Radiation emission by quasiclassical electrons in an atomic potential

V. I. Kogan and A. B. Kukushkin

I. V. Kurchatov Institute of Atomic Energy
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The bremsstrahlung, photorecombination, and line emission spectrum of quasiclassical electrons in a central attractive field and, in particular, in an atomic potential is considered. Two interrelated aspects pertaining to the region of fairly high frequencies are considered: 1) The high-frequency asymptotic form of the spectrum is obtained within the framework of classical electrodynamics, and the effective spatial localization of the emission region for a given frequency is demonstrated. 2) It is shown through direct computation of the quantum corrections to the classical limit of the radiative-transition matrix element that the classical nature of the spectrum of the radiation emitted at these high frequencies is secured by the quasiclassical nature of the electron motion, and needs not be supplemented (as is customarily done) by the condition $\hbar\omega \ll E$. A "rotational" approximation is constructed which is an extension of the Kramers approximations to the non-Coulomb case and to the region of low frequencies. The results are used to analytically describe the spectra of the bremsstrahlung emitted by electrons with energy of the order of several keV on many-electron atoms. This classical description allows the representation of the results of the corresponding quantum numerical calculations in the form of universal functions, and is in good agreement with recent experiments.

1. INTRODUCTION

Underlying the analysis of the processes of radiation emission by electrons in an atomic potential—below we shall be concerned with the emission of bremsstrahlung (BR), photorecombination radiation (PR), and radiation with a line spectrum (LR)—is, as is well known, the Coulomb potential model, for which the corresponding spectra (in the dipole approximation) can be computed exactly at both the classical and the quantum levels.[1–3] On going over to many-electron atoms (ions), we find the Coulomb model inapplicable, especially in the case of low-energy electrons, and what is more, the question of the applicability of the statistical model of the atomic potential becomes, on the whole, a subject of interest in itself (see Ref. 4 and the papers cited therein). The latter question was positively resolved recently in experiments[5] involving the measurement of the spectra of BR emitted by electrons with energy of the order of several keV on many-electron atoms, the measured spectra being in very good agreement (naturally, for not too low frequencies ω) with the corresponding quantum numerical computations.[6] As a result of this the problem of computing the spectra of the BR, PR, and LR emitted in a central attractive potential $U(r)$ has risen significantly in "status as a realistic problem." The exact solution to this problem, even at the classical level,[1] cannot be represented in an analytic form that would not require subsequent numerical calculations.

We shall, when dealing hot-plasma spectroscopy, be interested in the region of low, nonrelativistic electron energies E and fairly high atomic numbers Z (in, say, the region $E \lesssim 10$ keV, $Z \gtrsim 20$). In this region the well developed Born approximation turns out to be inapplicable (e.g., for the BR spectrum[7] it yields a form that is even qualitatively incorrect, i.e., that falls off with increasing ω). On the other hand, the elastic scattering of electrons of such energies by heavy atoms is well described—in accord with experiment—by the quasiclassical approximation.[8,9] This raises hopes of using the quasiclassical approach for the description of radiative processes as well.

By quasiclassicality of the electron we shall mean below the fulfillment of the condition $\lambda\!\!\!^{-} = \hbar/mv \ll a$, where v is the initial velocity of the electron and a is a characteristic distance determined by the equation $|U(a)| = E = mv^2/2$. (For the Coulomb field $U(r) = -Ze^2/r$ this condition reduces to $Ze^2/\hbar v \gg 1$.)

For electrons the atomic potentials are attractive potentials, in consequence of which we can expect that an important role will be played here by the radiation of high frequencies, due to the "close" highly curved trajectories. By high we shall mean below frequencies satisfying the condition $\omega \gg \tilde{\omega} \equiv v/a$.

In the particular case of the Coulomb field the high-frequency ($\omega \gg \tilde{\omega} \equiv mv^3/Ze^2$) classical BR spectrum is not exponential (specifically, it does not depend on ω at all[10,11]), and the quantum correction to it remains small right down to the short-wave limit $\omega_{\max} = E/\hbar$ of the BR spectrum,[11,12] so that, integrally, it is just the high-frequency region ($\omega_{\max}/\tilde{\omega} \sim Ze^2/\hbar v \gg 1$) that is important. We can see from this that, in the Coulomb case, the quasiclassical nature of the electron motion can alone guarantee the classicality of the entire BR spectrum, and the additional "kinematic" condition $\hbar\omega \ll E$ that is usually introduced in this connection[1,2,13–15] is actually superfluous.[16,17]

It is not *a priori* clear whether the indicated property of drawn-out—with respect to ω—classicality of the spectrum will be preserved when we go over to the case of a non-Coulomb potential or whether it is peculiar only to the Coulomb case, as is the case with the Rutherford formula approximated by the nondependence, on $Ze^2/\hbar v$, of the BR intensity integrated over the spectrum[16,17] etc.[1)] Only a direct compu-

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tation of the quantum corrections to the Coulomb spectrum can give an answer.

The foregoing predetermines the logical structure of the present paper. In §2 we find within the framework of classical electrodynamics the high-frequency ($\omega \gg \tilde{\omega}$) asymptotic form of the spectrum of the radiation emitted in attractive potential fields $U(r)$ of a certain class (specifically, potential fields that behave as $r \to 0$ like r^{-n}, $0 < n < 2$) encompassing, in particular, the atomic-potential fields. The obtained analytical description of the classical structures allows us to compute the high-frequency quantum correction to the classical spectrum (§3). The investigation of this correction in turn enables us to justify and generalize (in particular, to the entire class of potentials under consideration) the indicated decisive role played by the quasiclassicality of the motion in the securing of the classicality of the emission spectrum (§4). In the light of this, of great practical interest is an analytic description of the purely classical spectrum in a frequency range broader than the range considered in §2, specifically, in the range $\omega \gtrsim \tilde{\omega}$. A corresponding new "rotational" approximation (RA) is developed in §5. The RA generalizes in two respects—to the non-Coulomb case and to a spectral region broader than the $\omega \gg \tilde{\omega}$ region—the well-known Kramers approximation, which is the "standard" approximation for the entire theory of BR, PR, and LR.[10,1] In §5 we illustrate the application of the RA to the computation of the spectrum of the BR emitted by electrons with energy of the order of several keV on many-electron atoms, and in §6 we compare the results of the theory developed with experiment.[5] Some possible generalizations of the theory are outlined in §7.

§2. HIGH-FREQUENCY ASYMPTOTIC FORM OF THE CLASSICAL SPECTRUM OF THE RADIATION EMITTED BY AN ELECTRON IN A CENTRAL ATTRACTIVE FIELD

Let us consider the high-frequency spectrum of the dipole radiation emitted by an electron within the framework of classical electrodynamics. The results will pertain not only to BR—from the formal standpoint the only purely classical radiation-emission mechanism among the three—but also to PR and LR, since at high frequencies they all have the same physical basis,[2)] viz, the decisive role played by those sections of the highly curved trajectories which are closest to the field center. Here and everywhere below we use the usual term "spectrum" to designate the distribution of the radiation intensity over ω, *summed* over all the momenta $M = m\rho v$ (this distribution naturally plays the greatest practical role in the case of the BR), and for the intensity distribution at *fixed M* we shall, following Ref. 18, use the term "subline" (such a distribution is of greatest importance for the LR).

We shall, to begin with, carry out the Fourier analysis, necessary for the computation of the spectrum (and the subline), of the acceleration of an electron moving along a curvilinear trajectory at the classical level. On account of the general properties of the Fourier analysis, the radiation intensity for $\omega \gg \tilde{\omega}$ is not exponentially small only if by chance frequencies of the order of ω are present in the electron motion itself. In an attractive-potential field only the highest angular velocities $\omega_{\rm rot}$ of revolution of the electrons moving along the highly curved trajectories with impact parameters $\rho \ll a$ can be such frequencies.

The quantity $\omega_{\rm rot}$ at the turning point, i.e., at the distance r_0 of closest approach of the electron to the field center, is equal to

$$\omega_{\rm rot} = v_{max}/r_0 = M/mr_0^2 = [2(E+|U(r_0)|)/mr_0^2]^{1/2}, \qquad (1)$$

where $v_{\rm max}$ is the velocity at the turning point. From the relations $\omega_{\rm rot} \sim \omega$, (1), and the definition of $\tilde{\omega}$(§1) we can see that the trajectories with $|U(r_0)| \gg E$ are responsible for the emission in the region $\omega \gg \tilde{\omega}$ of interest to us. In this limit, according to (1), r_0 and $\omega_{\rm rot}$ depend only on M (but not on the initial energy E).

