Topological effect of fermion-pair production enhancement by a nonuniform magnetic field

I. V. Krive and S. M. Latinskiĭ

A. M. Gorki State University, Kharkov
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The effect of a nonuniform magnetic field on the rate of production of fermion pairs by a constant electric field is investigated. It is shown that, in quantum electrodynamics, the stimulation of e^+e^- pair production by a strong magnetic field is an essentially topological effect that is determined by the magnetic-field flux. The effect of a longitudinal magnetic field on the Zener current in the two-band model of tunnel phenomena in semiconductors is examined. The effect of boundaries on the probability of pair production by an electric field is discussed.

It is known that a constant electric field E gives rise to the production of charge pairs by tunneling. In the single-loop approximation, the mean number of particles created by the field in a state with momentum $\mathbf{p}$ and a given spin component is independent of the particle spin and is given by (see, for example, Ref. 1)

$$n_{\mathbf{p}}=\exp\{-\pi(m^2+\mathbf{p}_\perp{}^2)/eE\}, \tag{1}$$

where m is the particle mass, e is the modulus of its charge, and $\mathbf{p}_\perp$ is the particle momentum in the plane perpendicular to the direction of the electric field.

The situation changes radically when a magnetic field is applied.[2,3] As noted in Ref. 3, a uniform constant magnetic field H parallel to E gives rise to the production of bosons and to the enhancement of the production of fermions. In this paper, we shall discuss the physical reason for this pehnomenon and will show that the stimulation of fermion production by a magnetic field is due to the supersymmetry of the problem, and is of topological origin in the limit of strong fields $H\gg E$. This will enable us to obtain simple analytic expressions for the rate of production of e^+e^- pairs by uniform electric fields for a sufficiently wide class of nonuniform magnetic fields $H(x, y)$.

The recently published general results on the spectrum of a two-dimensional electron in a nonuniform magnetic field[4–6] will be the starting point for our analysis. It was shown in Refs. 4–6 that, both for fields with a finite flux[4] and for infinite magnetic fields,[5–6] the number (density) of zero modes of the Pauli Hamiltonian (ground-state degeneracy) does not reflect the local symmetry properties of the magnetic field but is due to its global characteristic, namely, its flux.

Let us consider pair production by an electric field (lying along the z axis) in the presence of a nonuniform magnetic field $H(x, y)$ parallel to it. The quantity $\mathbf{p}_\perp^2$ in (1) is an eigenvalue of the two-dimensional Laplace operator $-\Delta_{(2)} = \partial_x^2+\partial_y^2$. When the magnetic field is present, this operator must be replaced by the two-dimensional "Hamiltonian" as follows:

for scalar bosons

$$-\Delta_B=\{\partial_x-ieA_x(x,\ y)\}^2+\{\partial_y-ieA_y(x,\ y)\}^2; \tag{2}$$

for spin $\frac{1}{2}$ fermions

$$-\Delta_F=\{\partial_x-ieA_x(x,\ y)\}^2+\{\partial_y-ieA_y(x,\ y)\}^2+\sigma_3 eH(x,\ y), \tag{3}$$

where $\mathbf{A}$ is the vector potential, $H\equiv H_z(x,y) = \partial_y A_x - \partial_x A_y$ and σ_3 is a Pauli matrix.

For weak fields, $E\ll E_c = m^2/e$ we may consider the rate of production of charged-particle pairs by the field: $I=2\,\mathrm{Im}\mathscr{L}_{eff}$ ($\mathscr{L}_{eff}$ is the effective Lagrangian, $\mathrm{Im}\,\mathscr{L}_{eff} \ll \mathrm{Re}\,\mathscr{L}_{eff}$), which is readily derived from (1) by integrating over the momenta[1]

$$I=\frac{eE}{2\pi}\exp\left(-\frac{\pi m^2}{eE}\right)\sum_{\{\lambda\}}\nu(\lambda)\exp\left[-\frac{\pi\lambda(H)}{eE}\right], \tag{4}$$

where $\{\lambda\}$ is the set of eigenvalues of the operators given by (2) and (3), and $\nu(\lambda)$ is the degree of degeneracy of the spectrum.

We note that, in the limit of strong magnetic fields $H\gg E$, the principal contribution to the sum in (4) is due to terms with minimum λ. Since I is an exponential function of λ, we shall be interested in the case where the ground state is separated by a gap from excited states. This will probably occur for magnetic fields of constant sign and infinite flux [this question was examined in Ref. 5 for fields of the form $H(x, y) = H_0 + h(x, y)$, where H_0 is a uniform field and $h(x, y)$ is a doubly periodic magnetic field].

