The role of plasma compressibility in the gradient soliton problem

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We study the role of plasma compressibility in the gradient soliton problem. We show that the compressibility can affect both the linear and the nonlinear properties of gradient waves and thereby lead to the formation of gradient solitons. Some new kinds of gradient solitons caused by the compressibility are indicated, viz., electron fluted and oblique solitons and electron-ion short-wavelength drift solitons in a plasma with a finite ion pressure. We introduce the concepts of paramagnetic and diamagnetic solitons.

1. INTRODUCTION

One of the vital problems in the contemporary theory of nonlinear waves in a plasma is the problem of solitary gradient waves or, in other words, of gradient solitons. Waves in a magnetized plasma (i.e., a plasma in a magnetic field), which in the linear approximation are characterized by the gradients of the equilibrium parameters of the plasma and of the magnetic field, are called gradient waves. At one time linear gradient waves were studied extensively in connection with the problem of gradient (drift) instabilities. One can thus obtain an idea abouth these waves by turning to a monograph dealing with the theory of gradient instabilities.[1] The initial development of the theory of gradient solitons was connected with the work by Tasso, Oraevskiĭ, and Petviashvili.[2–4] In those papers, as in the majority of later papers (see, e.g., Refs. 5,6), one was dealing waves in relation to which the plasma behaved as an incompressible "two-dimensional" fluid with $\operatorname{div}\mathbf{V}_\perp = 0$, where $\mathbf{V}_\perp$ is the plasma velocity at right angles to the magnetic field (more precisely, $\mathbf{V}_\perp$ is the transverse velocity of the electron or ion component of the plasma, depending on the kind of wave). The aim of the present paper is a study of those kinds of gradient solitons whose existence is caused by the compressibility of the plasma.

Reference 7 is also related to our subject and we comment on that paper in what follows.

To get an idea under what circumstances the compressibility may be important we reason as follows. We note that in a number of cases of low-frequency waves (low-frequency in relation to the cyclotron rotation of the particles) the velocity of the plasma at right angles to the magnetic field is the same as the drift velocity of the particles in crossed electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, i.e., $\mathbf{V}_\perp \approx \mathbf{V}_E$, where

$$\mathbf{V}_E = c[\mathbf{E}\times\mathbf{B}]/B^2, \tag{1.1}$$

c is the velocity of light. It follows from (1.1) that

$$\operatorname{div}\mathbf{V}_E = -\frac{\mathbf{B}}{B^2}\frac{\partial\mathbf{B}}{\partial t} - c\mathbf{E}\operatorname{rot}\frac{\mathbf{B}}{B^2}. \tag{1.2}$$

In the problem of waves in a plasma the magnetic field is split into two parts, the equilibrium field $\mathbf{B}_0$ and the field of the wave $\tilde{\mathbf{B}}$, so that

$$\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}. \tag{1.3}$$

In the simplest case of a uniform equilibrium magnetic field, neglecting nonlinear terms, Eq. (1.2) becomes

$$\operatorname{div}\mathbf{V}_E = -\frac{1}{B_0}\frac{\partial\tilde{B}_z}{\partial t}. \tag{1.4}$$

It is clear that the plasma compressibility is connected with the compressibility of the magnetic field and as a consequence is important in the case of waves whose propagation is accompanied by the compression of rarefraction of the magnetic field.

The connection of the compressibility of the plasma with the compressibility of the magnetic field also follows from the freezing-in condition

$$\partial\mathbf{B}/\partial t = \operatorname{rot}[\mathbf{V}_\perp\times\mathbf{B}], \tag{1.5}$$

which is, in essence, nothing but another way of writing down Eq. (1.1). Taking the z-component of Eq. (1.5) in the case of the two-dimensional motions which are of interest to us we get

$$\partial B_z/\partial t = -B_z\operatorname{div}\mathbf{V}_\perp - (\mathbf{V}_\perp\nabla)B_z. \tag{1.6}$$

This result is in accordance with Eqs. (1.2), (1.4). It also follows from (1.6) that the magnetic field can be perturbed, $\tilde{B}_z \neq 0$, also in an incompressible plasma, $\operatorname{div}\mathbf{V}_\perp = 0$, if the equilibrium magnetic field is inhomogeneous, $\nabla B_0 \neq 0$. Such a situation occurs in particular in the case of the so-called inertial waves (see §16.3 of Ref. 1). The perturbation of the magnetic field is then caused by its convective transfer described by the second term on the right-hand side of (1.6). This fact must be taken into account when we look for the class of waves which are sensitive to the effect of the compressibility of the plasma.