We shall consider the class of monotonic potentials for which $U(r) \to -\alpha/r^n$ as $r \to 0$, where $\alpha > 0$, $0 < n < 2$ (the latter limitation is necessary to obviate a fall to the center). For this class the dependences $r_0(M)$ and $M_{\rm eff}(\omega)$ for small M are monotonic, so that for not too low ω (i.e., for the corresponding $r_{\rm eff}(\omega)$) this class encompasses, for example, the Thomas-Fermi atomic potential. For this same class of potentials the quasiclassical nature of the electron motion is destroyed at *small r* (see §4).

Thus, for $\omega \gg \tilde{\omega}$ and the indicated class of potentials the radiation frequency ω turns out to be roughly "embedded" in a chain of electron-trajectory characteristics, M, r_0, and $\omega_{\rm rot}$, interrelated in a one-to-one manner, which yields

$$r_{\rm eff}(\omega) \equiv r_\omega \sim \left(\frac{\alpha}{m\omega^2}\right)^{1/(2+n)},$$

$$M_{\rm eff}(\omega) \equiv M_\omega \sim \left(\frac{\alpha^2 m^n}{\omega^{2-n}}\right)^{1/(2+n)}. \qquad (2)$$

In this approximation the high-frequency asymptotic form of the emission spectrum of an electron stream is given simply by the distribution $d\varkappa(\rho)$ of the "effective radiation" emitted by this stream over $\omega_{\rm rot}$. For a homogeneous stream[1] $d\varkappa(\rho) = 2\pi\rho d\rho \Delta\mathscr{E}(\rho)$, where $\Delta\mathscr{E}(\rho)$ is the energy emitted from all the trajectories:

$$\Delta\mathscr{E}(\rho) = \frac{2e^2}{3c^3}\int_{-\infty}^{\infty} w^2(t)\,dt \sim \frac{e^2}{c^3} w_{max}^2 \Delta t_{\rm eff}. \qquad (3)$$

Here

$$w_{max} = \frac{1}{m}\frac{dU}{dr}(r_0)$$

is the acceleration of the electron at the turning point and $\Delta t_{\rm eff} \sim r_0/v_{\rm max} = 1/\omega_{\rm rot}(r_0)$ is the effective duration of the emission process for a given trajectory. Using (2) and (3), we find that

$$\frac{d\varkappa}{d\omega} \sim \frac{e^2}{c^3 v^2} n^2 \frac{2-n}{2+n}\left(\frac{\alpha}{m}\right)^{6/(2+n)} \omega^{4(n-1)/(2+n)}, \qquad \omega \gg \tilde{\omega}. \quad (4)$$

For the Coulomb case ($n = 1$, $\alpha = Ze^2$) we obtain from (4) the relation[3)]

$$d\varkappa/d\omega \sim Z^2e^6/c^3m^2v^2.$$

The foregoing qualitative analysis is confirmed by the results of the rigorous treatment. It is convenient to transform the expression[1]

$$\frac{d\varkappa}{d\omega}=\int_0^\infty \Delta\mathscr{E}_\omega(\rho)\,2\pi\rho\, d\rho,\qquad \Delta\mathscr{E}_\omega(\rho)=\frac{2}{3\pi c^3}|\ddot{\mathbf{d}}_\omega|^2 \tag{5}$$

for the spectrum into the form (obtained earlier by V. I. Gervids)

$$\frac{d\varkappa}{d\omega}=\frac{16e^2}{3q^2c^3m^2}\int_0^\infty (J_c^2+J_s^2)M\,dM, \tag{6}$$

$$J_c=\int_0^\infty \frac{dU}{dr}\cos\omega\tau\cos\varphi\, d\tau, \tag{7}$$

where J_s is obtained from J_c by replacing the cosines by sines in the integrand in (7), $q^2=2mE$,

$$\tau=\int_{r_0}^{r(\tau)}\frac{m\,dr'}{p(r',M,E)},\qquad \varphi=\int_{r_0}^{r(\tau)}\frac{M\,dr'}{r'^2p(r',M,E)},$$

$$p(r,M,E)=\left[q^2-2mU(r)-\frac{M^2}{r^2}\right]^{1/2}. \tag{8}$$

In the limit $\omega\gg\tilde{\omega}$, only the small values of τ are important in the integral (7). But it is not possible to carry out the calculation through expansion in a series of powers of τ, since the expansion severely distorts the behavior of the integrand at high τ values, so that the terms of the corresponding series turn out to be divergent integrals. This is explained by the fact that, for $\omega\to\infty$ and fixed M, the integral (7) is exponentially small.

As shown in Ref. 21, in this situation we must go over to contour integration in the plane of the complex variable τ, since the dominant contribution to (7) is made by the region around the singularity of the integrand, where the function $r(\tau)$ vanishes. From this it follows that the necessary expansion in powers of $1/\omega$ is guaranteed by going over in (6) and (7) to the new variables N and R (cf. (2)), where

$$N=\omega\left(\frac{M^{2+n}}{4\alpha^2m^n}\right)^{1/(2-n)}=\frac{\omega}{\omega_{\rm rot}},\qquad R=r\left(\frac{2m\alpha}{M^2}\right)^{1/(2-n)} \tag{9}$$

and the corresponding expansion of the potential energy ($\gamma>0$):

$$U(r)=-\frac{\alpha}{r^n}(1+Cr^\gamma+\ldots),\qquad 0<n<2. \tag{10}$$

The final result for the subline (the leading—in $1/\omega$—term and the corrections to it) has the form

$$\Delta\mathscr{E}_\omega(\rho)=\Delta\mathscr{E}_\omega^{(0)}(\rho)\,\{1+A_q(\omega)F_1(n,N)+A_c(\omega)F_2(n,N)\}, \tag{11}$$

$$\Delta\mathscr{E}_\omega^{(0)}(\rho)=\frac{2e^2n^2}{3c^3\pi}\left(\frac{2\alpha\omega^n}{mN^{2(n+1)}}\right)^{2/(2+n)}S(n,N), \tag{12}$$

$$S(n,N)=\left[\int_0^\infty\cos y\cos\varphi\frac{dy}{R^{n+1}}\right]^2+\left[\int_0^\infty\sin y\sin\varphi\frac{dy}{R^{n+1}}\right]^2, \tag{13}$$

where the functions $R(y,N)$ and $\varphi(R)$ are given by the relations

$$y=N\int_1^R x(x^{2-n}-1)^{-1/2}dx,\qquad \varphi=\frac{2}{2-n}\operatorname{arctg}(R^{2-n}-1)^{1/2}, \tag{14}$$

F_1 and F_2 are functions significantly more unwieldy than $S(n,N)$ (we shall not write them out here),

$$A_q=q^2(2\alpha m^{n+1}\omega^n)^{-2/(2+n)},\qquad A_c=C\left(\frac{2\alpha}{m\omega^2}\right)^{\gamma/(2+n)}. \tag{15}$$

Using (5), we obtain for the spectrum the expression

$$\frac{d\varkappa}{d\omega}=\frac{4e^2n^2}{3c^3v^2(2+n)}\left(\frac{2\alpha}{m}\right)^{6/(2+n)}\omega^{4(n-1)/(n+2)}\mathscr{P}_0(n) \tag{16}$$

$$\{1+A_q(\omega)\zeta_1(n)+A_c(\omega)\zeta_2(n)\},$$

$$\mathscr{P}_0(n)=(2-n)\int_0^\infty S(n,N)N^{-(7n+2)/(2+n)}dN, \tag{17}$$

where ζ_1 and ζ_2 are numerical coefficients corresponding to the terms with F_1 and F_2.

For an arbitrary n (let us recall that $0<n<2$) we cannot carry our an analytical calculation, nor can we even simplify the integral (17) because the values $y\sim N\sim 1$ are important in (13) and (17). In the particular cases $n=1$, $n\to 2$, and $n\to 0$ we arrive at the following results.

