Phonon-kinetics changes due to the dispersion law and to impurities

I. N. Adamenko and V. I. Tsyganok

Kharkov State University

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Kinetic phenomena due to phonons and impuritons in He^3–He^4 solutions are considered. The sound-velocity renormalization and the sound damping are calculated, and the times that determine the coefficients of viscosity and of thermal conductivity are found. These coefficients are found to have complicated dependences on density and on temperature. The results agree with the available experimental data. It is shown that the discrepancy between the earlier theory and experiment, which exceeds an order of magnitude for weakly concentrated solutions, is due to earlier failure to take three-particle phonon processes into account.

Khalatnikov and Zharkov[1] have shown in 1957 that the kinetic properties of weak He^3–He^4 solutions are determined by interactions of quasiparticles—phonons, rotons, and impuritons. Subsequent intensive experimental and theoretical investigations enabled Baym, Ebner, and Saam[2–4] ten years later to generalize and extend the theory of Ref. 1 to temperatures below 0.6 K, where the roton contribution can be neglected. Since that time and to this day, to our knowledge, the starting point for all theoretical calculations and for the experimental data reductions are the results of Refs. 2–4.

This theory has described quite well the experimental data in a number of cases. For weak solutions, however, when the phonon contribution is substantial, the theory differed from the experiment by more than a factor of ten (see, e.g., Refs. 5–7). Another decade later, Baym and Pethick[8] analyzed anew the equations of Refs. 2–4. The results of Ref. 8, however, did not differ in fact from the earlier ones, apart from some fine tuning of the parameters, which made the discrepancy between theory and experiment even greater. This disparity is particularly pronounced in recent precision measurements of the velocity of first sound in solutions containing $5\cdot10^{-4}$, $1.5\cdot10^{-3}$, and $5\cdot10^{-3}$ He^3.[9]

Analyzing the resultant situation, we arrived at the conclusion that this discrepancy, exceeding an order of magnitude, is due mainly to the fact that no account was taken of three-particle phonon processes in Refs. 2–4 and 8. These will be shown to change substantially the entire kinetics of the solutions. One must not conclude, however, that the earlier theory,[2–4] whose premises are used also in the present paper, is incorrect. All that is needed is a refinement of the region of its validity. It follows from the results presented below that the theory of Refs. 2–4 holds either at high pressures, when three-particle processes are forbidden, or at such low temperatures and high concentrations that the phonon contribution is negligible compared with that of the impurities. It is this which explains the good agreement between the theory of Refs. 2–4 and experiment in the indicated regions. In the remaining cases the experimental data cannot be explained without allowance for three-particle phonon processes, which alter the characteristic relaxation times by more than an order, as is indeed observed in experiment.

The purpose of the present paper is an analysis of the kinetic phenomena in He^3–He^4 solutions with account taken of three-particle phonon processes. The kinetic phenomena in solutions, just as in pure helium,[10,11] can be investigated by solving the problem of propagation and damping of sound of arbitrary frequency. The times that determine the viscosity and thermal conductivity coefficients are obtained in the hydrodynamic limit. The solution of our problem called for a calculation procedure different from those proposed earlier to solve analogous problems in pure helium[10,11] and in solutions.[4] The starting point in our scheme is introduction of projection operators that permit the use of a correct τ-approximation.

DIPERSION EQUATION. IMPURITON GAS

According to Refs. 1–4, the kinetic properties of solutions of He^3 in superfluid He^4 can be investigated by starting from the kinetic equation for impuritons

$$\frac{\partial f_3}{\partial t}+\frac{\partial f_3}{\partial \mathbf{r}}\frac{\partial H_3}{\partial \mathbf{p}_3}-\frac{\partial f_3}{\partial \mathbf{p}_3}\frac{\partial H_3}{\partial \mathbf{r}}=I_{33}(f_3)+I_{34}(f_3,f_4) \tag{1}$$

and the kinetic equation for the HeII-excitation gas

$$\frac{\partial f_4}{\partial t}+\frac{\partial f_4}{\partial \mathbf{r}}\frac{\partial H_4}{\partial \mathbf{p}_4}-\frac{\partial f_4}{\partial \mathbf{p}_4}\frac{\partial H_4}{\partial \mathbf{r}}=I_{44}(f_4)+I_{43}(f_4,f_3). \tag{2}$$

Here f_i and f_{ik} are the corresponding distribution functions and collision integrals,

$$H_3=\varepsilon_0+\frac{p_3^2}{2m}+\frac{\delta m}{m}\mathbf{p}_3\mathbf{v}_s-\frac{1}{2}\frac{\delta m}{m}m_3v_s^2$$

is the impuriton Hamiltonian, $H_4=\varepsilon_4+\mathbf{p}_4\cdot\mathbf{v}_s$ is the Hamiltonian of the excitations, $\mathbf{v}_s$ is the superfluid velocity, m is the impuriton effective mass, m_3 the mass of the He^3 atom, and $\delta m=m-m_3$.

The set of Eqs. (1) and (2) should be suplemented by a continuity equation best written in terms of the variables $\rho_4=m_4n_4$ (the He^4 density) and f_i:

$$\partial\rho_4/\partial t=-\operatorname{div}\mathbf{j}, \tag{3}$$

where

$$\mathbf{j}=\left(\rho_4-\frac{\delta m}{m}\rho_3\right)\mathbf{v}_s+\frac{\delta m}{m}\int\mathbf{p}_3f_3\,d\Gamma_3+\int\mathbf{p}_4f_4\,d\Gamma_4$$

is the momentum density and $\rho_3=m_3n_3$ is the He^3 density in the solution.