At the same time we must bear in mind that the presence of the plasma compressibility does, in general, not mean the presence of a perturbation of the plasma density. The reason is that according to the equation of continuity,

$$\partial n/\partial t = -n\operatorname{div}\mathbf{V}_\perp - (\mathbf{V}_\perp\nabla)n, \tag{1.7}$$

a change in the density, like a change in the magnetic field, depends also on the convective transfer fo the plasma described by the second term on the right-hand side of this equation (n is the density of any species of particles, electrons or ions). Noteworthy in this connection is the case of quasineutral purely electron waves in which the perturbation of the electron density is negligibly small, $\tilde{n} = 0$. In that case

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the change in the density due to the plasma compressibility is exactly compensated by its change caused by the convective transfer. In that case we have instead of (1.7)

$$\operatorname{div} \mathbf{V}_{\perp} = -V_x \varkappa_n, \tag{1.8}$$

where $\varkappa_n = [\partial \ln n_0/\partial x$ and n_0 is the equilibrium density which is assumed to be inhomogeneous in the x-direction.

Equation (1.8) is also useful to illustrate the fact, which is important for us, that for an actual manifestation of the effects of the plasma compressibility the presence of gradients in the equilibrium plasma parameters, in particular, a density gradient, is important. To perceive this fact we combine (1.8) and (1.6) and, as before, we put $\nabla B_0 = 0$. Using the fact that for the purely electron waves considered

$$V_x = j_x/e_e n_0 = (c/4\pi e_e n_0)\,\partial \tilde{B}_z/\partial y \tag{1.9}$$

(e_e is the electron charge), we then get in the approximation which is linear in the wave amplitude

$$\partial \tilde{B}_z/\partial t + u_n \partial \tilde{B}_z/\partial y = 0, \tag{1.10}$$

$$u_n \equiv -\varkappa_n c^2 \omega_{Be}/\omega_{pe}^2, \tag{1.11}$$

where $\omega_{Be} = e_e B_0/m_e c$ is the electron cyclotron frequency, $\omega_{pe}^2 = 4\pi e_e^2 n_0/m_e$ the square of the electron plasma frequency. It is clear that the presence of a gradient in the equilibrium density produces in the magnetic field a compression (or rarefaction) wave propagating along the y axis with a velocity u equal to

$$u = u_n. \tag{1.12}$$

Waves of the king (1.12) were first studied in connection with instabilities produced by a transverse current[8] and in connection with so-called drift-cone instabilities.[9] One can call them electron fluted waves ($\partial/\partial z = 0$).

Turning to Eq. (1.6) one can check that the compressibility may determine not only the linear but also the nonlinear properties of the gradient waves. For instance, taking into account in (1.6) the term of order $\tilde{B}_z/B_0$ we get instead of (1.10) the nonlinear equation

$$\frac{\partial \tilde{B}_z}{\partial t} + u_n \frac{\partial}{\partial y}\left(\tilde{B}_z + \frac{\tilde{B}_z^2}{2B_0}\right) = 0. \tag{1.13}$$

By adding an inertial term to the equation we shall show in Sec. 2 the existence of electron fluted solitons propagating with the velocity (1.12).

Are the compressible plasma perturbations discussed by us an alternative to rotational perturbations? To answer this problem one must find a relation between div $\mathbf{V}_\perp$ and $\operatorname{curl}_z \mathbf{V}_\perp$. We do this using the example of purely electron perturbations considered above. The expression for div $\mathbf{V}_\perp$ is in this case given by Eq. (1.8) in which we can take as an estimate for $\operatorname{curl}_z \mathbf{V}_\perp$ the quantity $\partial V_x/\partial y \approx \varkappa V_x$, where $\varkappa$ is the reciprocal characteristic size of the perturbation. Hence,

$$\operatorname{div} \mathbf{V}_\perp/\operatorname{rot}_z \mathbf{V}_\perp \approx \varkappa_n/\varkappa. \tag{1.14}$$

It is clear that in the case of small-scale perturbations, $\varkappa \gg \varkappa_n$, the quantity div $\mathbf{V}_\perp$ is small compared to $\operatorname{curl}_z \mathbf{V}_\perp$. The small-scale compressible perturbations considered by us are thus approximately rotational. Correspondingly the theme of our paper may be formulated as the problem of "compressible vortices."

The electron fluted solitons in a plasma with cold electrons in a uniform magnetic field, which are discussed here and in Sec. 2, are the simplest example of gradient solitons caused by the compressibility of the plasma. Other examples of such solitons are discussed in Secs. 3 to 6. We study in Sec. 3 the inhomogeneity and curvature of the equilibrium magnetic field, and in section 4 we consider oblique solitons ($\partial/\partial_z \neq 0$). As in Sec. 2, we assume in that case that the electrons are cold and that the perturbations are purely electronic. In Sec. 5 we consider electron fluted solitons in a plasma with hot electrons and in Sec. 6 we take into account the ion motion. A summary and discussion of the results are given in Sec. 7.

2. ELECTRON FLUTED SOLITONS IN A PLASMA WITH COLD ELECTRONS IN A UNIFORM MAGNETIC FIELD

We continue the analysis of purely electron perturbations of a plasma with an inhomogeneous density in a uniform magnetic field which we started in Sec. 1. As in that section, we assume the perturbations to be fluted, i.e., propagating at right angles to B_0, and the electrons to be cold and the perturbed electron density to be negligibly small. The latter is valid under the well known condition $\omega_{pe}^2 \gg \omega_{Be}^2$. We assume the perturbations to be low-frequency with respect to the electrons, $\partial/\partial t \ll \omega_{Be}$.