For the Coulomb case ($n=1$, the $K_\nu(x)$ are modified Bessel functions of imaginary argument)

$$S(1,N)=\frac{16}{3}N^4\left\{K_{1/3}^2\left(\frac{4N}{3}\right)+K_{2/3}^2\left(\frac{4N}{3}\right)\right\},$$

$$N=\frac{\omega M^3}{4mZ^2e^4},\qquad \mathscr{P}_0(1)=\pi\sqrt{3}=5.44, \tag{18}$$

which leads, after substitution into (12) and (16), to agreement with Ref. 1 and, in the latter case, to the Kramers high-frequency limit,[10] in units of which all the BR and PR spectra can, as is well known, be expressed (in terms of the Gaunt factors $g(\omega)$).

For the case $n\to 2$ ("advanced fall" to the center)

$$S(N)=\frac{\pi^2}{2(2-n)}\delta(N-1),$$

$$N=m\omega\left(\frac{M^2}{2m\alpha}\right)^{2/(2-n)}=\frac{\omega}{\omega_{\rm rot}},\qquad \mathscr{P}_0=\frac{\pi^2}{2}=4.93.$$

The physical characteristics of this case manifest themselves, first, in the narrowness of the subline (an obvious result of the multiple, $\sim 1/(2-n)$ times, twisting of the trajectories around the field center) and, second, in the stiffening of the condition that the energy E be negligible and, consequently, that (12) and (16) be applicable: $\omega\gg\tilde{\omega}e^{2/(2-n)}$ instead of $\omega\gg\tilde{\omega}$. For the case $n\to 0$ we have $\mathscr{P}_0=\pi^2/4$.

Notice that the expansion in powers of $1/\omega$ does not, in the general case, reduce simply to an expansion in $\tilde{\omega}/\omega$; the latter exists only for power-law potentials: here $A_q(\omega)=(\tilde{\omega}/\omega)^{2n/(2+n)}$.

The shape of the subline (12), which has a complicated functional form, admits of some model simplification. Analysis shows that, for $\omega\gg\tilde{\omega}$, the integrally (with respect to ω) important part of the subline can be qualitatively described by the function

$$\Delta\mathscr{E}_\omega(\rho)\propto\omega^{4(n-1)/(n+2)}M_{\rm eff}^{-2}(\omega)\,\eta(\omega_{\rm rot}-\omega), \tag{19}$$

where $M_{\rm eff}(\omega)$ is given by the formula (2) and $\eta(x)$ is the unit step function. The truncation, corresponding to (19), of the subline at the frequency $\omega_{\rm rot}$ reflects the fact that the profile of the subline slopes down exponentially in the region $\omega \gg \omega_{\rm rot}$. The indicated form of the subline models the subline's principal property: the integral concentration in the region $\omega \sim \omega_{\rm rot}$.

Three important conclusions can be drawn from the foregoing. First, in the region $\omega \gg \tilde{\omega}$ the subline (see (12)) depends functionally on only one parameter N, (9). It is this "self-similarity" that allowed us to determine the parametric structure of the spectrum (16) without computing the subline's form itself, which still remains fairly complicated in the limit $\omega \gg \tilde{\omega}$. Second, the existence of the indicated self-similarity parameter, together with the fact that only the section of the trajectory close to the turning point is integrally important, indicates the realization in the region $\omega \gg \tilde{\omega}$ of an approximate unique relationship between the emitted frequency ω and the electron trajectory or, equivalently, the distance $r_{\rm eff}(\omega) \equiv r_0(M_{\rm eff}(\omega))$ of the electron from the field center (cf. the qualitative analysis performed above). Finally, in the region $\omega \gg \tilde{\omega}$, the analytic properties of the formulas for the spectrum and the subline are determined by the "competition" between the centrifugal energy of the particle and only its potential energy at small r, as a result of which the functional dependence on the total energy E (and, hence, on the type of trajectory, i.e., on whether the motion is a finite or an infinite one) disappears. The last property allows us in §3 to analytically describe the high-frequency quantum spectrum in the quasiclassical case, in which the analytic properties of the corresponding matrix element are entirely determined by its essentially classical structure.

§3. QUASICLASSICAL DESCRIPTION OF THE HIGH-FREQUENCY SPECTRUM. CRITERIA FOR CLASSICALITY OF THE SPECTRUM AND THE SUBLINE

1. Let us proceed to compute the limits of applicability of the classical description of the spectrum of the radiation emitted by quasiclassical electrons in an attractive field of the class under consideration on the basis of a formal expansion in $\hbar$. The rigorous criterion for the applicability of such an expansion can naturally be determined only from the final result, but a preliminary condition is the inequality $\lambda\!\!\!^{-} \ll a$ (in the Coulomb case, $Ze^2/\hbar v \gg 1$) indicated in §1. This inequality is the condition for the scattered-electron motion to be quasiclassical in the entire region $r \gg r_1$, where r_1 is the characteristic distance from the field center of the electron in the ground (bound) state (in the Coulomb case this is the Bohr radius (see §49 in Ref. 22)).

To find the quasiclassical emission spectrum, we use its representation in the form of a partial-wave expansion.[20,23] Let us transform the corresponding sum into an integral over the continuous parameter $M = \hbar(l + \frac{1}{2})$ (see, for example, Ref. 24)[4)]:

$$\frac{d\varkappa}{d\omega} = \hbar\omega \frac{d\sigma}{d\omega} = \frac{2e^2m^2\omega^4}{3c^3q^2} \int\limits_0^\infty \{R^2(E, l \to E', l+1) + R^2(E, l+1 \to E', l)\}\hbar(l+1)\,dM. \tag{20}$$

Here $E' = E - \hbar\omega$, R is the radial-coordinate matrix element connecting quasiclassical radial wave functions (WF):

$$R(\beta \to \beta') = \int\limits_{r_m}^\infty \chi_\beta \chi_{\beta'} r\,dr,$$

$$\chi_{E,l} = p^{-1/2}(r, l, E)\left[\exp\left(\frac{i}{\hbar}\int\limits_{r_0(E,l)}^{r} p(r', l, E)\,dr' - \frac{i\pi}{4}\right) + \text{c. c.}\right], \tag{21}$$

where

$$r_m = \max\{r_0(\beta),\ r_0(\beta')\},$$
$$p(r,\ l,\ E) = [q^2 - 2mU(r) - \hbar^2 l(l+1)/r^2]^{1/2}, \quad q^2 = 2mE.$$

Such a representation will allow us to investigate at the same time the classicality criterion for the quantum sublines corresponding to the transitions $nl \to n'l'$, with $l' = l \pm 1$, and having $\Delta\varepsilon_\omega(\rho)$ (5), (11) as their classical analog.

Let us go over in (21) to the limit $\hbar \to 0$, $\omega \to \infty$, and let us do this in such a way that $E - E' = \hbar\omega$ remains arbitrary. This is admissible because, according to §2, the small quantities will be the E and E' themselves, so that the need for the imposition of the limitation $\hbar\omega \ll E$ does not arise when carrying out analytical simplifications. We then arrive at the result

$$R(E, l+1 \to E', l) = R_-^{(0)} + R_-^{(1)},$$
$$R(E, l \to E', l+1) = R_+^{(0)} + R_+^{(1)}, \tag{22}$$

$$R_\pm^{(0)} = 2\int\limits_{r_0}^\infty \cos\Phi_\pm(r, M)\frac{r\,dr}{p}, \qquad \Phi_\pm = \int\limits_0^{t(r)} (\omega_{\rm rot}(r') \pm \omega)\,dt', \tag{23}$$

$$R_\pm^{(1)} = -2\int\limits_{r_0}^\infty \sin\Phi_\pm(r, M)\frac{r\,dr}{p}\left\{\frac{\hbar}{2}\int\limits_{r_0}^{r}\frac{dr'}{pr'^2} + \frac{\hbar m^2}{(\partial p^2/\partial r)_{r_0}}\Omega_\pm(r_0)\right.$$
$$\left.\times\left[\frac{1}{p} + \int\limits_{r_0}^{r}\frac{dr'}{2p^3}\left(\frac{\partial p^2}{\partial r'} - \left(\frac{\partial p^2}{\partial r}\right)_{r_0}\frac{\Omega_\pm(r')}{\Omega_\pm(r_0)}\right)\right]\right\}$$
$$-\lim_{r^*\to r_0}\left\{\hbar m\int\limits_{r^*}^\infty \cos\Phi_\pm(r, M)\right.$$
$$\left.\times(2\omega_{max} - \omega - \omega_{\rm вр}(r))\frac{r\,dr}{p^3} + \frac{2\hbar m r_0(2\omega_{max} - \omega - \omega_{\rm rot}(r_0))}{(\partial p^2/\partial r)_{r_0} p(r^*, M)}\right\}. \tag{24}$$

Everywhere here $r_0 = r_0(M)$ is a root of the equation $p = 0$,

$$p \equiv p(r,\ M) = [2m|U(r)| - M^2/r^2]^{1/2}, \quad \omega_{\rm rot}(r) = M/mr^2,$$
$$\Omega_\pm(r) = (\omega_{\rm rot}(r) \pm \omega)\,(2\omega_{max} - \omega - \omega_{\rm rot}(r)),$$

where $\omega_{\rm rot}(r, M)$ is the "local" angular velocity of revolution [when $r = r_0(M, E)$ it goes over into (1)]; the function $t(r)$ is found from the equation

$$t = \int\limits_{r_0}^{r} m\,dr/p.$$

The quantity $R^{(0)}$ is the classical limit of the radial matrix element for high frequencies, and it is obtained as such in Ref. 25.