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Equations (1)–(3) must be considered jointly with the equation for $\mathbf{v}_s$:

$$\partial \mathbf{v}_s/\partial t = -\nabla \mu_4, \tag{4}$$

where μ_4 is the chemical potential of He^4 in the solution. At low concentrations $x = n_3/n_4$ the variation of this potential can be written in the form

$$\begin{aligned}\delta\mu_4 &= \frac{c^2}{\rho_4}\rho_4' + \int \frac{\partial E_3}{\partial \rho_4}\delta f_3\, d\Gamma_3 + \int \frac{\partial \varepsilon_4}{\partial \rho_4}\delta f_4\, d\Gamma_4,\\ \frac{c-c_0}{c_0} &= -u\left[xa + \frac{P_f}{\rho_4 c^2}\left(1-\frac{3}{2}w\right)\right]\\ &+ \frac{\rho_4}{2c^2}\left[\int \frac{\partial^2 E_3}{\partial \rho_4^2} f_3\, d\Gamma_3 + \int \frac{\partial^2 \varepsilon_4}{\partial \rho_4^2} f_4\, d\Gamma_4\right],\end{aligned} \tag{5}$$

c_0 is the speed of sound in pure He^4 at $T = 0$, P_f is the pressure of the impurity gas,

$$E_3 = \varepsilon_0 + \frac{p_3^2}{2m},\quad u = \frac{\rho_4}{c}\frac{\partial c}{\partial \rho_4},\quad a = \frac{\rho_4}{m_4 c^2}\frac{\partial \varepsilon_0}{\partial \rho_4},\quad w = \frac{\rho_4}{m}\frac{\partial m}{\partial \rho_4}.$$

To obtain the dispersion equation that characterizes small oscillations in the solutions, the set (1)–(4) must be linearized. In the upshot we have for the Fourier components of the independent variables ρ_4^0, $\mathbf{v}_s$, δf_3, and δf_4, which describe the deviations from equilibrium,

$$(\omega - \mathbf{k}\mathbf{v}_3)\delta f_3 + \mathbf{k}\mathbf{v}_3 f_3'\left(\frac{\partial E_3}{\partial \rho_4}\rho_4' + \frac{\delta m}{m}\mathbf{p}_3\mathbf{v}_s\right) = iI_{33}(f_3) + iI_{34}(f_3, f_4), \tag{6}$$

$$(\omega - \mathbf{k}\mathbf{v}_4)\delta f_4 + \mathbf{k}\mathbf{v}_4 f_4'\left(\frac{\partial \varepsilon_4}{\partial \rho_4}\rho_4' + \mathbf{p}_4\mathbf{v}_s\right) = iI_{44}(f_4) + iI_{43}(f_4, f_3), \tag{7}$$

$$-\omega\rho_4' + \left(\rho_4 - \frac{\delta m}{m}\rho_3\right)\mathbf{k}\mathbf{v}_s + \frac{\delta m}{m}\int \mathbf{p}_3\mathbf{k}\,\delta f_3\, d\Gamma_3 + \int \mathbf{p}_4\mathbf{k}\,\delta f_4\, d\Gamma_4 = 0, \tag{8}$$

$$-\omega\mathbf{v}_s + \mathbf{k}\frac{c^2}{\rho_4}\rho_4' + \mathbf{k}\int \frac{\partial E_3}{\partial \rho_4}\delta f_3\, d\Gamma_3 + \mathbf{k}\int \frac{d\varepsilon_4}{\delta \rho_4}\delta f_4\, d\Gamma_4 = 0, \tag{9}$$

where $\mathbf{v}_i = \partial\varepsilon_i/\partial\mathbf{p}_i$ is the quasiparticle velocity, $\varepsilon_3 = p_3^2/2m$, and the prime at f_i denotes the derivative with respect to energy.

It is convenient to introduce new variables:

$$\begin{aligned}\delta\tilde f_3 &= \delta f_3 - f_3'\left(\frac{\partial E_3}{\partial \rho_4}\rho_4' + \frac{\delta m}{m}\mathbf{p}_3\mathbf{v}_s\right),\\ \delta\tilde f_4 &= \delta f_4 - f_4'\left(\frac{\partial \varepsilon_4}{\partial \rho_4}\rho_4' + \mathbf{p}_4\mathbf{v}_s\right).\end{aligned} \tag{10}$$

Using the conservation laws, we can rewrite Eqs. (6) and (7) in terms of the new variables:

$$(\omega - \mathbf{k}\mathbf{v}_3)\delta\tilde f_3 + \omega f_3'\left(\frac{\partial E_3}{\partial \rho_4}\rho_4' + \frac{\delta m}{m}\mathbf{p}_3\mathbf{v}_s\right) = iI_{33}(\delta\tilde f_3) + iI_{34}(\delta\tilde f_3, \delta\tilde f_4), \tag{11}$$

$$(\omega - \mathbf{k}\mathbf{v}_4)\delta\tilde f_4 + \omega f_4'\left(\frac{\partial \varepsilon_4}{\partial \rho_4}\rho_4' + \mathbf{p}_4\mathbf{v}_s\right) = iI_{44}(\delta\tilde f_4) + iI_{43}(\delta\tilde f_4, \delta\tilde f_3). \tag{12}$$

Here I_{ik} are the linearized collision integrals and depend on $\delta\tilde f_i$.