Our first aim is to obtain for $\tilde{B}_z$ an equation analogous to (1.13) but supplemented by the inertial term. To do that we turn to the electron equation of motion

$$m_e \frac{d\mathbf{V}}{dt} = e_e\left(\mathbf{E} + \left[\frac{\mathbf{V}}{c} \times \mathbf{B}\right]\right), \tag{2.1}$$

where $d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$ and $\mathbf{V}$ is the electron velocity. We apply the curl operator to both sides of this equation and express the velocity and the electric field in terms of the magnetic field using the relation [cf.(1.9)]

$$\mathbf{V} = (c/4\pi e_e n_0)\operatorname{rot}\mathbf{B} \tag{2.2}$$

and the Maxwell equation curl $\mathbf{E} = -\partial \mathbf{B}/\partial t$. We then get (cf. Ref. 7)

$$\frac{\partial \mathbf{B}}{\partial t} - \frac{c}{4\pi e_e}\operatorname{rot}\left[\frac{\operatorname{rot}\mathbf{B}}{n_0}, \times \mathbf{B}\right] - \frac{c^2}{\omega_{pe}^2}\frac{d}{dt}\Delta_\perp \mathbf{B} = 0, \tag{2.3}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In the case of perturbations depending on x and $\zeta = y - ut$ it follows from this equation that [cf. (1.13)]

$$\tilde{D}\Delta_\perp h - \Lambda \partial h/\partial \zeta + S \partial h^2/\partial \zeta = 0. \tag{2.4}$$

Here $h = \tilde{B}_z/B_0$,

$$\tilde{D} = \frac{\partial}{\partial \zeta} - \frac{1}{u}[\nabla \chi, \times \nabla]_z, \tag{2.5}$$

$$\Lambda = \frac{\omega_{pe}^2}{c^2}\left(1 - \frac{u_n}{u}\right), \tag{2.6}$$

$$S = \frac{\omega_{pe}^2}{2c^2}\frac{u_n}{u}, \tag{2.7}$$

x is the stream function defined by the equation $V_x = -\partial x/\partial y$ and connected with h through the relation

$$\chi = -c^2 \omega_{Be} h/\omega_{pe}^2. \tag{2.8}$$

Equation (2.4) contains two kinds of nonlinearities: a vector one caused by the term $[\nabla x \times \nabla]_z$ in the operator $\tilde{D}$

[see (2.5)] and a scalar one described by the term with S. Correspondingly two kinds of solitons are possible, the so-called "vector" and "scalar" ones. The vector solitons were studied in Ref. 10. Neglecting the vector nonlinearity Eq. (2.4) reduces to the form

$$\Delta_\perp h - \Lambda h + S h^2 = 0. \tag{2.9}$$

From the condition that h be bounded at infinity it follows that $\Lambda > 0$. Using this and establishing with the aid of (2.9) the integral form

$$\int [(\nabla_\perp h)^2 + \Lambda h^2 - S h^3] d\mathbf{r} = 0, \tag{2.10}$$

we conclude that if h has constant sign we must have

$$h > 0. \tag{2.11}$$

This means that the total magnetic field in the region of the soliton is larger than the equilibrium field. In this sense the solitons considered may be called paramagnetic.

To illustrate this we consider "one-dimensional" solitons, putting $\partial^2/\partial\zeta^2 \gg \partial^2/\partial x^2$. In that case Eq. (2.9) has the solution

$$h = h_0/\mathrm{ch}^2(\varkappa\zeta/2), \tag{2.12}$$

where

$$\varkappa^2 = \Lambda, \qquad h_0 = 3\Lambda c^2/\omega_{pe}^2. \tag{2.13}$$

By assumption $h_0 \ll 1$. Hence, the propagation speed u of the solitons is determined by the approximate Eq. (1.12). It also follows from (2.12) and (2.13) that the characteristic size of the soliton l_c as a function of h_0 is of the order of[7]

$$l_c \approx c/\omega_{pe} h_0^{1/2}. \tag{2.14}$$

We have similar estimates also in the case of a circular soliton, i.e., one for which $h = h(\rho)$ with $\rho = (x^2 + \zeta^2)^{1/2}$.

3. THE ROLE OF THE INHOMOGENEITY AND CURVATURE OF THE MAGNETIC FIELD IN THE PROBLEM ELECTRON FLUTED SOLITONS IN A PLASMA WITH COLD ELECTRONS

3.1. Rectilinear magnetic field. We assume as above that $\mathbf{B}_0 \| z$, but that now there is in the plasma an equilibrium electric field E_{x0} causing an equilibrium drift of the electrons in crossed fields $V_{y0} \equiv V_0 = -cE_{x0}/B_0$ and a corresponding equilibrium electric current $j_{y0} = e_e n_0 V_0$. The equilibrium magnetic field then turns out to be inhomogeneous, $\partial B_0/\partial x \neq 0$. The relative gradient of this field $\varkappa_B \equiv \partial \ln B_0/\partial x$ is given by the relation

$$\varkappa_B = -\frac{\omega_{pe}^2}{c^2\omega_{Be}} V_0 = \frac{\omega_{pe}^2}{c\omega_{Be}} \frac{E_{x0}}{B_0}. \tag{3.1}$$

The equilibrium considered is assumed to exist only for a relatively short time during which the ions cannot be dragged along by the field E_{x0}.