The radial-motion wave function (21) used by us in the computation led to a quasiclassical asymptotic form with a quantum correction to it:

$$\psi = A \exp(iS/\hbar)[1+i\hbar\Delta] + \text{c. c.}, \qquad (25)$$

where A, S, Δ do not contain $\hbar$. In the wave function (21) the correction Δ stems only from the expansion of the centrifugal energy ($\hbar^2 l(l+1) \approx M^2 - \hbar^2/4$). As to that correction to the wave function (21) itself which is the next term of the series in $\hbar$ for a fixed (in terms of $\hbar$ and l) effective potential energy,[5] it is not reflected in (21) at all, since it does not make a contribution to (24).

The quantum correction, stemming from the quantity Δ in the radial wave function, to the spectrum is the contribution of the non-classicality of the radial motion. The remaining quantum corrections to the spectrum result from the presence of $\hbar$ in the final energy E' and from the corresponding expansion of the term outside the curly brackets in (20).

2. The expression (20) with R given by (22) is quite complicated, but for $U(r)$ of the form (10) it is possible to establish (using a procedure similar to the one used in §2) the functional forms of the subline $I_l(\omega)$ and the spectrum $d\varkappa/d\omega$:

$$I_l(\omega) \equiv \frac{\omega q^2}{2\pi M}\frac{d\sigma}{d\omega}(l, E\to E')$$

$$= \Delta\mathscr{E}_\omega^{(0)}(\rho)\left\{1 + A_q F_1 + A_c F_2 + \frac{\hbar}{M}\Pi(n,N)\right\}, \qquad (26)$$

$$\frac{d\varkappa}{d\omega} = \left(\frac{d\varkappa}{d\omega}\right)^{(0)}_{\text{class}}$$

$$\times\{1 + A_q\zeta_1 + A_c\zeta_2 + B(n)\hbar\alpha^{-2/(2+n)}m^{-n/(2+n)}\omega^{(2-n)/(2+n)}\},$$

$$0<n<2. \qquad (27)$$

Here the function $\Pi(n, N)$ is obtained from (20), (22)–(24) as the leading term of the expansion in $1/\omega$ of the term proportional to $\hbar$ (we do not write it out because of its extreme unwieldiness) and $B(n)$ is a numerical coefficient corresponding to the term with $\Pi(n, N)$; all the remaining quantities have the same meanings as in the "classical" formulas (9), (11)–(17).

In the particular case of the Coulomb field all the computations can be carried out to the end. Thus, from (23), (24) we obtain the correction $R^{(1)}$ to the classical high-frequency limit $R^{(0)}$ of the radial matrix element:

$$R_\mp^{(0)} = \frac{4(2Ze^2)^{1/3}}{\sqrt{3}\,m^{4/3}\omega^{5/3}}N^{2/3}\left[K_{2/3}\left(\frac{4N}{3}\right) \pm K_{1/3}\left(\frac{4N}{3}\right)\right], \quad (28)$$

$$R_\mp^{(1)} = \frac{4\hbar}{\sqrt{3}(2Ze^2)^{1/3}m^{5/3}\omega^{4/3}}\left[K_{1/3}\left(\frac{4N}{3}\right)\left(2N\mp\frac{1}{2}\right)\right.$$

$$\left.\pm 2NK_{2/3}\left(\frac{4N}{3}\right) + \left(\frac{2E}{\hbar\omega}-1\right)W_\mp\right], \qquad (29)$$

$$W_\mp = [-1\pm 12N - 16N^2]\frac{K_{1/3}(4N/3)}{5} + \frac{8N}{5}[1\mp 2N]K_{2/3}\left(\frac{4N}{3}\right). \qquad (30)$$

We can isolate in (29) the above-defined contribution of the nonclassicality of the radial motion. It turns out to be equal to

$$(R_\mp^{(1)})_{\text{rm}} \propto \left[\frac{5N}{2}K_{1/3}\left(\frac{4N}{3}\right) + (\pm 2N - N^2)K_{2/3}\left(\frac{4N}{3}\right)\right.$$

$$\left.- W_\mp - \sqrt{3}N^2\varphi(N)\right], \qquad (31)$$

where

$$\varphi(N) = \int_0^\infty \sin\left[2N\left(x + \frac{x^3}{3}\right)\right]\frac{dx}{x}$$

and we have dropped the factor in front of the square brackets in (29).

The expression (28) coincides, as it should, with the expression obtained from the corresponding formulas of the classical BR theory[1] in the limit $\omega \gg \tilde\omega$. This same result for $R^{(0)}$ is obtained by the quasiclassical approach in Ref. 25 (the quantum correction $R^{(1)}$ is not considered in that paper). The agreement of (28) with the result obtained in Ref. 25 is up to the "academically" necessary replacement of l in Ref. 25 by $(l+\frac{1}{2})$, and up to the normalization factor, proportional to $(EE')^{-3/2}$, which we took into account earlier in (20). (Notice that the quantum character of this factor is due only to the arbitrariness in the normalization of the wave function, and therefore it is not at all important in the "quantumness-classicality" aspect under consideration.)

We must emphasize in connection with the results (23) and (28) that the leading term of the quasiclassical expansion of the radial matrix element, i.e., the matrix element of the quantity $r \equiv |\mathbf{r}|$, (21), does not reduce to the Fourier transform of this same quantity r at any ω (including low ω!). Nor is the correspondence principle in any way contradicted here, since this principle pertains only to the *total* transition matrix element, but the radial matrix element (21) is only a part (i.e., is only the radial factor) of the total $nl \to n'l' \pm 1$ transition matrix element and besides this transition matrix element is for another quantity, the vector $\mathbf{r}$. (For the quantity r the total matrix element of such a transition is generally equal to zero.)

The final result for the "near-classical" Coulomb (BR + PR) spectrum has, in terms (as usual) of the Gaunt factor $g(\omega)$, the form

$$g(\omega) = 1 + \frac{3^{1/6}\Gamma^2(1/3)}{20\pi 2^{1/6}}\nu^{-2/3}\left(2 - \frac{\hbar\omega}{E}\right). \qquad (32)$$

The formula (32) was obtained by Babikov[11] through a corresponding expansion of the hypergeometric function in Sommerfeld's exact result for the spectrum of the BR emitted in the Coulomb field.[2] The result (32) is also confirmed by the expansion of the exact Coulomb formulas for the PR spectrum,[26,27] which are an analytic continuation of the Sommerfeld formula.

3. Let us now determine the criteria for classicality of the emission subline (26) and the spectrum (27). For the subline a "practical" classicality criterion is, as can be shown, the inequality $l \gg 1$, since it guarantees the smallness of the quantum correction in (26) in the entire region $\tilde\omega \ll \omega \lesssim \omega_{\text{rot}}(M)$ where the intensity of the (classical) subline itself is not yet exponentially small. For example, in the Coulomb case, according to (29), the frequency ω_l^* at which the classicality condition for the subline is substantially violated is of the order of $mZ^2e^4/(\hbar M^5)^{1/2}$, and here, because

$$\omega_l^* M^3/mZ^2e^4 \sim (M/\hbar)^{1/2} \gg 1,$$

the subline is already exponentially small. Notice that the "academic" classicality condition $\omega \ll \omega_l^*$ for the subline also does not reduce to the criterion $\hbar\omega \ll E$; for example, in the Coulomb case $\hbar\omega_l^*/E \sim (a/\rho)^2 l^{-1/2}$, and is by no means ~ 1.

