discontinuity in the form (see Ref. 1, §81)

$$(v_0-v_2)n_2=v_0n_1, \qquad (v_0-v_2)^2/2+w(n_2)+g(\Delta)=v_0^2/2. \tag{29}$$

Here v_0 is the velocity of the equivalent discontinuity, $\omega(n) = c_s^2\ln(n/n_1)$ is the thermal function of isothermal hydrodynamics (Ref. 1, §2), and

$$g(\Delta)=c_s^2\left(\frac{1}{2}+\frac{1}{\Delta-1}-\frac{1}{\ln(\Delta)}\right)\ln^2(\Delta), \qquad \Delta=\frac{n_2}{n_1} \tag{30}$$

is a correction to the thermal function and expresses the effective momentum change due to the self-similar expansion of the discontinuity. It follows from (29) and (30) that

$$v_0=c_s[1+1/(\Delta-1)]\ln\Delta, \qquad v_2=c_s\ln\Delta \tag{31}$$

which is in accord with (28). According to (31), the velocity v_0 is always greater than c_s and less than $c_s\tau_+$.

§4. DECAY OF INITIAL DISCONTINUITIES

The asymptotic solutions of dispersive hydrodynamics of nonisothermal plasmas can be constructed, as shown above, within the framework of the Euler equations. They differ from the usual dissipative hydrodynamics by the following: (a) instead of a strong discontinuity, we now have an expanding self-similar discontinuity; (b) the equation of state of the gas in the region behind the self-similar discontinuity does not change; (c) there is no tangential discontinuity (by this, we mean a degenerate tangential discontinuity in one-dimensional motion).

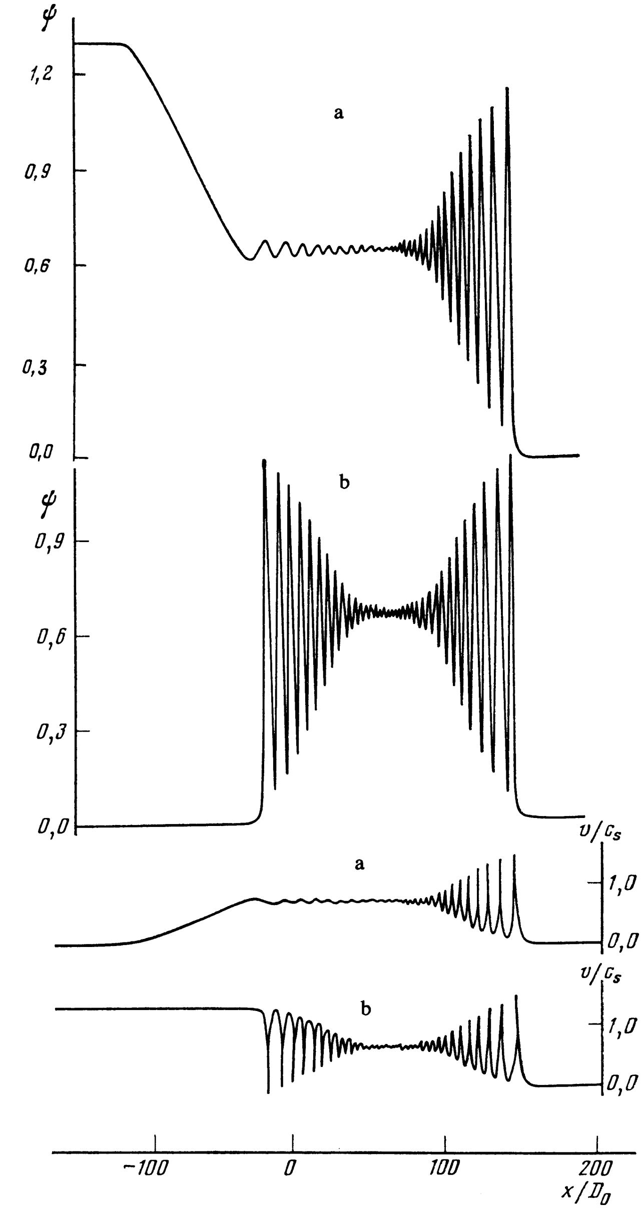

FIG. 7. Distribution of the potential ψ and velocity v in plasma at time $t = 100\ \Omega^{-1}$ for: a—influx of plasma into plasma ($n_0 = 3.67n_1$, $v_0 = v_1 = 0$), b—collision between two plasmas ($n_0 = n_1$, $v_0 = 1.3c_s$, $v_1 = 0$).