We study the fluted perturbations of this equilibrium. The analysis of such perturbations is performed starting from Eq. (2.3). We then get instead of (2.4)

$$\tilde{D}_e \Delta_\perp h - \Lambda \partial h/\partial\zeta + S \partial h^2/\partial\zeta = 0, \tag{3.2}$$

where Λ and S are given by Eqs. (2.6) and (2.7) while the operator $\tilde{D}_e$ has the form

$$\tilde{D}_e = \left(1 - \frac{V_0}{u}\right)\frac{\partial}{\partial\zeta} - \frac{1}{u}[\nabla\chi, \nabla]_z. \tag{3.3}$$

Similarly to (2.4), Eq. (3.2) describes two kinds of solitons: vector and scalar ones. Neglecting the vector nonlinearity, it follows from (3.2) that [cf. (2.9)]

$$\Delta_\perp h - \frac{\Lambda h}{1 - \varkappa_B/\varkappa_n} + \frac{\omega_{pe}^2}{2c^2(1 - \varkappa_B/\varkappa_n)} h^2 = 0. \tag{3.4}$$

We used here the approximate relation $1 - V_0/u = 1 - \varkappa_B/\varkappa_n$ obtained by using Eq. (1.12).

Using the analysis of Sec. 2 we conclude that, in contrast to the case $\varkappa_B = 0$, when $\varkappa_B/\varkappa_n > 1$ the solitons are diamagnetic rather than paramagnetic,

$$h < 0. \tag{3.5}$$

The propagation soliton speed u remains approximately the same as when $\varkappa_B = 0$, i.e., it is determined by the approximate Eq. (1.12) but its shift, caused by finite h, now has a sign which is the opposite of that in the case $\varkappa_B = 0$. We note also that in terms of E_{x0} the condition $\varkappa_B/\varkappa_n > 1$ means

$$E_{x0} > \varkappa_n B_0^2/4\pi e_e n_0. \tag{3.6}$$

Solitons caused by the vector nonlinearity (vector solitons) can conveniently be characterized by the parameter $\varkappa$, the exponent of the decrease of the field as $\rho \to \infty$, defined by the relation $h \propto \exp(-\varkappa\rho)$ [cf. (2.12), (2.13)]. It follows from (3.2) that for fixed $\varkappa$ the propagation speed of the vector solitons equals

$$u = \frac{V_0 + \varkappa_n \omega_{Be}/\varkappa^2}{1 - \omega_{pe}^2/\varkappa^2 c^2}. \tag{3.7}$$

This result is valid both when $\varkappa < \omega_{pe}/c$ and when $\varkappa > \omega_{pe}/c$. When $\varkappa \gg \omega_{pe}/c$ it follows from (3.7) that

$$u = V_0 + \varkappa_n \omega_{Be}/\varkappa^2. \tag{3.8}$$

This result was obtained in Ref. 9 for the case $V_0 = 0$.

3.2. Solitons in a cylindrical z-pinch. We now assume that the equilibrium state of the plasma has cylindrical symmetry and that the equilibrium magnetic field is directed along the angular coordinate ϑ of a cylindrical system of coordinates r, ϑ, z, i.e., $\mathbf{B}_0 \| \boldsymbol{\vartheta}$. We assume also that at equilibrium there is a radial electric field $\mathbf{E}_0 \| \mathbf{r}$. This field leads to motion of the electrons along the z axis with a velocity V_{z0}, and to a magnetic field gradient $\varkappa_B \equiv \partial \ln B_0/\partial r$ given by the relation [cf. (3.1)]

$$\varkappa_B + \frac{1}{r} = \frac{\omega_{pe}^2}{c^2\omega_{Be}} V_{z0} = \frac{\omega_{pe}^2 E_{r0}}{c\omega_{Be} B_0}. \tag{3.9}$$

We consider perturbations of this equilibrium assuming them to depend on r and $\zeta = z - ut$. We introduce a function χ defined by the relation $V_r = \partial\chi/\partial z$. If we then take the ϑ component of Eq. (2.3), we are led to an equation of the form (3.4) with expressions for Λ and S in the form (2.6) and (2.7) but with the substitution

$$u_n \to u_n\left(1 + \frac{2}{r\varkappa_n}\right) \tag{3.10}$$

and with an expression for D_e of the form (3.3) with $[\nabla\chi \times \nabla]_z$ replaced by $[\nabla\chi \times \nabla]_\vartheta$.

It is clear from what we have said that the results obtained in Secs. 2 and 3.1 for the case of a magnetic field with $\mathbf{B}_0 \| \mathbf{z}$ can also be used in the case $\mathbf{B}_0 \| \boldsymbol{\vartheta}$ if we make in the corresponding formula the notation change that follows from (3.9) and (3.10).