For the spectrum, the characteristic frequency ω^* corresponding to a significant destruction of its classical nature is, according to (27), approximately given by the expression

$$\omega^* \sim (\alpha^2 m^n/\hbar^{2+n})^{1/(2-n)} \tag{33}$$

(in the Coulomb case this is the Rydberg frequency $Z^2me^4/\hbar^3$).

When the basic assumption $\lambdabar \ll a$ is realized, we have $\omega^* \gg \omega_{\max} (= E/\hbar) \gg \tilde{\omega}$, which guarantees the existence of a broad region where the spectrum is simultaneously a high-frequency ($\omega \gg \tilde{\omega}$) and a classical ($\omega \ll \omega^*$) one. In particular, in the case of the purely power-law potential $U = -\alpha/r^n$, for which there are explicit expressions for a and $\tilde{\omega}$:

$$a \sim (\alpha/mv^2)^{1/n}, \quad \tilde{\omega} \sim (mv^{2+n}/\alpha)^{1/n}, \tag{34}$$

we have

$$\omega^*/\tilde{\omega} \sim (a/\lambdabar)^{(2+n)/(2-n)} \gg 1. \tag{35}$$

It follows from the foregoing that the broadening, following from Refs. 11 and 12, of the classicality region for the BR emission spectrum of quasiclassical electrons in comparison with the usually cited condition $\hbar\omega \ll E$ is not limited to the Coulomb cse, but is the case for all the potentials of the class under consideration. Here the classicality region encompasses not only the entire BR spectrum, but also almost the entire PR spectrum, since, in the case when $\lambdabar \ll a$, the latter terminates precisely at frequencies[6)] $\omega \sim \omega^*$.

Here we were right to treat the BR and PR emission processes as processes describable by a single formula, since the virtual discreteness of the PR spectrum for a fixed initial electron energy E reflects only the quantization of the energy of the electron in the final state, a quantization which does not destroy the classical nature of the emission spectrum (this spectrum is obtained by the standard additional continuous-spectrum "discretization" procedure, which can be traced back to Ref. 10). This quantization is due to the fact that the probability for radiative transition in the case when $\omega \gg \tilde{\omega}$ functionally depends not on the energies E and E' themselves, but only on their difference.

It should be noted again that the quantum correction in (27), which was obtained above through simultaneous passages to the limits $\hbar \to 0$ and $\omega \to \infty$ in the case of arbitrary $\hbar\omega/E$, can also be obtained through a "universal" power series expansion in only $\hbar$ (and, thus, in $\hbar\omega/E$) with subsequent passage to the limit $\omega \to \infty$ (as is done in Ref. 21). This is due to the nontrivial fact that the expansion in $\hbar$ is also automatically realized when we perform the expansion (which is therefore more general) in $\tilde{\omega}/\omega$. Thus, for the case of the power-law potential the formula (27) with allowance for (34) can be represented in the form

$$\frac{d\varkappa}{d\omega} = \left(\frac{d\varkappa}{d\omega}\right)^{(0)}_{\text{clas}} \times \left\{1 + \left(\frac{\tilde{\omega}}{\omega}\right)^{2n/(2+n)} \left[\zeta_1(n) + 2^{-n/(2+n)} B(n) \frac{\hbar\omega}{E}\right]\right\}. \tag{36}$$

This fact can be explained by the fact that $\tilde{\omega}$ is an increasing function of the velocity v, which, for the class of potentials in question, in turn enters into the key parameter $a/\lambdabar$ of the problem in "cooperation" with $\hbar$ (for example, in the case of the power-law potentials $a/\lambdabar \sim (\alpha m^{n-1}/\hbar^n v^{2-n})^{1/n}$).

§4. ON THE RELATION BETWEEN THE QUASICLASSICALITY OF THE MOTION OF THE ELECTRONS AND THE CLASSICALITY OF THEIR EMISSION SPECTRUM

1. Above we showed that the quasiclassical nature of the initial electron (in the sense that $\lambdabar \ll a$) is sufficient for the emission spectrum in the region $\omega \ll \omega^*$ to be classical (this was demonstrated for $\omega \gg \tilde{\omega}$; for $\omega \lesssim \tilde{\omega}$ the spectrum is all the more classical). This is a generalization to the non-Coulomb case of the corresponding assertion made in Refs. 16 and 17, and essentially exhausts the practical aspect of the problem. For the subline the classicality criterion is the condition $l \gg 1$.

The above-formulated relation between the quasiclassicality of the motion and the classicality of the spectrum pertains to the entire frequency range. At the same time, from (27) we can obtain the corresponding relation for a fixed ω, interpreting the quantum correction in (27) in terms of a departure from quasiclassicality of the electron motion at small distances from the field center (including here the condition for the nondeformability of the trajectory (along its emissive section) by the effect of the emission of $\hbar\omega$). To do this, let us represent the quantum correction in (27) in the following mutually equivalent forms:

$$\left[\frac{d\varkappa/d\omega}{(d\varkappa/d\omega)_{\text{class}}} - 1\right] \sim \frac{\lambdabar(r_\omega)}{r_\omega} \sim \frac{\hbar}{M_\omega} \sim \frac{\hbar\omega}{E_{\text{kin}}(r_\omega)}, \tag{37}$$

where r_ω and M_ω are given by the formulas in (2),

$$\lambdabar(r_\omega) = \hbar/[q^2 + 2m|U(r_\omega)|]^{1/2} \sim \hbar/[m|U(r_\omega)|]^{1/2}$$

is the local wavelength of the electron (not to be confused with $\lambdabar = \hbar/q$!), and $E_{\text{kin}}(r)$ is the local kinetic energy of the electron. Indeed, the condition $\lambdabar(r_\omega) \ll r_\omega$ is the quasiclassicality criterion for the motion, which can be obtained directly from the criterion

$$\hbar p^{-2}(r)\,\mathrm{div}\,\mathbf{p}(\mathbf{r}) \ll 1, \tag{38}$$

where $\mathbf{p}(\mathbf{r})$ is the total momentum of the electron (a generalization of the standard criterion $|d\lambdabar(x)/dx| \ll 1$ to the three-dimensional case (see, for example, Ref. 28)), as applied to the "rotational" sections of the trajectories (along these sections the momentum for the radial motion is small and the local radius of curvature $\sim r_\omega$). Further, on account of (2), the condition $\hbar/M_\omega \ll 1$ is practically equivalent to the standard criterion $l \gg 1$, on the basis of which the correct limitation on the classicality of the Coulomb spectrum of the BR is obtained in Refs. 20 and 23. Finally, the condition $\hbar\omega \ll E_{\text{kin}}(r_\omega)$, which has a "dynamical" character, replaces the usually cited (and, as we have seen, superfluous) "kinematic" condition $\hbar\omega \ll E$.

Let us note further that the classicality limitation $\omega \ll \omega^*$ for the spectrum is related through the condition $E - E' = \hbar\omega$ with the condition for the electron to be quasiclassical in the final state as well. Thus, for example, in the Coulomb case, when $Ze^2/\hbar v \gg 1$ and $\hbar\omega \gg E$, the quantum correction in (27) reduces to a quantity $\sim (n')^{-2/3}$, where n' is the principal quantum number of the final (bound) state.

2. The principal meaning of what we have said in connection with (37) amounts to the assertion that the spectrum of the radiation emitted at the frequency ω is classical to the extent that the electron motion in the region of space responsible for the emission of this frequency is quasiclassical. (Notice that this assertion is valid not only for the above-considered "spatially localized" frequencies $\omega \gg \tilde{\omega}$, at which the nonclassicality parameter for the spectrum is $\lambda\!\!\!^{-}(r_\omega)/r_\omega$, but also for the "delocalized" frequencies $\omega \lesssim \tilde{\omega}$).