We seek the solutions of Eqs. (11) and (12) in the form

$$\delta\tilde f_i = -f_i' g_i. \tag{13}$$

We rewrite (11) and (12) in matrix form

$$(\omega - \mathbf{k}\mathbf{v} - iI)g = \omega\delta U, \tag{14}$$

with the matrices

$$\omega - \mathbf{k}\mathbf{v} = \begin{vmatrix} \omega - \mathbf{k}\mathbf{v}_3 & 0 \\ 0 & \omega - \mathbf{k}\mathbf{v}_4 \end{vmatrix},\quad I = \begin{vmatrix} I_{33}+I_{34} & I_{34} \\ I_{43} & I_{44}+I_{43} \end{vmatrix},$$

$$g = \begin{vmatrix} g_3 \\ g_4 \end{vmatrix},\quad \delta U = \begin{vmatrix} \dfrac{\partial E_3}{\partial \rho_4}\rho_4' + \dfrac{\delta m}{m}\mathbf{p}_3\mathbf{v}_s \\ \dfrac{\partial \varepsilon_4}{\partial \rho_4}\rho_4' + \mathbf{p}_4\mathbf{v}_s \end{vmatrix}.$$

The formal solution of (14) can be expressed as

$$g = R\omega\delta U, \tag{15}$$

with the operator matrix

$$R = (\omega - \mathbf{k}\mathbf{v} - iI)^{-1}. \tag{16}$$

Using (10), (13), and (15) we rewrite (8) and (9) in the form

$$(k\langle V_p|R|V_s\rangle - 1)\omega\rho_4' + \left(\rho_4 - \frac{\delta m}{m}\rho_3 - \langle V_p|V_p\rangle + \omega\langle V_p|R|V_p\rangle\right)kv_s = 0 \tag{17}$$

$$\left(\frac{c^2}{\rho_4} - \langle V_s|V_s\rangle + \omega\langle V_s|R|V_s\rangle\right)k\rho_4' + (k\langle V_s|R|V_p\rangle - 1)\omega v_s = 0. \tag{18}$$

Here $\langle\psi|$ and $|\varphi\rangle$ are bra and ket row and column vectors, for which the scalar product is defined as

$$\langle\psi|\varphi\rangle = -\int \psi_3^*\varphi_3 f_3'\, d\Gamma_3 - \int \psi_4^*\varphi_4 f_4'\, d\Gamma_4, \tag{19}$$

and the matrices as

$$V_p = \begin{vmatrix} (\delta m/m)\mathbf{p}_3\mathbf{n} \\ \mathbf{p}_4\mathbf{n} \end{vmatrix},\quad V_s = \begin{vmatrix} \partial E_3/\partial\rho_4 \\ \partial\varepsilon_4/\partial\rho_4 \end{vmatrix}; \tag{20}$$

the unit vector $\mathbf{n}$ is directed along $\mathbf{k}$.

Equating the determinant of the set (17) and (18) to zero we get the following dispersion equation for the first sound ($\omega \approx ck$):

$$\left(\frac{\omega}{ck}\right)^2 - 1 = -\frac{\delta m}{m}\frac{\rho_3}{\rho_4} - \frac{1}{\rho_4}\langle V|V\rangle + \frac{1}{\rho_4}\langle V|R\omega|V\rangle, \tag{21}$$

where the matrices are

$$V = \begin{vmatrix} \dfrac{\rho_4}{c}\dfrac{\partial E_3}{\partial\rho_4} + \dfrac{\delta m}{m}\mathbf{p}_3\mathbf{n} \\ \dfrac{\rho_4}{c}\dfrac{\partial\varepsilon_4}{\partial\rho_4} + \mathbf{p}_4\mathbf{n} \end{vmatrix} \equiv \begin{vmatrix} V_3 \\ V_4 \end{vmatrix}. \tag{22}$$

In the derivation of (21) we used only the first terms of the expansions in the small parameters $\rho_{n3}/\rho_4 \ll 1$ and $\rho_{n4}/\rho_4 \ll 1$, where

$$\rho_{ni} = -\frac{1}{3}\int p_i^2 f_i'\, d\Gamma_i \tag{23}$$

are the densities of the normal components.

The entire difficulty of the solution has been reduced in practice to calculation of the expression

$$K=-\langle V|V\rangle+\langle V|R\omega|V\rangle, \tag{24}$$

that enters in (21) an defines the renormalization of the velocity and absorption coefficient of the sound:

$$\frac{\Delta c}{c}=\mathrm{Re}\,\frac{\omega-c_0k}{ck}=\frac{c-c_0}{c}-\frac{\delta m}{2m}\frac{\rho_3}{\rho_4}+\frac{1}{2\rho_4}\mathrm{Re}\,K, \tag{25}$$

$$\alpha=-\frac{\omega}{2\rho_4 c}\,\mathrm{Im}\,K. \tag{26}$$

We define the vector $|\chi\rangle$ by the relation

$$|\chi\rangle=R\omega|V\rangle. \tag{27}$$

Then

$$K=-\langle V_3|V_3\rangle-\langle V_4|V_4\rangle+\langle V_3|\chi_3\rangle+\langle V_4|\chi_4\rangle. \tag{28}$$

In accordance with (27), the vectors $|\chi_3\rangle$ and $|\chi_4\rangle$ can be written in the form

$$|\chi_3\rangle=(\omega-\mathbf{k}\mathbf{v}_3-iI_{33}-iI_{34}+I_{34}R_4I_{43})^{-1}(\omega|V_3\rangle+iI_{34}R_4\omega|V_4\rangle), \tag{29}$$

$$|\chi_4\rangle=(\omega-\mathbf{k}\mathbf{v}_4-iI_{44}-iI_{43}+I_{43}R_3I_{34})^{-1}(\omega|V_4\rangle+iI_{43}R_3\omega|V_3\rangle), \tag{30}$$

where

$$R_3=(\omega-\mathbf{k}\mathbf{v}_3-iI_{33}-iI_{34})^{-1},\qquad R_4=(\omega-\mathbf{k}\mathbf{v}_4-iI_{44}-iI_{43})^{-1}. \tag{31}$$

We confine ourselves to temperatures lower than 0.6 K, where the roton influence can be neglected. The operators I_{34} and I_{34} are known[1-4] then to describe Rayleigh scattering. In this case, the actions of the indicated linear operators on the ket vector are given by

$$I_{34}|\varphi(\mathbf{p}_i)\rangle=\left|\int w_{34}f_4'[\varphi(\mathbf{p}_i')-\varphi(\mathbf{p}_i)]\,d\Gamma_3'\,d\Gamma_4\,d\Gamma_4'\right\rangle, \tag{32}$$

$$I_{43}|\varphi(\mathbf{p}_i)\rangle=\left|\int w_{34}f_3'[\varphi(\mathbf{p}_i')-\varphi(\mathbf{p}_i)]\,d\Gamma_3\,d\Gamma_3'\,d\Gamma_4'\right\rangle, \tag{33}$$

where $w_{34}(\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_3'\mathbf{p}_4')$ is the transition pobability density.