4. OBLIQUE ELECTRON SOLITONS IN A PLASMA WITH COLD ELECTRONS

In contrast to Secs. 2 and 3 we now consider oblique perturbations, i.e., perturbations for which $\mathbf{B}_0 \cdot \nabla\chi \neq 0$. For the sake of simplicity we restrict ourselves to the case of a uniform magnetic field $\mathbf{B}_0 \| \mathbf{z}$ and to only the scalar nonlinearity. It is necessary to take into account not only the perturbed longitudinal magnetic field $\tilde{B}_z$ but also the perturbed transverse field $\tilde{\mathbf{B}}_\perp$ when $\mathbf{B}_0 \cdot \nabla\chi \neq 0$.

We perform the analysis starting from Eq. (2.3). Taking the z-component of that equation we get

$$\frac{\partial \tilde{B}_z}{\partial t} - \frac{c\varkappa_n}{4\pi e_e n_0}\frac{\partial}{\partial y}\left(B_0\tilde{B}_z + \frac{\tilde{B}_z^2}{2}\right) - \frac{c^2}{\omega_{pe}^2}\frac{\partial}{\partial t}\Delta_\perp \tilde{B}_z - \frac{cB_0}{4\pi e_e n_0}\frac{\partial}{\partial z}\operatorname{rot}_z \tilde{\mathbf{B}}_\perp = 0. \quad (4.1)$$

Moreover applying the operator curl_z to Eq. (2.3) we get

$$\frac{\partial}{\partial t}\operatorname{rot}_z \tilde{\mathbf{B}}_\perp = -\frac{cB_0}{4\pi e_e n_0}\frac{\partial}{\partial z}\Delta_\perp \tilde{B}_z. \quad (4.2)$$

We assume that the perturbations depend on x and $\zeta = y + \alpha z - ut$, where α is a constant such that $|\alpha| \ll 1$. In that case we get from (4.1) and (4.2)

$$(1-\alpha^2 c_{Ae}^2/u^2)\Delta_\perp h - \Lambda h + \omega_{pe}^2 h^2/2c^2 = 0, \quad (4.3)$$

where $c_{Ae}^2 = B_0^2/4\pi n_0 m_e$ is the square of the electron Alfvén speed.

Equation (4.3) has the same structure as the Eqs. (2.4) and (3.3) which we analyzed before. It describes paramagnetic solitons when $\alpha < |\varkappa_n| c/\omega_{pe}$ and diamagnetic ones when $\alpha > |\varkappa_n| c/\omega_{pe}$.

5. ELECTRON FLUTED SOLITONS IN A PLASMA WITH HOT ELECTRONS

The assumption of cold electrons made above in the problem of the compressible perturbations means that $\beta_e \to 0$, where $\beta_e = 8\pi n_0 T_{0e}/B_0^2$ is the ratio of the electron pressure to the magnetic field pressure, and T_{0e} is the equilibrium electron temperature. We now consider electron fluted solitons in a plasma with finite β_e.

To describe the electrons we use the hydrodynamic equations of motion and heat balance[11]

$$m_e n\frac{d\mathbf{V}}{dt} + \nabla\pi = -\nabla p + e_e n\left(\mathbf{E} + \left[\frac{\mathbf{V}}{c}\mathbf{B}\times\right]\right), \quad (5.1)$$

$$\frac{\partial p}{\partial t} + \mathbf{V}\nabla p + \gamma_0 p \operatorname{div}\mathbf{V} = -\frac{2}{3}\operatorname{div}\mathbf{q}. \quad (5.2)$$

Here p, $\mathbf{q}$, and π are the pressure, heat flux,a nd viscosity tensor of the electrons, $\gamma_0 = 5/3$ is the adiabatic exponent. We consider the case $\mathbf{B}\|\mathbf{z}$. In that case[11]

$$\mathbf{q} = \frac{5}{2}\frac{cp}{e_e B}[\mathbf{e}_z, \times \nabla T_e], \quad (5.3)$$

and the components of the viscosity tensor which we need have the form[1]

$$\pi_{yy} = -\pi_{xx} = \frac{p}{2\omega_B}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + \frac{1}{5\omega_B}\left(\frac{\partial q_x}{\partial y} + \frac{\partial q_y}{\partial x}\right)$$
$$\pi_{xy} = \pi_{yx} = \frac{p}{2\omega_B}\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right) + \frac{1}{5\omega_B}\left(\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y}\right). \quad (5.4)$$

Here $\omega_B = e_e B/m_e c$, where B is the total magnetic field.