The simplicity of this relation is explained by the fact that, as it turns out, the nonclassicality of the radiative-transition matrix element (its deviation from the corresponding classical Fourier coefficient) is characterized by the same parameter that characterizes the contribution of the nonclassicality of the electron motion itself [i.e., the contribution of the deviation of the wave function from the quasiclassical asymptotic form (see (25))]. The separation of the indicated contribution from the total quantum correction to the spectrum can be carried out only within the framework of the *two-dimensional* quasiclassical description. This description consists[21] in the fact that, in the matrix element, we reduce the total wave functions (and not just their radial components, as is the case in §3) to the form (25), after which we compute the matrix element by the two-dimensional stationary-phase method. This, in particular, furnishes an explicit demonstration, not found in the literature, of the correspondence principle for the matrix element of an arbitrary inelastic (not necessarily radiative) transition between continuous-spectrum states that are described by wave functions of the type ψ^+, ψ^- given in §136 of Ref. 22:

$$|\langle \mathbf{p}'|A|\mathbf{p}\rangle| = \frac{2\pi\hbar}{m}\left[\frac{d\sigma}{d\Omega'}\right]^{1/2} |A_\omega|, \tag{39}$$

where $d\sigma/d\Omega'$ is the classical cross section for scattering from a state with momentum $\mathbf{p}$ into a state with momentum $\mathbf{p}'|\mathbf{p}|/|\mathbf{p}'|$, A_ω is the Fourier transform of the quantity A for the trajectory corresponding to the indicated scattering act, and $\omega = (E - E')/\hbar$ (for more details, see (41) in Ref. 21).

The quantum correction, obtained by the indicated method, to the classical limit of the matrix element contains as a separate term a contribution (due to corrections of the type Δ from (25) in the wave functions of the initial and final states) of the nonclassicality of the two-dimensional motion, so that the remaining part of the correction can be interpreted as the contribution of the nonclassicality of the emission event itself. (Notice that the above-described two-dimensional approach enables us to investigate the classicality of the matrix element of an arbitrary operator; on the other hand, it does not allow us to use the summability over l for a particular case, namely, the case of the dipole moment operator (see (20)). This summation simplifies the final result for the spectrum, but then it leads to the mixing of the contributions of the nonclassicality of the azimuthal motion and the nonclassicality of the emission event.)

The expression for the quantum correction in the two-dimensional quasiclassical formalism has an extremely unwieldy form. Nevertheless, it is possible to establish[21] the fact that, for the class of potentials in question and for $\omega \gg \tilde{\omega}$, the formal structures of the contributions of the nonclassicality of the motion and the emission event (which together give, as they should, the quantum correction in (27)) are the same.

Moreover, a situation is possible in which the initially "motional" nonclassicality parameter determines first and foremost the contribution of the nonclassicality of the emission event, since the numerical coefficient attached to the indicated parameter in the contribution of the nonclassicality of the motion turns out (in this order) to be equal to zero. This is true for all fields within the class under consideration that satisfy the condition $n = 2 - 1/N$, where N is a whole number ($N = 1$ corresponds to the Coulomb field), and is connected with the symmetry properties of the trajectory with zero angular momentum.

The vanishing of the indicated numerical coefficient indicates the following: for the Coulomb field—manifestation of a general inherent "tendency toward classicality," for the remaining fields of this subclass—the possibility of their possessing an additional (classical) symmetry of the Coulomb type, and on the whole—the unimportance of the indicated "classicality" of the field for the classicality of the spectrum, which predetermines the very possibility of that the property of drawn-out—with respect to ω—classicality of the Coulomb emission spectrum (see §1) can be extended to a broad class of fields.

3. The foregoing conclusion that the quasiclassicality of the motion is sufficient for the classicality of the emission spectrum pertains, strictly speaking, to the case of radiation intensity that is not exponentially small. For those frequencies ω at which this intensity is itself exponentially small, its classical nature can be destroyed when the motion is quasiclassical. For the class of potentials under consideration such a situation arises only for the subline. Thus, when the condition $l \gg 1$ (which, for $\lambda\!\!\!^{-} \ll a$, is the quasiclassicality condition for the motion along the entire corresponding trajectory) is fulfilled, the subline is nonclassical at frequencies lying even in the region of exponential smallness (see Subsec. 3 of §3).

§5. THE "ROTATIONAL" APPROXIMATION. SPECTRUM OF THE BR EMITTED ON A MANY-ELECTRON ATOM

The results of the analysis of the limitations imposed on the emission spectrum by the quantum effects (§§3 and 4) make the description of the spectra within the purely classical framework quite expedient.

The physical picture of radiation of frequencies $\omega \gg \tilde{\omega}$ emitted in an attractive-potential field (the spatial localization of the region responsible for the emission at a given frequency), together with the fact that the unsimplified (in the sense that E is neglected and a power law is assumed for $U(r)$—as opposed to what is done in §2) structure of ω_{rot} (1) actually enters into the exact classical formulas (6)–(8), allows us to construct a new, model approximation that describes the classical BR spectrum in a significantly broader frequency range, namely, in the range $\omega \gtrsim \tilde{\omega}$. It is natural to call this approximation, which is a development of the approach proposed in Ref. 16, the "rotational" approximation (RA).

To the indicated approach corresponds the replacement of the actual subline by $\delta(\omega-\omega_{\rm rot})$ given by (1). To find the corresponding spectrum (the superposition of all the sublines), it is convenient to use the exact formula for the integrated effective radiation $\varkappa$, namely, the formula (1) from Ref. 16. Interpreting the dummy integration variable r in this formula as r_0, and introducing the above-indicated δ function under the integral sign, we obtain

$$\frac{d\varkappa}{d\omega}\sim\frac{8\pi e^2}{3c^3m^2v}\int\left(\frac{dU}{dr}\right)^2\left(1-\frac{U}{E}\right)^{1/2}r^2\delta(\omega-\omega_{\rm rot}(r))\,dr. \quad (40)$$

Although this formula pertains, from the standpoint of its derivation, to fairly high ω, because of the "indestructibility" of its functional structure (in the sense indicated above), it is equivalent to the entire series in $1/\omega$, and not just to the leading term in it. It is precisely this property that guarantees the broadened region of applicability of the RA: $\omega\gtrsim\tilde{\omega}$, and not just $\omega\gg\tilde{\omega}$. The first terms of this series in $1/\omega$ have the form

$$\frac{d\varkappa}{d\omega}\sim\frac{4\pi e^2n^2}{3c^3v^2(2+n)}\left(\frac{2\alpha}{m}\right)^{6/(2+n)}\omega^{4(n-1)/(2+n)}\left\{1+\frac{2-n}{2+n}A_q(\omega)+\frac{6n-4\gamma-\gamma n}{n(n+2)}A_c(\omega)\right\}, \quad (41)$$

where A_q and A_c are given by the expressions (15), which pertain to the *exact* expansion (16).[7)] The remaining as yet undetermined numerical coefficient should be extracted from a comparison of the leading term in (41) with the corresponding term of the exact asymptotic form, and turns out to be equal to $\mathscr{P}_0(n)/\pi$. Such a procedure reflects the fact that the accurate replacement of the actual subline by a model one should be carried out at a fixed value of the subline area (i.e., of the integral over the frequencies).

For the practically most important potentials of the class under consideration (§2), namely, the atomic potentials (which go over into the Coulomb potential $U=-Ze^2/r$ as $r\to 0$) we finally obtain from (40) the Gaunt factor

$$g_{\rm rot}\equiv\frac{(d\varkappa/d\omega)_{\rm rot}}{(d\varkappa/d\omega)_{\rm Kram}}=\frac{6}{Z^2e^4}\frac{D_\omega^2}{2+D_\omega}\frac{[E+|U(r_\omega)|]^3}{m\omega^2},$$
$$D_\omega=-\frac{d\ln(E+|U(r_\omega)|)}{d\ln r_\omega}, \quad (42)$$

where the effective radius r_ω is a root of the equation (cf. (1))

$$[E+|U(r_\omega)|]/r_\omega^2=m\omega^2/2. \quad (43)$$

Let us consider two examples of the application of the RA (42), (43) to the BR spectrum.