Using (29)–(33), we rewrite (38), accurate to terms linear in the small parameter $\rho_{n4}/\rho_{n3}\ll 1$, in the form

$$K=K_{33}+K_{44}+K_{43}+K_{34}. \tag{34}$$

The impurity contribution is

$$K_{33}=-\langle V_3|V_3\rangle+\langle V_3|R_{33}\omega|V_3\rangle, \tag{35}$$

the phonon contribution

$$K_{44}=-\langle V_4|V_4\rangle+\langle V_4|R_4\omega|V_4\rangle, \tag{36}$$

and the phonon + impurity contribution

$$K_{43}=\langle V_4|R_4iI_{43}R_{33}\omega|V_3\rangle, \tag{37}$$

$$K_{34}=\langle V_3|R_{33}iI_{34}R_4\omega|V_4\rangle+\langle V_3|R_{33}(iI_{34}-I_{34}R_4I_{43})R_{33}\omega|V_3\rangle, \tag{38}$$

$$R_{33}=(\omega-\mathbf{k}\mathbf{v}_3-iI_{33})^{-1}. \tag{39}$$

We proceed now to calculate the impurity contribution. The eigenfunctions of the Hermitian operator I_{33} are known to form a complete orthonormal basis. The eigenfunctions with zero eigenvalues form the subspace $|c_{3j}$ of the collision invariants: the number of particles, the energy, the momentum ($j=1, 2, 3, 4, 5$):

$$|c_{31}\rangle=\frac{|1\rangle}{\langle 1|1\rangle^{1/2}};\quad |c_{32}\rangle=\frac{|\Delta\varepsilon_3\rangle}{\langle\Delta\varepsilon_3|\Delta\varepsilon_3\rangle^{1/2}},\quad |c_{33}\rangle=\frac{|p_{3z}\rangle}{\langle p_{3z}|p_{3z}\rangle^{1/2}}, \tag{40}$$

where $\Delta\varepsilon_3=\varepsilon_3-\langle\varepsilon_3|1\rangle/\langle 1|1\rangle$. The vectors $|c_{34}\rangle$ and $|c_{35}\rangle$ drop out of the calculations, since the z axis is chosen along k.

We introduce the operators of projection on the subspace of the collision invariants and the subspace orthogonal to it:

$$\mathscr{P}_{3c}=\sum_\tau |c_{3j}\rangle\langle c_{3j}|,\quad \mathscr{P}_{3n}=1-\mathscr{P}_{3c}. \tag{41}$$

Since $|v_3\rangle$ belongs to the subspace of the collision invariants, we can replace R_{33} in (35) by $\mathscr{P}_{3c}R_{33}\mathscr{P}_{3c}$. Using the definition (39) we have

$$(\omega-\mathbf{k}\mathbf{v}_3-iI_{33})R_{33}=1. \tag{42}$$

We multiply (42) by $\mathscr{P}_{3c}$ from the left and from the right, and then from the left by $\mathscr{P}_{3n}$ and from the right by $\mathscr{P}_{3c}$. From the two equations obtained in this manner we get

$$\mathscr{P}_{3c}R_{33}\mathscr{P}_{3c}=\mathscr{P}_{3c}(\omega-\mathscr{P}_{3c}\Omega\mathscr{P}_{3c})^{-1}, \tag{43}$$

where

$$\Omega=\mathbf{k}\mathbf{v}_3+\mathbf{k}\mathbf{v}_3\mathscr{P}_{3n}(\omega-\mathscr{P}_{3n}\mathbf{k}\mathbf{v}_3\mathscr{P}_{3n}-i\mathscr{P}_{3n}I_{33}\mathscr{P}_{3n})^{-1}\mathscr{P}_{3n}\mathbf{k}\mathbf{v}_3. \tag{44}$$

That term of (44) which contains the small parameter

$$(kv_3/\omega)^2=(v_3/c)^2=\beta, \tag{45}$$

can be omitted.

To calculate the second term of (35) we must find the matrix elements $\langle C_{3j}|\theta|C_{3j'}\rangle$ and the determinant $(\omega-\mathscr{P}_{3c}\Omega\mathscr{P}_{3c})$ of the operator. To calculate the matrix elements we use the τ approximation, according to which $\mathscr{P}_{3n}I_{33}\mathscr{P}_{3n}$ can be replaced by $-\mathscr{P}_{3n}/\tau_{33}$. We emphasize specially that the τ approximation can be used only after separating the subspace of the collision invariants, since $I_{33}|c_{3j}\rangle$ is equal to zero and not to $-|c_{3j}\rangle/\tau_{33}$.

The zeros of the determinant $(\omega-\mathscr{P}_{3c}\Omega\mathscr{P}_{3c})$ give the collective modes in the impuriton gas at $\omega\tau_{33}\ll 1$ the zero determinant leads to the following dispersion law:

$$\left(\frac{\omega}{k}\right)^2=u_3^2\left[1-\frac{4}{5}i\omega\tau_{33}-\frac{m}{2w_3}\frac{\langle(\varepsilon_3-w_3)v_3|(\varepsilon_3-w_3)v_3\rangle}{\langle\varepsilon_3|\varepsilon_3\rangle}i\omega\tau_{33}\right], \tag{46}$$

where

$$u_3^2=\frac{2}{3m}\frac{\langle\varepsilon_3|\varepsilon_3\rangle}{\langle\varepsilon_3|1\rangle}=\frac{5}{3}\frac{P_f}{mn_3},\quad w_3=\frac{\langle\varepsilon_3|\varepsilon_3\rangle}{\langle\varepsilon_3|1\rangle}. \tag{47}$$

The result has a simple physical meaning: sound propagates in the impurity gas ("impurity second sound") with velocity u_3 equal to the speed of sound in an ideal gas of particles of mass m if the properties of the solution are determined only by the impuritons. The second and third terms in (46) describe the damping of the impurity second sound due respectively to viscosity and heat conduction. The result (46) differs from that given in Ref. 2 in that it contains a third term equal to $-(2/5)i\omega\tau_{33}$ in the nondegenerate case.