According to (5.1) in the equilibrium state the electrons move along y with a speed

$$V_{y0} = V_{E0} + V_p, \quad (5.5)$$

where $V_{E0} = -c E_{x0}/B_0$, $V_p = cT_{0e}\varkappa_p/e_e B_0$ is the so-called electron drift velocity under the action of the pressure gradient, $\varkappa_p = \varkappa_n + \varkappa_T$, $\varkappa_T = \partial \ln T_{0e}/\partial x$. In the case of fixed ions there is connected with this motion an equilibrium electric current $j_{y0} + e_e n_0 V_{y0}$ and a gradient of the equilibrium magnetic field determined by the relation [cf. (3.1)]

$$\varkappa_B = -\frac{\beta_e}{2}\varkappa_p - \frac{\omega_{pe}^2}{c^2\omega_{Be}}V_{E0}. \quad (5.6)$$

Moreover, at equilibrium there is an electron heat flux equal to

$$q_{y0} = \frac{5}{2}n_0 T_{0e} V_T, \quad (5.7)$$

where $V_T = cT_{0e}\varkappa_T/e_e B_0$ is the drift velocity of the electrons under the action of a temperature gradient.

By analogy with (2.3) and (4.1) we get from (5.1)

$$\frac{\partial \tilde{B}_z}{\partial t} + u_n\frac{\partial}{\partial y}\left(\tilde{B}_z + \frac{\tilde{B}_z^2}{2B_0}\right) - \frac{c\varkappa_n}{e_e}\frac{\partial \tilde{T}_e}{\partial y} + \frac{c}{e_e}\operatorname{rot}_z\left(m_e\frac{d\mathbf{V}}{dt} + \frac{\nabla\pi}{n_0}\right) = 0, \quad (5.8)$$

where $\tilde{T}_e$ is the perturbation of the electron temperature.

It follows from (5.2) and (5.3) that

$$\tilde{T}_e = \frac{B_0\tilde{B}_z}{4\pi n_0}\left(\delta_1 + \frac{\tilde{B}_z}{2B_0}\delta_2\right), \quad (5.9)$$

where

$$\delta_1 = \frac{1}{\tilde{u}}\left[V_T\left(1 - \frac{5}{6}\beta_e\right) - \frac{2}{3}V_n\right], \quad (5.10)$$

$$\delta_2 = \frac{1}{\tilde{u}}\left\{\frac{5}{3}\beta_e V_T + \delta_1\left[\left(\frac{2}{\beta_e} - \frac{5}{3}\right)V_\delta - \frac{5}{3}V_T + \frac{2}{\beta_e}V_B\left(1 - \frac{5}{2}\beta_e\right) - \frac{10}{3}V_n\left(\frac{1}{\beta_e} - 1\right)\right] + \frac{10}{3\beta_e}\delta_1^2(V_B - V_n)\right\}. \quad (5.11)$$

We have introduced here the notation:

$$\tilde{u} = u - \frac{5}{3}(V_B - V_n) + \frac{2}{\beta_e}V_B, \quad V_n = V_p - V_T,$$
$$V_\delta = \frac{cT_{0e}}{e_e B_0}\frac{\partial \ln \delta_1}{\partial x}, \quad V_B = \frac{cT_{0e}}{e_e B_0}\varkappa_B. \quad (5.12)$$

Using (2.2), (5.3), (5.4), and (5.9) we find

$$\operatorname{rot}_z\left(m_e\frac{d\mathbf{V}}{dt} + \frac{\nabla\pi}{n_0}\right) = \frac{uc^2 m_e \xi}{4\pi e_e n_0}\frac{\partial}{\partial \zeta}\Delta_\perp \tilde{B}_z, \quad (5.13)$$

where

$$\xi = 1 - \frac{1}{u}\left[V_{E0} + V_n + \frac{\beta_e}{4}V_T + V_B\left(1 - \frac{3}{2}\delta_1\right)\right]. \quad (5.14)$$

Substituting (5.9), (5.13) into (5.8) we get a nonlinear equation for h of the form (2.9) with Λ and S equal to

$$\Lambda = \frac{1}{\xi}\frac{\omega_{pe}^2}{c^2}\left[1 - \frac{u_n}{u}(1+\delta_1)\right], \quad (5.15)$$

$$S = \frac{\omega_{pe}^2}{2c^2\xi}\frac{u_n}{u}(1+\delta_2). \quad (5.16)$$

It follows from this equation that, if we neglect the nonlinearity and dispersion, $\Lambda = 0$. With Λ of the form (5.15) this means that we are dealing with waves with a phase velocity which satisfies the relation

$$1-\frac{u_n}{u}(1+\delta_1)=0. \tag{5.17}$$

Substituting here δ_1 from (5.10) we get for u a quadratic equation from which we can find two values of u, denoted by u_1 and u_2. It then turns out that in contrast to the case of a cold plasma we have either two traveling waves (if u_1 and u_2 are real) or we have no travelling waves at all (when u_1 and u_2 are complex). Complex u mean the presence of an instability. Such solutions are possible when $\beta \approx 1$ and $\nabla T_{0e} \neq 0$. We shall not dwell upon the problem of instabilities and restrict ourselves to the case $\nabla T_{0e} = 0$. Moreover, for the sake of simplicity we take $V_{E0} = 0$. We then get a dependence of the phase velocities of the waves u_1 and u_2 on the plasma density which is shown qualitatively in Fig. 1. We note that [cf. (1.12)]

$$u_1=u_n, \quad u_2=5\beta^2 u_n/12, \quad \beta\ll 1, \tag{5.18}$$

$$u_1=5\beta^2 u_n/12, \quad u_2=u_n, \quad \beta\gg 1. \tag{5.19}$$

It follows from an analysis of the solitons corresponding to these oscillation branches that both for small and for large β both kinds of solitons are paramagnetic (cf. Sec. 2).