Let us being by considering the simple case of a uniform magnetic field ($H=\mathrm{const}$) for which the spectrum of the operators (2) and (3) is known exactly (Landau spectrum):

$$\lambda_B^{(0)}=eH(2n+1),\quad \lambda_F^{(0)}=eH(2n+1+\mu) \tag{5}$$

($\mu = \pm 1, n = 0, 1,\ldots$). Each "energy" level characterized by a particular (n,μ) is in addition infinitely degenerate: $\nu(H) = eH/2\pi$. The simple form of (5) enables us to find the explicit form of the pair-production rate as a function of the magnetic field.[2,3]

The question is: what is the physical reason for the suppression of the boson-production probability by a magnetic field and of the enhancement of fermion production? It is readily seen that when the magnetic field is applied, all low-momentum states that make the principal contribution to (4) in the absence of quantization of $p_\perp$ have "condensed" in the

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momentum plane on the $n = 0$ orbit having a major radius $p_H = (eH)^{1/2}$. Boson pair production for $H \gg E$ is therefore additionally suppressed by the small factor $(H/E)\exp(-H/E)$. On the contrary, for fermions, condensation of states from a circle of radius p_H takes place to the zero-energy level $(N = 0, \mu = -1)$, and the rate of fermion production is increased by a factor of H/E (because of the high degree of degeneracy $\nu(H)$.

Thus, stimulation by a magnetic field of e^+e^- pair production is due to the zero modes of the operator Δ_F. It has recently been noted[7,6] that this operator $\Delta_F/2m$ is identical with the two-dimensional Pauli Hamiltonian for electrons) determines the $N = 2$ (N is the number of supercharges) supersymmetric quantum mechanics.[8] Hence, according to the general properties of supersymmetric theories,[8] all levels with $\lambda_F > 0$ turn out to be doubly degenerate in this case, and the existence of zero modes and the degree of their degeneracy are determined by the global characteristics of the field.

For the two-dimensional Pauli Hamiltonian in flat space, this characteristic is the magnetic field flux. The existence of zero modes of Δ_F is closely related to the supersymmetry of the Dirac particle (normal magnetic moment of the Pauli Hamiltonian), and their degree of degeneracy

$$\nu = \frac{e}{2\pi}\int H(x,y)\,dS \Big/ \int dS \tag{6}$$

turns out to be the same for both the uniform field H_0 and the nonuniform field $H(x, y) = H_0 + h(x, y)$ if (x, y) does not alter the total flux.[5,6]

Hence, the enhancement, linear in the magnetic field H, of fermion pair production is of "topological" origin (it is independent of the local symmetry properties of the field) and applies to a wide class of nonuniform magnetic fields. In particular, in fields $H(x, y)$ having the same flux density as the effective uniform field H_0, the rate of production of fermion pairs by the electric fields E for $H_0 \gg E$ is

$$I_F = \frac{e^2EH_0}{4\pi^2}\exp\left(-\frac{\pi m^2}{eE}\right). \tag{7}$$

We note that fermions produced by the electric field at the rate given by (7) are completely polarized. This is also a rigorous consequence of the supersymmetry of the "Hamiltonian" (3). In fact, all states with $\lambda_F > 0$ are doubly degenerate in spin, and the zero modes do not have a superpartner and correspond to one direction of polarization, i.e., the direction of the magnetic field.

Thus, enhancement of fermion pair production by a strong magnetic field $H_0 \gg E$ is due entirely to the flux—a global characteristic of the two-dimensional magnetic field. However, according to Maxwell's equation, this particular magnetic-field configuration $H_x = H_y = 0$, $H_z = H(x, y)$ can exist only when there are local currents $\mathbf{J} \sim \operatorname{curl} \mathbf{H}$. On the other hand, when we solve the problem, we implicitly assume the existence of a nonuniform magnetic field $H_z(x, y)$ in vacuum. We must therefore ensure the mutual consistency of the two conditions $H_x, H_y \ll H_z$ (necessary to reduce the problem to the two-dimensional case) and $j_x \approx j_y \approx 0$. It is readily verified that this can be achieved for weakly nonuniform fields $H_z(x, y)$.

The problem of the effect of a magnetic field on e^+e^- pair production by an electric field in a vacuum is quite abstract from the point of view of a laboratory experiment. It is more natural to look for manifestations of this effect in atomic physics and solid-state physics.

It was noted in Ref. 9 that a strong uniform magnetic field leads to the saturation of ionization of atoms and negative ions by a constant electric field. When the long-range Coulomb forces are ignored, the enhancement in high fields $H \gtrsim E/v$ (v is the velocity of the electron in the atom in units of the velocity of light) is linear in the field and is entirely due to the existence of the zero-energy level of the two-dimensional Pauli Hamiltonian. Stimulation of ionization of negative ions by a magnetic field[9] is therefore also topological in origin. Since the electric field producing the ionization of an atomic system is relatively low, the experimental verification of this prediction is a practical proposition.

Another example is provided by interband (Zener) tunneling in semiconductors. In narrow-gap semiconductors in which $\varepsilon_g \ll W_{c,v}$, $W_{c,v}$ is the width of the conduction band and the valence band), direct quantum transitions can be satisfactorily described by the so-called two-band model (see, for example, Ref. 10) in which the spectrum of electrons (holes) has the Dirac shape and the quantity $s = (\varepsilon_g/2m^*)^{1/2}$ plays the role of the velocity of light (m^* is the effective band mass of the electron). Tunneling calculations based on this model are analogous, subject to slight modification, to calculations of quantum-electrodynamic processes in strong fields. However, in contrast to quantum electrodynamics (QED), para- and diamagnetic interactions between electrons in crystals are characterized by different masses. The effective band mass m^* and the effective spin mass m_s of an electron are not equal and differ widely for different materials. In the two-band model, the difference between the orbital and spin effective masses can formally be taken into account by introducing the anomalous magnetic moment $\mu = m^*/m_s$ into the integraged Dirac equation.