Knowing the matrix elements Ω and the determinant

$(\omega - \mathscr{P}_{3c}\Omega\mathscr{P}_{3c})$ we can easily calculate (35). From the result and from (5), (25), and (26) we find the impurity part of the velocity renormalization and the sound absorption coefficient:

$$\left(\frac{\Delta c}{c}\right)_3 = \frac{x}{2}\left[\rho_4\frac{\partial a}{\partial\rho_4}+\lambda(\lambda-1)\frac{m}{m_4}\right]+\frac{P_f}{2\rho_4 c^2}\left[3(w-\lambda)^2+w\lambda + (3w-2)u-\frac{3}{2}w_2-\frac{{}^4/_3\lambda^2}{1+\omega^2\tau_{33}^2}\right], \quad (48)$$

$$\alpha_3 = \frac{2\lambda^2\omega^2}{3\rho_4 c^3}\frac{P_f\tau_{33}}{1+\omega^2\tau_{33}^2}, \quad (49)$$

where

$$\lambda=\left(a+\frac{\delta m}{m_4}\right)\frac{m_4}{m}, \quad w_2=\frac{\rho_4^2}{m}\frac{\partial^2 m}{\partial\rho_4^2}.$$

The result (48) was first obtained in Ref. 12 by perturbation theory with allowance for thermal expansion, while (49) was obtained in Ref. 4 by the method of variational derivatives. In the present paper we determined (48) and (49) by a single procedure that permitted the problem to be solved also for a phonon + impurity system.

PHONON GAS IN THE PRESENCE OF IMPURITIES

We proceed to calculate the phonon impurity contribution to the renormalization of the sound velocity and of the absorption. Starting from the definitions (22) and (39) and recognizing that $I_{33}|c_{31}\rangle = 0$, the ket vector in (37) can be rewritten as

$$\omega R_{33}|V_3\rangle = -|\delta m c\rangle + \omega R_{33}\left|\lambda\mathbf{p}_3\mathbf{n} - w\frac{\varepsilon_3}{c}\right\rangle. \quad (50)$$

Substitution of the first term of (50) in (37) yields no contribution, since $I_{43}|\text{const}\rangle = 0$. The second term of (50) can be written, accurate to terms linear in β, as

$$\begin{aligned}\omega R_{33}&\left|\lambda\mathbf{p}_3\mathbf{n}-w\frac{\varepsilon_3}{c}\right\rangle \\ &= \left(\lambda - w\frac{w_3}{mc^2}\right)|\mathbf{p}_3\mathbf{n}\rangle - \left(w-\frac{2}{3}\lambda\right)\left|\frac{\varepsilon_3}{c}\right\rangle \\ &\quad - \frac{i\lambda k\tau_{33}}{1-i\omega\tau_{33}}\left|(\mathbf{p}_3\mathbf{n})(\mathbf{v}_3\mathbf{n})-\frac{1}{3}p_3 v_3\right\rangle \\ &\quad + \frac{i}{c}\frac{wk\tau_{33}}{1-i\omega\tau_{33}}|(\varepsilon_3 - w_3)(\mathbf{v}_3\mathbf{n})\rangle. \end{aligned} \quad (51)$$

The vectors in the last two terms of (51) are orthogonal to the subspace of the collision invariants, so that the τ approximation can be used. The parameter $w_3 = \langle\varepsilon_3|\varepsilon_3\rangle/\langle\varepsilon_3|1\rangle$ is determined from the orthogonality condition $\langle(\varepsilon_3 - w_3)(\mathbf{v}_3\cdot\mathbf{n})|(\mathbf{p}_3\cdot\mathbf{n})\rangle = 0$.

It is necessary next, in accordance with (37), to apply the operator I_{43} to (51). We use here conservation laws according to which

$$I_{43}|c_{3j}\rangle = -I_{43}|c_{4j}\rangle\left[\frac{\langle c_{4j}|c_{4j}\rangle}{\langle c_{3j}|c_{3j}\rangle}\right]^{1/2}, \quad (52)$$

where $|c_{4j}\rangle$ are vectors of the subspace of the phonon-phonon invariants (energy, phonon momentum). As a result we change over from impurity to phonon variables. Taking (52) into account, it is easy to verify that the contribution of the third and fourth terms of (51) to (37) is small compared with the contribution of the first in terms of the parameter β. The contribution of the second term, which is also proportional to β, turns out to be significant in the region $\omega\tau_{phi} \ll \beta$ (τ_{phi} is the characteristic time to establish equilibrium between the phonon and the impurity subsystems). The third and fourth terms contain in the indicated frequency region and additional small factor $\omega\tau_{33} \ll 1$. Equation (37) takes as a result the form

$$K_{43} = -\langle V_4|R_4 iI_{43}|\lambda\mathbf{p}_4\mathbf{n} - (w-{}^2/_3\lambda)p_4\rangle. \quad (53)$$

Similar calculations yield

$$K_{34} = \frac{1}{\omega}\left\langle \lambda\mathbf{p}_4\mathbf{n} - \left(w-\frac{2}{3}\lambda\right)p_4\left|iI_{43} - I_{43}R_4 I_{43}\right|\lambda\mathbf{p}_4\mathbf{n} - \left(w-\frac{2}{3}\lambda\right)p_4\right\rangle. \quad (54)$$