6. SHORT-WAVELENGTH DRIFT SOLITONS IN A PLASMA WITH A FINITE ION PRESSURE.

In the equilibrium state the ions move along the y axis with a macroscopic velocity $V_{pi} = cT_i\varkappa_p/e_iB_0$. We assume that there is no equilibrium electric field, $E_{x0} = 0$. The equilibrium condition (the equation giving the balance between the plasma and magnetic field pressures) has the form

$$\varkappa_B=-\beta_i\varkappa_p/2, \tag{6.1}$$

where $\beta_i = 8\pi n_0 T_{0i}/B_0^2$.

To describe the electrons we use the continuity equation (1.7) and the equation of motion (2.1). Assuming that $d/dt \ll \omega_{Be}$ we write the solution of Eq. (2.1) in the form

$$\mathbf{V}=\mathbf{V}_E+\mathbf{V}_I, \tag{6.2}$$

where $\mathbf{V}_I$ is the velocity of the electron inertial motion, given by the relation

$$\mathbf{V}_I=\frac{1}{\omega_{Be}}\left[\mathbf{e}_z\times\frac{d_0\mathbf{V}_E}{dt}\right], \tag{6.3}$$

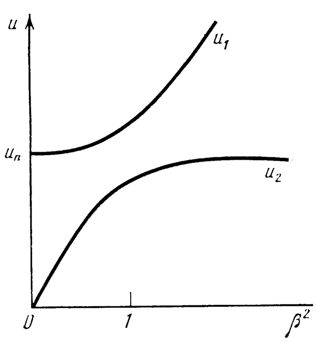

FIG. 1.

where $d_0/dt = \partial/\partial t + \mathbf{V}_E\nabla$. Substituting (6.2) into (1.7) we get [cf. (1.8)]

$$(\partial/\partial t+\mathbf{V}_E\nabla)\ln n+\operatorname{div}\mathbf{V}_E+\operatorname{div}\mathbf{V}_I=0. \tag{6.4}$$

We write the perturbed electric field in the form

$$E_y=-\frac{\partial\varphi}{\partial y}, \quad E_x=-\frac{\partial\varphi}{\partial x}-\frac{u}{c}\tilde{B}_z. \tag{6.5}$$

Here, as above, we used that fact that the perturbations depend on x and $\zeta = y - ut$. In the case of interest to us of short-wavelength perturbations $\rho_i\nabla_\perp \gg 1$ (ρ_i is the ion Larmor radius), the magnetic field $\tilde{B}_z$ weakly affects the ion motion. One can thus approximately assume that the ion density n_i is given by the Boltzmann formula

$$n_i=n_0\exp(-e_i\varphi/T_i). \tag{6.6}$$

To find the connection between φ and $\tilde{B}_z$ we use Eq. (1.9) (the contribution from the ions to the current j_x is negligibly small compared with the electron contribution). We then use the approximate relation $V_x \approx V_{Ex} \equiv cE_y/B$ and the fact that $B = B_0 + \tilde{B}_z$. Using the first Eq. (6.5) we then get

$$\varphi=-\frac{c\omega_{Be}}{\omega_{pe}^2}\tilde{B}_z\left[1+\frac{\tilde{B}_z}{2B_0}\left(1+\frac{2}{\beta_i}\right)\right]. \tag{6.7}$$

Expressing all quantities in Eq. (6.4) in terms of $\tilde{B}_z$ we reduce that equation to the form (2.9) with coefficients Λ and S equal to

$$\Lambda=\frac{1}{\rho_{ei}^2}\left[1-\frac{V_{ni}}{u}+\frac{\beta_i}{2}\left(1-\frac{V_{pi}}{u}\right)\right], \tag{6.8}$$

$$S=\frac{1}{2\rho_{ei}^2}\,\frac{\tau(2/\beta_i+1-\beta_i/2)}{1+\beta_i/2+\tau\beta_i/2}, \tag{6.9}$$

where $\rho_{ei}^2 = T_i/m_e\omega_{Be}^2$ is the square of the electron Larmor radius at the ion temperature,

$$V_{ni}=cT_i\varkappa_n/e_iB_0, \quad \tau=\partial\ln T_i/\partial\ln n_0.$$

By analogy with (1.12), (5.17) we conclude that in this case we are dealing with waves which in the linear approximation have a phase velocity [cf. Eq. (15.24) of Ref. 1]

$$u=\frac{V_{ni}+\beta_i V_{pi}/2}{1+\beta_i/2}. \tag{6.10}$$

Solitons corresponding to such waves were considered in Ref. 6 for $\beta_i \ll 1$ and $\tilde{B}_z \to 0$. It is clear from (6.9) that in case it is necessary to have a temperature gradient present, $\tau\neq 0$, for solitons with $B_z \to 0$ to exist. For finite β_i the role of the temperature gradient is also important. In accordance with what was said in Sec. 2, solitons may then be either diamagnetic, $h<0$ or paramagnetic, $h>0$.