When these changes are taken into account, it is possible to perform a precise classification of the decay of arbitrary initial discontinuities (4), in dispersive hydrodynamics as illustrated in Fig. 8. Discontinuities arising after decay should move away from the point at which they are formed, i.e., from the position of the initial discontinuity. Repeating the reasoning given in Ref. 1 (§93), it can then be readily verified that either a self-similar discontinuity or a rarefaction wave bounded by a pair of weak discontinuities can propagate in each of the two directions. Figure 8 shows all possible variants of this type of motion. The first case (Fig. 8a) occurs when two gases collide. Two self-similar discontinuities (i.e., two nondissipative shock waves) are formed from the initial discontinuity and move in opposite direc-

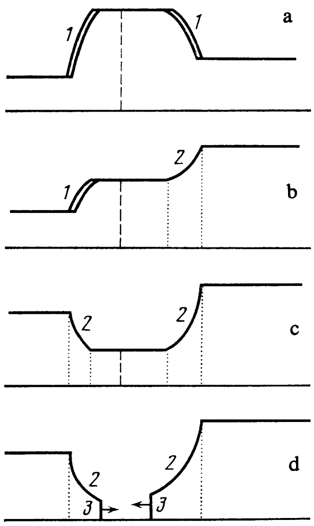

FIG. 8. Classification of decays of initial discontinuities in accordance with the density distribution: 1—self-similar discontinuity; 2—rarefaction wave: dotted line—weak discontinuity, dashes—tangential discontinuity; 3—accelerating strong discontinuity.[10]

tions away from the point $\tau_p = v_p$. In the second case (Fig. 8b), which occurs, for example, when plasma flows into plasma with zero initial velocity, we have a rarefaction wave on one side of the point τ_p and a self-similar discontinuity on the other. Finally, in the third case (Fig. 8c), which occurs when a plasma expands, we have rarefaction waves flowing in both directions away from the point τ_p. We note that, as $t \to \infty$, a vacuum region is not produced asymptotically as the plasma expands (this is in contrast to ordinary hydrodynamics; Ref. 1, (§93). However, for finite values of t and high expansion velocities, there may be a vacuum region that contracts logarithmically with time (in terms of the variables τ) and is bounded by singular accelerating strong discontinuities on the plasma-vacuum boundary that appears in the dynamics of nonisothermal plasmas[10] (Fig. 8d).

We shall show that the above relationships can readily be used to find all the basic characteristics of flow produced during the decay of an arbitrary initial discontinuity (4). Without loss of generality, we may suppose that $v_0 = 0$ for $x < 0$ (we are transforming to a coordinate frame moving with velocity v_0). Moreover, if we normalize to the velocity c_s, we find from (28) for the self-similar discontinuity on the left

$$v_2 = -\ln(n_2/n_0) \tag{32}$$

and on the right

$$v_2 = v_1 + \ln(n_2/n_1). \tag{33}$$

The rarefaction wave is described by (7b). Matching (32), (33), and (7b), we can find all the basic characteristics.

For example, in the case of collision between two gases [Fig. 8a, $v_1 < \ln(n_1/n_0)$], equating the velocities v_2 on the plateau in accordance with (32) and (33), we find that

$$-\ln(n_2/n_0) = v_1 + \ln(n_2/n_1),$$

and hence

$$n_2 = (n_1 n_0)^{1/2} \exp(-v_1/2), \quad v_2 = [-\ln(n_1/n_0) + v_1]/2. \tag{34}$$

The points τ_+ and τ_- for both discontinuities are determined with the aid of the curves shown in Fig. 6, remembering that $\Delta = n_2/n_0$, for the left discontinuity and $\Delta = n_2/n_1$ for the right discontinuity. There is thus a net transport with velocity v_1. Knowing the velocity $u_s = \tau_+$ of the leading soliton, we can find its amplitude from (16). Equations (17), (18), and (25) describe the structure of the soliton front, and (19), (22), (23), and (24) the structure of oscillations in the region of the trailing edge. The tangential discontinuity point $\tau_p = v_2$, the structure of the oscillations in the region of the plateau, and the structure of the weak discontinuities are described by the expressions given in §1.