Using the identity

$$R_4 iI_{43}|c_{4j}\rangle = -|c_{4j}\rangle + R_4(\omega - \mathbf{k}\mathbf{v}_4)|c_{4j}\rangle,$$

we obtain the total phonon contribution $k_{ph} = K_{44} + K_{34} + K_{43}$. Substitution of this contribution in (25) and (26) yields the phonon part of the renormalization of the sound velocity, and also the absorption coefficient:

$$\left(\frac{\Delta c}{c}\right)_{ph} = \frac{3}{2}\frac{\rho_{n4}}{\rho_4}\left[\operatorname{Re}M + \frac{u_2}{4} - \left(u+w-\frac{2}{3}\lambda\right)^2 - \frac{1}{3}(1-\lambda)^2 - \frac{2}{3}\lambda\left(w-\frac{2}{3}\lambda\right)\right], \quad (55)$$

$$\alpha_{ph} = -\frac{3}{2}\frac{\rho_{n4}}{\rho_4}\frac{\omega}{c}\operatorname{Im}M, \quad (56)$$

where

$$\begin{aligned} M &= \frac{1}{3\rho_{n4}}\langle\psi(y)p_4|R_4\omega|p_4\psi(y)\rangle, \quad \psi(y) \\ &= \left(u+w-\frac{\lambda}{3}\right)P_0(y) + \left(1-w-\frac{\lambda}{3}\right)P_1(y) + \frac{2}{3}\lambda P_2(y), \end{aligned} \quad (57)$$

$$u_2 = \frac{\rho_4^2}{c}\frac{\partial^2 c}{\partial\rho_4^2}.$$

$F_l(y)$ are Legendre polynomials, $y = (\mathbf{p}_4\cdot\mathbf{n})/p_4$.

To calculate the matrix element (57) we must write out explicitly the operators I_{44} and I_{43}, which act differently in the angle and energy subspaces. It is convenient to introduce in this connection vectors that are defined in the spaces of the angles $|a\rangle_y$ and of the energies $|b\rangle_\varepsilon$, and the corresponding scalar products

$$\begin{aligned} {}_y\langle a_1|a\rangle_y &= \frac{1}{2}\int_{-1}^{1} a_1^*(y)a(y)\,dy, \quad {}_\varepsilon\langle b_1|b\rangle_\varepsilon \\ &= -\int_0^\infty b_1^*(p_4)b(p_4)f_4'\frac{p_4^2\,dp_4}{2\pi^2}. \end{aligned} \quad (58)$$

We represent the operator I_{44}, in accordance with Refs. 13–15, in the form

$$I_{44} = I_{\parallel} + I_{\perp},$$

where $I_{\parallel}$ describes fast relaxation due to the three-particle

processes and having a characteristic time $\tau_\parallel$ in a given direction, while $I_\perp$ describes the slow establishment of complete equilibrium in the phonon gas, with a characteristic time $\tau_\perp$. It follows from the subsequent calculation that at the temperatures and concentrations considered by us establishment of complete equilibrium in the phonon gas is governed by the phonon-impurity collisions ($\tau_{phi} \ll \tau_\perp$), so that $\tau_\perp$ can be omitted.

The operator $I_\parallel$ conserves energy in the specified direction, i.e.,

$$I_\parallel |p_4\rangle_\varepsilon = 0. \tag{59}$$

Starting from the last condition, the correct τ approximation can be written in the form

$$I_\parallel = -\frac{1}{\tau_\parallel}(1-\mathscr{P}_\parallel), \tag{60}$$

where

$$\mathscr{P}_\parallel = |p_4\rangle_{\varepsilon\varepsilon}\langle p_4|/_\varepsilon \langle p_4|p_4\rangle_\varepsilon \tag{61}$$

is a projector on the subspace of the eigenvectors of the operator $I_\parallel$ with zero eigenvalues. In accordance with the assumed definition of the scalar product (19), $\tau_\parallel$ should be regarded as independent of the momentum, since the following two conditions must be simultaneously satisfied:

$$\langle p_4|(1-\mathscr{P}_\parallel)|\varphi_4\rangle = 0, \quad \left\langle p_4 \left| \frac{1}{\tau_\parallel}(1-\mathscr{P}_\parallel) \right| \varphi_4 \right\rangle = 0.$$

According to (57), we must find the matrix element of the operator

$$\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel = \mathscr{P}_\parallel (\omega - \mathbf{k}\mathbf{v}_4 + i/\tau_\parallel - iI_{43} - i\mathscr{P}_\parallel/\tau_\parallel)^{-1}\mathscr{P}_\parallel. \tag{62}$$

By iteration we obtain

$$\mathscr{P}_\parallel R_4 \mathscr{P}_\parallel = (1-\mathscr{P}_\parallel R_\parallel \mathscr{P}_\parallel i/\tau_\parallel)^{-1}\mathscr{P}_\parallel R_\parallel \mathscr{P}_\parallel, \tag{63}$$

where

$$R_\parallel = (\omega - \mathbf{k}\mathbf{v}_4 + i/\tau_\parallel - iI_{43})^{-1}. \tag{64}$$

Using the result of Ref. 3, we express the operatore I_{43}, accurate to terms linear in β, in the form

$$I_{43} = -\sum_{l=1}^{\infty} \frac{1}{t_l}\mathscr{P}_l + \beta I', \tag{65}$$

where

$$\mathscr{P}_l = |P_l\rangle_{yy}\langle P_l|/_y\langle P_l|P_l\rangle_y, \tag{66}$$

and the times t_l are, according to Ref. 8,

$$t_1 = \frac{7{,}7\rho_4}{m_4 c x p_4^4}, \quad t_2 = \frac{11{,}3\rho_4}{m_4 c x p_4^4}, \quad t_{l\geqslant 3} = t_{phi} = \frac{10{,}7\rho_4}{m_4 c x p_4^4}. \tag{67}$$

The second term in (65) describes slow processes, particularly establishment of energy equilibrium between the phonons and the impuritons of frequency defined by $t_0^{-1} = \beta I'$.