7. CONCLUSIONS

The analysis given here indicates the existence of gradient solitons caused by the plasma compressibility. We considered several types of electron solitons and one of the simplest kinds of electron-ion solitons. Of course, the class of "compressible" gradient solitons is broader. Such solitons are particularly interesting in connection with the problem of magnetic confinement of a finite pressure plasma and their study is necessary to elucidate transport processes in such a plasma.

In the analysis of the electron solitons we assumed that the role of the ion motion was negligibly small. Such an assumption is justified for a sufficiently strong plasma inhomogeneity. It follows from this also that electron solitons may be experimentally studied in relatively simple laboratory setups. According to Sec. 2, this calls only for a sufficiently large density gradient, and it is not necessary to produce in the plasma a transverse electric current. Moreover, according to Sec. 3 electron solitons may arise in current systems such as the φ and z pinches (see, respectively, subsections 3.1 and 3.2). We note that electron solitons in a z-pinch configuration were considered earlier in a paper referred to above.[7] The nonlinear equation obtained in Ref. 7 differs from ours which is, apparently, explained by the additional assumptions made in Ref. 7. The exposition of Ref. 7 is, however, too sketchy and does not contain explanations of the assumptions made.

We note also that in the case of a z-pinch configuration the fluted solitons considered by us (and also in Ref. 7) correspond to so-called "constrictions." Constrictions are characteristic for experiments of the plasma-focus type.[12] One may thus think that the concept of compressible solitons may turn out to be useful for the interpretation of these experiments. For this it is, however, necessary to consider electron-ion rather than electron fluted solitons with characteristic sizes larger than the ion Larmor radius and this has so far not been done.

Oblique solitons considered by us in Sec. 4 may be of interest for the case of a plasma contained in a tapered magnetic field when the existence of purely fluted waves is impossible.

The analysis of the role of finite β_e in the problem of electron fluted solitons which we gave in Sec. 5 may be of interest for ϑ- for z-pinch type systems. We used in Sec. 5 a hydrodynamic description of the electron component and showed that it is possible for two varieties of solitons to exist for finite β_e. The kinetic description is, however, more appropriate for the case of a collisionless plasma. One should therefore consider the results of Sec. 5 basically as only indicative.

The contents of Sec. 6 are important both for the problem of nonlinear short-wavelength drift oscillations of a plasma with finite β_i to which Sec. 6 was specifically devoted and for the problem of long-wavelength drift oscillations of a plasma with finite β_e which are similar to them. The latter problem is more complicated as the corresponding waves are three-dimensional. However, its analysis is also necessary in connection with the special role of long-wavelength oscillations in transfer processes in a plasma across the magnetic field.

[1] A. B. Mikhaĭlovskiĭ, Teoriya plazmennykh neustoĭchivosteĭ (Theory of plasma instabilities) Vol. 2, Atomizdat, Moscow, 1977, Chs. 4,6 [English translation published by Consultants Bureau, New York].
[2] H. Tasso, Phys. Lett. **24A**, 618 (1967)
[3] V. N. Oraevskii, H. Tasso, and H. Wobig, Plasma Phys. Contr. Fus. Res., Vol. II, *IAEA*, Vienna, 1969, p.671.
[4] V. I. Petviashvili, Fiz. Plazmy **3**, 270 (1977) [Sov. J. Plasma Plys. **3**, 150 (1977)].
[5] J. D. Meiss and W. Horton, Phys. Fluids **25**, 1838 (1982).
[6] A. B. Mikhaĭlovskiĭ, G. D. Aburdzhaniya, O. G. Onishchenko, and S. E. Sharapov, Phys. Lett. **104A**, 94 (1984).
[7] D. V. Fillipov, A. A. Chernov, and V. V. Yankov, Preprint *IAE*-3838/6, Kurchatov Inst., Moscow, 1983.
[8] V. I. Aref'ev, A. V. Gordeev, and L. I. Rudakov, Plasma Phys. Contr. Nucl. Fus. Res., Vol. II, *IAEA*, Vienna, 1969, p.165.
[9] A. B. Mikhaĭlovskiĭ, Voprosy Teor. Plazmy **6**, 70 (1972) [Rev. Plasma Phys. **6**, 77 (1975)].
[10] A. B. Mikhaĭlovskiĭ, V. P. Lakhin, L. A. Mikhaĭlovskaya, and O. G. Onishchenko, Zh. Eksp. Teor. Fiz. **86**, 2061 (1984) [Sov. Phys. JETP **59**, 1198 (1984)].
[11] S. I. Braginskiĭ, Voprosy Teor. Plazmy **1**, 189 (1963) [Rev. Plasma Phys. **1**, 205 (1965)].
[12] N. V. Fillipov, Fiz. Plazmy **9**, 82 (1983) [Sov. J. Plasma Phys. **9**, 55 (1983)].

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