Similarly, when plasma flows into plasma [Fig. 8b, $n_1 < n_0$, $\ln(n_1/n_0) < v_1 < -\ln(n_1/n_0)$], we find from (7b) and (33) that n_2 and v_2 are given by (34), as before. The location of the weak discontinuities bounding the region of the rarefaction wave is given by

$$\tau_1 = -1, \quad \tau_2 = -1 + v_1/2 - \ln(n_1/n_0)/2. \tag{35}$$

The points τ_+ and τ_- are determined in accordance with Fig. 6 for $\Delta = n_2/n_1$, using the transport velocity v_1. In the case of expanding plasmas [Fig. 8c, $v_1 > -\ln(n_1/n_0)$], we find from (7b) that the values of n_2 and v_2 on the plateau are given by (34), as before. The locations of the weak discontinuities bounding the rarefaction wave were determined from (35) on the left and from $\tau_1 = 1 + v_1$, $\tau_2 = 1 + v_1/2 + \ln(n_1/n_0)/2$ on the right. A vacuum region bounded by the accelerating strong discontinuities (Fig. 8d) exists for certain finite intervals of time. Numerical calculations are in complete agreement with these solutions.

It is important to note the breaking of the leading soliton of a quasistationary wave, which occurs when the soliton amplitude becomes sufficiently large (16), so that the velocity $u_s = \tau_+$ reaches the value 1.6. Using (34) and the results of numerical calculations, we may determine the limit for the parameters of the initial discontinuity for which breaking occurs:

$$v_{1c} = \ln(n_0/n_1) - 1{,}38.$$

Hence, it is clear that, in particular, when $v_1 = 0$, the critical value is $(n_0/n_1)_c \approx 4.0$. When $v_1 > v_{1c}$, there is no breaking and the single-flux solution examined above exists. If, however, $v_1 < v_{1c}$, the soliton breaks and this means that a second flux, that of the particles reflected from the solitons, appears.[4]

It is important, however, that the soliton breaking changes only the width of the region of the self-similar discontinuity region, increasing the value of τ_+. Conditions (27) and (28) remain unchanged in this case. This means that breaking does not restrict the range of validity of the hydrodynamic solution constructed above, which is therefore valid for arbitrary values of the initial parameters. Of course, the width and structure of the self-similar discontinuity change as a result of the break. However, the analysis of this question is outside the framework of the present research.

We note that although we have confined our attention to the case of isothermal dispersive hydrodynamics, described by (2), (3), and (6), at a gas pressure $P \propto n$, analogous results can readily be obtained for an arbitrary dependence of the form $P \propto n^{\gamma}$. We then have $c_s = c_0 n^{(\gamma-1)/2}$, $c_0^2 = T_e/M$, and instead of (28) we obtain

$$\Delta v = \frac{2c_0}{\gamma-1}\left(n_+^{(\gamma-1)/2} - n_-^{(\gamma-1)/2}\right).$$

Instead of (34), we then obtain the following expression for the decay of an arbitrary initial discontinuity:

$$n_2 = \left[\frac{1}{2}\left(n_0^{(\gamma-1)/2} + n_1^{(\gamma-1)/2}\right) - \frac{\gamma-1}{4c_0}v_1\right]^{2/(\gamma-1)},$$

$$v_2 = \frac{c_0}{\gamma-1}\left(n_0^{(\gamma-1)/2} - n_1^{(\gamma-1)/2}\right) + \frac{v_1}{2}.$$

Numerical calculations show complete agreement with these results.

The authors are indebted to V. L. Ginzburg and L. P. Pitaevskiĭ for useful discussions and to O. A. Gel'fond for help with the numerical calculations.