It is necessary next to susbtitute (63) in (57) and integrate with respect to the modulus of the phonon momentum. In the upshot we get for the sought matrix element

$$M = {}_y\langle \psi(y)|\omega\langle R_4\rangle|\psi(y)\rangle_y, \tag{68}$$

where

$$\langle R_4\rangle = \left(1 - \langle R_\parallel\rangle \frac{i}{\tau_\parallel}\right)^{-1}\langle R_\parallel\rangle, \tag{69}$$

$$\langle A\rangle = \frac{1}{3\rho_{n4}}{}_\varepsilon\langle p_4|A|p_4\rangle_\varepsilon. \tag{70}$$

Using the identity

$$1 - \langle R_\parallel\rangle \frac{i}{\tau_\parallel} = \omega\langle R_\parallel\rangle - \langle R_\parallel \mathbf{k}\mathbf{v}_4\rangle - i\langle R_\parallel I_{43}\rangle, \tag{71}$$

we can rewrite (69) in the form

$$\langle R_4\rangle = (\omega - \langle \mathbf{k}\mathbf{v}_4\rangle_R - i\langle I_{43}\rangle_R)^{-1}, \tag{72}$$

$$\langle A\rangle_R = \langle R_\parallel\rangle^{-1}\langle R_\parallel A\rangle. \tag{73}$$

To continue the calculations we must be more specific about the considered frequency range. Thus, in the hydrodynamic limit we get

$$\langle R_4\rangle = i\frac{\langle(\tau_\parallel^{-1} - I_{43})^{-1}\rangle}{\langle I_{43}(\tau_\parallel^{-1} - I_{43})^{-1}\rangle}. \tag{74}$$

According to (65), the operator I_{43} is diagonal in a Legendre-polynomial basis, so that calculation of the matrix element (68) with (74) entails no difficulty. Substituting the obtained M in (55) and (56) we obtain the renormalization of the velocity and the sound absorption in the hydrodynamic limit:

$$\left(\frac{\Delta c}{c}\right)_{ph} = \frac{3\rho_{n4}}{2\rho_4}\left[\frac{u_2}{4} - \left(u + w - \frac{2}{3}\lambda\right)^2 - \frac{1}{3}(1-\lambda)^2 - \frac{2}{3}\lambda\left(w - \frac{2}{3}\lambda\right)\right], \tag{75}$$

$$\alpha_{ph} = \frac{\omega^2\rho_{n4}}{2\rho_4 c}\left[3\left(u + w - \frac{\lambda}{3}\right)^2\tau_0 + \left(1 - w - \frac{\lambda}{3}\right)^2\tau_1 + \frac{4}{15}\lambda^2\tau_2\right], \tag{76}$$

where

$$\tau_l = \langle t_l(t_l + \tau_\parallel)^{-1}\rangle / \langle (t_l + \tau_\parallel)^{-1}\rangle, \tag{77}$$

the times τ_2, τ_0, and τ_1 determine respectively the first and second viscosities and by the thermal conductivity. The time $\tau_{l>3} \equiv \tau_{phi}$, as follows from (67), is close to τ_2 and τ_1. We note that Eqs. (75) and (76), just as the hydrodynamic result in Ref. 4, are valid at $1 \gg \omega\tau_0 \gg C_4/C_3$, where C_i is the heat capacity of the quasiparticles. The ;times τ_1 and τ_2 are not subject to the indicated constraint. Thus, the first-viscosity and thermal conductivity coefficients are respectively

$$\eta_{ph} = {}^1/_5\rho_{n4}c^2\tau_2, \qquad \varkappa_{ph} = {}^1/_3 C_4 c^2 \tau_1. \tag{78}$$

All the phonon times were omitted in Refs. 2–4; this corresponds to the limit $\tau_\parallel \to \infty$. In this case we have

$$\tau_l^\infty = \langle t_l\rangle. \tag{79}$$

This result must be modified, for when (67) is substituted in (79) the integrals diverge at the lower limit. The cause of this divergence is that the free-path time of the long-wave phonons increases rapidly with decreasing energy ($t_l \propto \varepsilon_4^{-4}$) and their contribution to the kinetic coefficients becomes predominant. The divegence is eliminated either by taking into account the fact that the system is finite, or by introducing some other mechanism that cuts off the free path of the long-wave phonon. The mechanism used in Refs. 2–4 was phonon absorption because of the viscosity of the impurity gas. It is

necessary then to replace t_l^{-1} in the corresponding equations by $t_l^{-1}+t_a^{-1}$, where t_a is the characteristic time of the indicated process. The second and third terms in (76) coincide then with the result of Ref. 4. As for the first term, the time obtained in Ref. 4 was $\tau_0^\infty=\langle t_a\rangle$, as against our $\langle(t_a^{-1}+t_0^{-1})^{-1}\rangle$. This difference is due only to omission in Ref. 4 of terms containing small β, yet $\langle t_0^{-1}\rangle\sim\beta/\tau_{phi}$. It follows from the estimates, however, that t_a is so long, that t_0 and τ_0^∞ must be taken into account in a wide range of temperatures and concentrations. The second viscosity is thus determined by the slow establishment of energy equilibrium between the phonons and the impuritons.