For the purely Coulomb case we find from (42) and (43) that

$$g_{\rm rot}(\nu)=\frac{(2\nu)^{4/3}+R(\nu)}{(2\nu)^{2/3}+{}^2/_3R(\nu)},\qquad \nu=\frac{Ze^2}{mv^3}\omega, \quad (44)$$

where

$$R(\nu)=2^{-1/3}\left[1+\left(1-\frac{1}{27\nu^2}\right)^{1/2}\right]^{1/3}+2^{-1/3}\left[1-\left(1-\frac{1}{27\nu^2}\right)^{1/2}\right]^{1/3},$$
$$\nu\geqslant\frac{1}{3\sqrt{3}}=0.192, \quad (45)$$
$$R(\nu)=\frac{2^{2/3}}{\sqrt{3}}\nu^{-1/3}\cos\left[\frac{1}{3}\operatorname{arctg}\left(\frac{1}{27\nu^2}-1\right)^{1/2}\right],\qquad \nu\leqslant\frac{1}{3\sqrt{3}}.$$

A comparison of (44) with the numerical tabulation of the exact classical Coulomb BR spectrum[29] shows that, even for $\nu=\frac{1}{2}$, there is only a 5% difference.

For a many-electron atom, let us use the Thomas-Fermi (TF) model:

$$U=-\frac{Ze^2}{r}\chi\left(\frac{r}{a_{\rm TF}}\right),\qquad a_{\rm TF}=\frac{b\hbar^2}{me^2Z^{1/3}},$$

where $\chi(x)$ is the TF universal function[22] and $b=0.885$. In this case the exact classical Gaunt factor depends on two dimensionless parameters[8)]:

$$\Omega=\omega\left(\frac{ma_{\rm TF}^3}{2Ze^2}\right)^{1/2}=\left(\frac{b^3}{2}\right)^{1/2}\frac{\hbar^3}{me^4}\frac{\omega}{Z}, \quad (46)$$

$$\varepsilon=\frac{Ea_{\rm TF}}{Ze^2}=\frac{b\hbar^2}{me^4}\frac{E}{Z^{4/3}}=32.6\frac{E\,({\rm keV})}{Z^{4/3}} \quad (47)$$

and the RA (42), (43) yields

$$g_{\rm rot}(\Omega,\varepsilon)=3(\chi-y\chi')^2\left[2+\frac{\chi-y\chi'}{\chi+\varepsilon y}\right]^{-1},\quad \Omega\gtrsim\tilde{\Omega}\sim\frac{\sqrt{\varepsilon}}{\bar{x}(\varepsilon)}, \quad (48)$$

where $\chi\equiv\chi(y)$, the prime denotes differentiation, and $y(\Omega,\varepsilon)$ and $\bar{x}(\varepsilon)$ are roots of the equations

$$\chi+\varepsilon y=\Omega^2y^3,\qquad \chi(\bar{x})=\varepsilon\bar{x}. \quad (49)$$

The role of the quasiclassicality parameter $a/\lambda\!\!\!^-$ (§3) is played in the present case by [21] $1/\varepsilon$, so that the smaller ε is the more classical the spectrum should be.

The effectiveness of the RA can clearly be seen from the good agreement between (48) and the results of the corresponding numerical quantum calculations[30,31] in the region $Z\gtrsim 20$, $\varepsilon\leqslant 2$. Here even at the (lower—with respect to Ω) limit of applicability of the RA, a limit which in its simplified form is described by the straight line $\Omega=2\varepsilon$, there is only a 10–20% difference.

To describe the remaining "nonrotational" part of the BR spectrum ($\Omega\lesssim\tilde{\Omega}$), it is sufficient in practice to carry out a linear inter-polation between $g_{\rm rot}(2\varepsilon,\varepsilon)\equiv g_1(\varepsilon)$ and the value of the exact classical Gaunt factor at the point $\omega=0$: $g(0,\varepsilon)\equiv g_0(\varepsilon)$ (the "transport" limit).[32]

The natural variables, Ω and ε, of the classical spectrum are (for the indicated E and ε regions) approximate scaling parameters of the initially quantum, more multiparametric spectrum. This allows us to represent all the corresponding spectra[30,31] in a universal form by reconstructing them in terms of the variables Ω and ε. It is sufficient here to illustrate the overall comparison of the quantum results with the classical results at the level of the "reference" functions $g_1(\varepsilon)$ and $g_0(\varepsilon)$ (Fig. 1).

Moreover, even within the classical framework the RA is able to furnish the scaling law for the short-wave limit $\omega_{\max}=E/\hbar$ of the BR spectrum[32]: $g(\Omega_{\max},\varepsilon)\approx G(E/Z)$. As

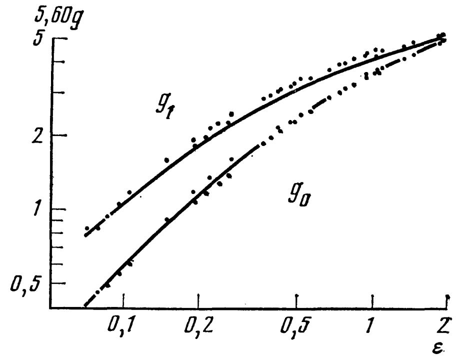

FIG. 1. The universal critical functions $g_0(\varepsilon)$ and $g_1(\varepsilon)$ (curves) and a comparison with them of the corresponding (reconstructed) results of the numerical quantum calculations of Lee *et al.*[30]

ω decreases to zero, a transition occurs to the scaling law $G_0(E/Z^{4/3})$.

It should be noted separately that in the emission of low frequencies there occur those quantum effects which are characteristic of the elastic scattering of slow particles in a potential field that falls off faster than the centrifugal potential. We have in mind the nonuniqueness of the dependence of the scattering angle θ on the impact parameter ρ, a circumstance which leads to the interference of the classical branches of the scattering and, consequently, to oscillations in the differential scattering cross section. In the case of the Thomas-Fermi potential such a situation (including, in particular, the interference in the scattering of trajectories in terms of ρ—the "rainbow") already obtains, as the numerical calculation shows, when $\varepsilon \sim 0.2$. But in the θ-integrated quantity, i.e., in the transport scattering cross section (to which, in the general quantum case, the low-frequency limit of the BR spectrum is proportional), the indicated quantum effect turns out to be suppressed in the ε region under consideration which can be seen from the good agreement between the purely classical $g_0(\varepsilon)$ and the results of the numerical quantum calculation (Fig. 1).

Thus, the classical description, in the indicated region of its applicability, provides at the same time a simple physical explanation of the main features of the quantum spectra of the radiation: the degree of effective screening of the nucleus, the overall sluggish behavior of the spectra, their increase with increasing ω, to be replaced by a decrease when ε attains a value ~ 2 (Fig. 1). This shows the adequacy of the classical approach precisely as a description method, whereas the specific choice of a model for the atomic potential (the Hartree-Fock potential in Ref. 30 and the Thomas-Fermi potential in Ref. 31) does not play an important role (see Figs. 2 and 3 in Ref. 32).

§6. COMPARISON WITH EXPERIMENT: THE ANGULAR DISTRIBUTION OF THE SPECTRUM OF BR OF AN ATOM

Let us carry out the comparison of the developed theory with experiment for the particular case of the spectra of BR of many-electron atoms. Such spectra have been measured only for a fixed photon-emission angle $\theta = 90°$, measured relative to the direction of incidence of the electrons.[5]

The formulas for the differential—with respect to θ—spectrum that are needed for the comparison with Ref. 5 can be obtained on the basis of the results of §§2–5. Indeed, the structure (the separation of the angular dependence $P_2(\cos\theta)$) of the classical formula (68.5) in Ref. 1 for the angular distribution of the spectrum of the dipole radiation emitted in a central field, a formula which preserves its form on going over to the quantum case to within a replacement of quantities in accordance with the correspondence principle (39), itself guarantees the applicability of the classicality criteria obtained in §3 to the differential—with respect to θ—spectrum too.

The normalized angular distribution $S(\theta, \omega)$ ($\int S(\omega, \theta) dO = 1$, $dO = 2\pi \sin\theta d\theta$) that follows from the formula (68.5) in Ref. 1 has the form

$$S(\theta, \omega) = \frac{3}{8\pi}[1 - \Delta(\omega) + (3\Delta(\omega) - 1)\cos^2\theta], \tag{50}$$

$$\Delta(\omega) = \frac{1}{2}\left[1 - \frac{\int_0^{\infty'} (J_c^2 \cos^2\varphi_\infty + J_s^2 \sin^2\varphi_\infty) M\, dM}{\int_0^{\infty} (J_c^2 + J_s^2) M\, dM}\right], \tag{51}$$

where J_c (J_s) is given by the formula (7); $\varphi_\infty \equiv \varphi(r = \infty)$, by the formula (8).