APPENDIX

For small initial discontinuities, $\Delta n = (n_2 - n_0)/n_0 \ll 1$, the change of variables

$$x_1 = \frac{(x - c_s t)(2\Delta n)^{1/2}}{D_p}, \quad t_1 = \frac{t\Omega}{2^{1/2}}(\Delta n)^{3/2},$$
$$\eta = \frac{2(n-n_0)}{n_0 \Delta n} = \frac{2v}{c_s \Delta n} = \frac{2\psi}{\Delta n} \tag{A.1}$$

reduces Eqs. (2) and (3) as is well known, to the KdV equation (Ref. 2, §15):

$$\frac{\partial \eta}{\partial t_1} + \eta \frac{\partial \eta}{\partial x_1} + \frac{\partial^3 \eta}{\partial x_1^3} = 0.$$

Consider the region near the trailing edge of a quasistationary wave in the case of the KdV equation. The dispersion relation for linearized KdV waves is

$$\omega = k(1-k^2).$$

Here we have taken into account the fact that the parameter η is normalized to unity on the plateau, in accordance with (A.1). From (21) we than have

$$u = (2+\tau_1)/3, \quad k = (1-\tau_1)^{1/2}/3^{1/2}, \quad \tau_1 = x_1/t_1. \tag{A.2}$$

We have also taken into account the fact that, in the decay of the initial discontinuity, u and k depend asymptotically only on the self-similar variable τ_1. For the average quantity $\bar{\eta}$ we find by analogy with (23) that, in the approximation that is quadratic in the wave amplitude a, we have

$$\bar{\eta} = 1 - a^2/4(1-\tau_1). \tag{A.3}$$

Stationary nonlinear KdV waves are defined by three parameters. We shall take them to be $\bar{\eta}$, a and s^2, where the last parameter determines the nature of the oscillations in the waves. The phase velocity u and the wave vector k for nonlinear waves are then given by[8]

$$u = \frac{2a}{3s^2}\left(2 - s^2 - \frac{3E(s)}{K(s)}\right) + \bar{\eta} = \bar{\eta} - \frac{2}{3}\frac{a}{s^2} + \frac{a}{3} + \frac{as^2}{8} + \ldots,$$

$$k = \frac{\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2} = 2\left(\frac{a}{6s^2}\right)^{1/2}\left(1 - \frac{s^2}{4} - \frac{5}{64}s^4 + \ldots\right), \tag{A.4}$$

where K(s) and E(s) are the complete elliptic integrals of the first and second kind, respectively. The last two expansions are written so as to indicate that we are interested only in the neighborhood of the trailing edge of the discontinuity τ_{1-}, where the amplitude a of the oscillations and the parameter s^2 are small. Comparison of (A.2) with (A.4) for $a, s^2 \to 0$ then shows that the parameters a and s^2 are not independent near τ_{1-}, but are related by

$$s^2 = 2a/(1-\tau_{1-}). \tag{A.5}$$

If we use (A.5) and (A.3), we can rewrite (A.4) in the form

$$u = \frac{2+\tau_{1-}}{3} + \frac{a}{3} + O(a^3),$$

$$k = \left(\frac{1-\tau_{1-}}{3}\right)^{1/2}\left[1 - \frac{a}{2(1-\tau_{1-})} - \frac{5a^2}{16(1-\tau_{1-})^2}\right].$$

Substituting these expressions in (21), and taking the terms of the principal order in a and a^2, we reduce it to the form

$$\left[\frac{\tau_1 - \tau_{1-}}{2} - \frac{3}{4}a\right]\frac{da}{d\tau_1} = 0.$$

Hence it follows that $da/d\tau_1 \neq 0$ if $a = 2/3(\tau_1 - \tau_{1-})$, which corresponds to (24), to the lower branch of the $C(\tau_-)$ curve for $\tau_- \to 1$ in Fig. 5, and to the results of the exact solution.[5]

[1)]Apart from the KdV equation, for which the Whitham system becomes in principle much simpler.[5]

[2)]In fact, if we introduce the new variable $\Delta n = n/n_1$ instead of n, where $n_1 = n(\tau_+)$, we can write (2) and (3) in the form

$$\frac{\partial}{\partial t}\ln \Delta n + v\frac{\partial}{\partial x}\ln \Delta n + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \psi}{\partial x} = 0,$$

$$D^2(n_1 + \Delta n)\frac{\partial^2 \psi}{\partial x^2} = -1 + e^{\psi}.$$

From this and (28) it follows that the solutions of the average equations, which do not include the Debye length,[6] depend only on the ratio n_2/n_1, whereas the change of $D(n_1)$ determines the scale of the oscillations.

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