Standard QED calculations of the rate of production of electron-hole pairs by a constant electric field (Zener current) in the presence of a longitudinal ($\mathbf{H}\|\mathbf{E}$) nonuniform magnetic field yield

$$j(E, H) = \frac{e^2EH_0}{4\pi^2\hbar c}\exp\left\{-\frac{\pi\varepsilon_g^2}{4s\hbar eE}\right\} \times \operatorname{ch}\left(\mu\pi\frac{s}{c}\frac{H}{E}\right) \Big/ \operatorname{sh}\left(\pi\frac{s}{c}\frac{H}{E}\right). \tag{8}$$

When $H = 0$, this expression becomes identical with the well-known formula for the tunneling current.[11] In QED, we have $\mu = 1$, $s = c$, and we obtain the rate of production of e^+e^- pairs by an electric field in a vacuum in a constant magnetic field.[2,3]

In strong magnetic fields $H \gtrsim (c/s)E$,

$$\frac{j(E,H)}{j(E,0)} \approx \pi\frac{s}{c}\frac{H}{E}\exp\left\{\pi\frac{s}{c}\frac{H}{E}\left(\frac{m^*}{m_s}-1\right)\right\} \tag{9}$$

and the effect of the longitudinal magnetic field on tunneling depends on the ratio of the orbital to spin effective masses. A different situation arises in transverse fields $\mathbf{H}\perp\mathbf{E}$. When tunneling takes place ($H < (c/s)E$), the magnetic field can be eliminated by transforming to a moving coordinate frame, the tunneling current is independent of m_s in the first approximation, and the magnetic field always leads to the exponential suppression of the Zener current.[12]

Experiments on the effect of a longitudinal magnetic field on Zener tunneling in a number of compounds, indicate the presence of an exponential suppression effect. Theoretical papers explaining this effect either ignore the interaction between the electron spin and the magnetic field altogether,[13] or find that calculations of the spectrum of particular compounds indicate that the contribution of the paramagnetic interaction between electrons is smaller than that of the diamagnetic interaction.[1] It is therefore common to find in the literature[12–14] the statement that a strong longitudinal magnetic field will always produce the exponential suppression of the tunneling current. Here, we merely wish to emphasize that, in materials in which interband tunneling is satisfactorily described by the two-band model (see the discussion in Ref. 12) and $m^* > m_s$, a longitudinal magnetic field will produce an exponential rise in the Zener current.

According to (8), for $m^* > m_s$ and $H \to H_c$, we have

$$H_c = \frac{c}{2e\hbar}\frac{\varepsilon_g}{(m_s^{-1} - m^{*-1})} \tag{10}$$

and even arbitrarily low electric fields will produce considerable tunneling currents. This means that the semiconductor-metal phase transition will take place in such ultraquantum magnetic fields.[15]

In conclusion, let us consider the effect of boundary conditions on the rate of pair production by a constant electric field. In the simplest case, when the motion is quantized only in the plane perpendicular to the direction of the electric field, and boundaries define a rectangular region, the spectrum of the mass operator is known exactly, and the problem can be solved analytically. However, we shall be interested in only the "topological" aspect of the problem.

We begin by considering the conditions for the confinement of particles within the boundary Γ surrounding a region with characteristic linear dimension L (such conditions are usually imposed on particles in the "bag" model).[16] It is physically clear that, in this case, the minimum particle momenta in the system are $p_\perp \sim L^{-1}$ and pair production in the presence of "two-dimensional confinement" will be exponentially suppressed independently of the particle spin for $eEL^2 \ll 1$, i.e., it will be described by the factor $\exp(-C/eEL^2)$, $C \sim 1$. The absence of zero modes of the two-dimensional Laplace operator $\Delta_{(2)}$ for boundaries of arbitrary shape in the example that we are considering is an elementary consequence of a theorem on the maximum and minimum of a function that is harmonic in a bounded region (see, for example, Ref. 17), which is well-known in complex analysis.

On nontrivial manifolds, the Laplace operator may acquire normalizable zero modes separated by a gap from the positive spectrum. (In this case, it is more natural in a physical formulation to consider the creation of particles not by the electric field but by the metric.) For example, let us replace the xy plane by a two-dimensional torus, and impose periodic (antiperiodic for fermions) boundary conditions on the fields. For bosons, the zero mode [reflecting the $U(1) \times U(1)$ symmetry of the problem] will then appear in the spectrum of $\Delta_{(2)}$, and boson production on this particular compact manifold will occur with a much higher rate than fermion production.

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[1)]For example, the expression for the tunneling current found in Ref. 14 in the classical approximation follows from (8) for $\mu = \frac{1}{2}$.

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