For the class of fields under consideration (§2) we find in the limit $\omega \gg \tilde{\omega}$ that $\Delta(\infty) = 1/2[1 - \mathscr{P}_{0z}(n)/\mathscr{P}_0(n)]$, where $\mathscr{P}_{0z}(n)$ is obtained from $\mathscr{P}_0(n)$ (17), (13) by multiplying the first term in (13) by $\cos^2[\Pi/(2-n)]$ and the second term by $\sin^2[\Pi/(2-n)]$. For atomic potentials (in the ω limit under consideration, these are effectively Coulomb potentials) we have $\Delta(\infty) = \frac{1}{6}$. In the limit $\omega = 0$ we obtain

$$\Delta(0) = \frac{1}{4\sigma_{tr}} \int \sin^2\theta'\, d\sigma,$$

where σ_{tr} and $d\sigma$ are the transport and differential cross sections for elastic scattering of an electron and θ' is the scattering angle.

Investigation of the function $\Delta(\omega)$ in the case of the TF atom shows that in the ε region under consideration

$$\Delta_{TF}(\Omega, \varepsilon) \approx \Delta_{TF}(\Omega = \infty) \approx 0.2. \tag{52}$$

For the $\theta = \pi/2$ case of interest to us we find from (50) and (52) that

$$\left(\frac{dg}{dO}\right)_{\pi/2} \approx \frac{3}{8\pi} 0.2 g(\Omega, \varepsilon), \tag{53}$$

where $g(\Omega, \varepsilon)$ is given by the formula (3) in Ref. 32. In Fig. 2 we compare (53) with the experiment reported in Ref. 5 (as well as with the numerical quantum calculations performed in Ref. 6) for E and Z values falling within the region $\varepsilon \leqslant 2$ (see §5). The agreement is good. The observed intensity peaks are the result of the super-position of the characteristic x-ray lines on the BR spectrum (this has been experimentally verified by the authors of Ref. 5 themselves). The steep drop in the spectra at the short-wavelength limit (and, consequently, the destruction of the "rotational" character of the spectrum on going beyond the framework of the BR) is due simply to the fact that here we are dealing (in view of the absence of

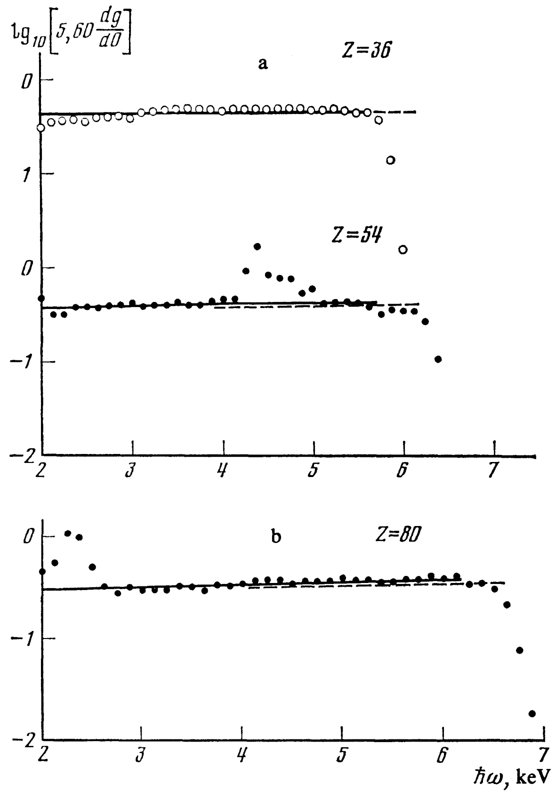

FIG. 2. The Gaunt factor of the BR emitted by electrons of energy E on neutral Kr, Xe, and Hg atoms for a photon-emission angle of $\theta = 90°$ and for: a) $E = 6$ keV and b) $E = 6.5$ keV. The points indicate the results of the measurements performed by Semaan and Quarles[5]; the continuous curves, the results of the numerical quantum calculations of Tseng *et al.*[6]; and the dashed lines, the results of calculations performed with the use of the classical formula (53).

vacancies in the electron shell of the neutral atom) not with PR, but with photoattachment.

Let us also note that, even though the relativistic effects begin to show up in the angular distribution of the radiation (in contrast to the θ-integrated spectrum) already at $E \sim 1$ keV, for the angle $\theta = 90°$ under consideration their influence is minimal, and is insignificant at $E \approx 6$ keV (see Ref. 6).

§7. CONCLUSION

The results of the present paper are, on the whole, a generalization, to the non-Coulomb case, of the analytic description of the classical and quasiclassical high-frequency spectra of the radiation emitted by electrons in a central attractive field. It is shown that the high-frequency radiation emitted by quasiclassical electrons in the fields of a broad class of attractive potentials (§2) have a common physical basis consisting in the dominant role played by the electron motion along the highly curved trajectories, of which effective spatial localization of the region of emission of a given frequency is characteristic. In classical electrodynamics this leads to the emission by an electron stream of high-frequency radiation (BR, PR) of high (nonexponential) intensity. In the quantum formalism the quantum correction to the classical limit of the radiative-transition matrix element remains small in the high-frequency region under consideration. Analysis of this correction shows that the quasiclassicality of the electron motion in the region of space responsible for the emission of a given frequency ω is a sufficient condition for the corresponding part of the spectrum to be classical (with the exception of only the case when ω lies in the region where the classical intensity is itself exponentially small (see Subsec. 3 of Sec. 4), and therefore does not need to be supplemented (as is usually done) by the "kinematic" condition $\hbar\omega \ll E$. In the successful scheme the formulated relation between the classicality of the motion and the classicality of the emission spectrum amounts, in particular, to the fact that the fulfillment of the condition $\bar{\lambda} \ll a$ is sufficient for practically the entire BR + PR spectrum to be classical.

As applied to the spectra of BR emitted on many-electron atoms (ions), the foregoing allows us to draw the conclusion that the classical description is applicable in the case of low incoming-electron energies. Within the framework of the classical description, the broad (ω) region of applicability of the "rotational" approximation (§5) provides a basis for an analytical description of these spectra[32] that agrees well with the numerical quantum calculations.[30,31] For the differential—with respect to the photon-emission angle—BR spectra, the classical description (§6) is in good agreement with the experimental data reported in Ref. 5.

In conclusion, let us note that the potentialities of the method developed in the present paper can be extended. Thus, within the framework of classical electrodynamics, we can, on the basis of (3), describe the low-frequency ($\omega \ll \tilde{\omega}$) region of the spectrum as well, and also establish a new physical analogy between the mechanisms underlying BR (PR, LR) emission and the broadening of the spectral lines by "extraneous" particles.[21] In the quantum aspect of the paper the above-formulated relation between the quasiclassicality of the motion and the classicality of the emission spectrum can be generalized in two directions. First, we can, in the case of an attractive central potential, treat not only radiative, but also arbitrary inelastic, transitions (e.g., the excitation of atoms by electrons). Secondly, within the general framework of attractive potentials, the validity of the above-discussed relation is not limited to the case of central fields.

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1) Another more subtle effect of this type is indicated in Subsec. 2 of Sec. 4.

2) Let us emphasize that this basis is essentially not affected by the possible quantization of the electron energy (see Sec. 3, Subsec. 3).

3) Similar qualitative derivations of this formula (i.e., the Kramers formula) are given in Refs. 19 and 20 in treatments of just the Coulomb case.

4) In the general case the sum over l goes over into a sum of integrals of the type (20). For the purposes of interest to us one term is sufficient, since the rest are exponentially small in $\hbar$.

5) We are talking about the quantity σ_2 in (46.11) in Ref. 22, which assumes, after a transformation, the form (20).[21]

6) For example, in the Coulomb case, when $Ze^2/\hbar v \gg 1$, the classical Kramers formula is, even at the short-wavelength limit of the PR, accurate to within a factor of 1.5 (Refs. 3 and 23).

7) The functional accuracy of the RA is illustrated by the fact that, for the Coulomb case ($n = 1$), the numerical coefficient attached to A_q differs

from the corresponding exact coefficient by only 3%.

[8)]Here they essentially have a classical nature, since the $\hbar$ entering into them is simply a parameter of the strictly fixed potential.

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