Let us consider now the opposite limiting case $\tau_\parallel\to 0$. Then, according to (77), the reciprocal times average out:

$$\tau_l^0=\langle t_l^{-1}\rangle^{-1}. \tag{80}$$

In contrast to (79) the integrals converge and the times obtained are approximately shorter by one order, for according to (67) a substantial contribution to the integrals of (80) is made by phonons with $\varepsilon_4\approx 2\pi T$. The situation here is similar to that in a phonon-roton system, for which it was assumed in the expression for the distribution function in Ref. 11 that $\tau_\parallel=0$. We emphasize, that this analogy was first noted already in Ref. 1. A detailed physical interpretation of this phenomenon in a phonon-roton system was given in Refs. 16 and 17. As applied to a phonon-impurity system, the gist of the interpretation is the following. First, Rayleigh scattering ($t_l\propto\varepsilon_4^{-4}$) establishes an equilibrium between the energetic phonons and the impuritons. We have then $\tau_l^0>\tau_{33}$ in the entire considered region of temperatures and concentrations. Next, all the remaining phonons enter instantaneously ($\tau_\parallel\to 0$) into equilibrium with the energetic ones. The time t_a can be left out in this case, since it is long at all concentrations and temperatures. If, however, $\tau_\parallel\to\infty$, the establishment of equilibrium in the phonon system is determined by Rayleigh scattering and by long-wave-phonon absorption on account of the viscosity of the impurity gas. The estimates that follow indicate that the above limiting cases are applicable in a rather wide range of temperatures and concentrations. It is necessary then to start from the general relation (77).

We shall assume for estimates that[2,8,15]

$$\tau_\parallel\approx 2{,}6\cdot 10^{-10}\,T^{-5} \tag{81}$$

(hereafter the temperature is in K and the times in sec), and

$$t_a\approx 9{,}6\tau_{33}x^{-1}T^{-1}. \tag{82}$$

Above the degeneracy temperature we have for the time

$$\tau_{33}\approx 10^{-11}x^{-1}T^{-1/2}. \tag{83}$$

Averaging in accordance with (80) yields for the characteristic relaxation time

$$\tau_{phi}^0=7\cdot 10^{-12}x^{-1}T^{-4}. \tag{84}$$

The time τ_{phi}^∞ cited in Refs. 2–4 is longer by more than an order of magnitude than the time τ_{phi}^0 and can be used if $\tau_\parallel>t_a$. As expected from (81)–(83), at saturated-vapor pressure this inequality is satisfied at very low temperatures and high concentrations, when the phonon contributions can in fact be neglected. This is the reason why an attempt to reduce the experimental data on the phonon thermal conductivity[5] and on second-sound absorption[6] by assuming times τ_{phi}^∞ resulted in a deviation from the theory by more than an order of magnitude. The times that follow from these experiments are of the order of τ_{phi}^∞.

We note that according to (77) the dependence of the phonon-impurity time on the concentration and temperature is quite complicated. Actually the relation $\tau_l^{-1}\propto x$ is realized at $\tau_\parallel\to 0$ and is close to linear if $\tau_\parallel\to\infty$. The coefficients in the different limiting cases differ then by more than one order.

The situation changes substantially if the spectrum no longer decays. Three-particle processes are then forbidden, the inequality $t_a\ll\tau_\parallel$ can be satisfied, so that the data of Ref. 5 on phonon thermal conductivity at high pressures agrees well with the theory of Refs. 2–4.

As $\tau_\parallel\to$ the procedure proposed here can solve the problem at all $\omega\tau_{phi}$. In this case, as noted above, it is necessary to make in (65) the substitution $t_l^{-1}\to t_l^{-1}+t_a^{-1}$, let $\tau_\parallel$ tend to infinity in (69), and obtain a relation such as (63) for $R_\parallel$. The result obtained in this case for the sound absorption coincides then with the result of Ref. 4. It is valid when the temperature are low enough and the concentrations high enough to be able to neglect the phonon contribution compared with that of the impurties.

We consider now the kinetic regime. To simplify the procedure and the final results, we put $t_1=t_2=t_{phi}$ and confine ourselves to the case $\omega\tau_{phi}\gg\beta$. The last inequalility, which is valid in a wide range of frequencies and concentrations, allows us to discard in the calculations the terms that contain the small parameter β. The first assumption is in fact inessential, for according to (67) the times t_l differ little.

The results can be obtained in terms of elementary functions at arbitrary $\omega\tau_{phi}$ if $\tau_\parallel\ll\tau_{phi}$ and $\langle\delta\rangle\omega\tau_\parallel\gg 1$, where $\delta=(v_4-c)/c$. In this case we have for (72)

$$\langle R_k\rangle=(\omega-\langle\mathbf{kv}_k\rangle-i\langle I_{k3}\rangle)^{-1}. \tag{85}$$

Starting from the definition (65) we obtain in the equal-time approximation

$$\langle R_k\rangle=\left(\omega-\langle\mathbf{kv}_k\rangle+\frac{i}{\tau_{phi}}-\frac{i}{\tau_{phi}}\mathcal{P}_0\right)^{-1}. \tag{86}$$

By iteration we reduce (86) to the form

$$\langle R_k\rangle=\mathcal{R}\left[1+\left(1-\frac{iQ}{\omega\tau_{phi}}\right)^{-1}\frac{i}{\tau_{phi}}\mathcal{P}_0\mathcal{R}\right], \tag{87}$$

where

$$Q=\frac{1}{2}\ln\frac{q+1}{q-1},\quad q=1-\langle\delta\rangle+\frac{i}{\omega\tau_{phi}},\quad \omega\mathcal{R}=(q-y)^{-1},$$

(87) must next be substituted in (68) and the result integrated over the angles. Substituting the calculated M in (55) and (56) we obtain the velocity renormalization and the speed of sound:

$$\left(\frac{\Delta c}{c}\right)_{ph}=\frac{3}{2}\frac{\rho_{nk}}{\rho_k}\left[\frac{u_2}{4}-\frac{1}{3}+\mathrm{Re}\left(u+1+\frac{i\lambda}{\omega\tau_{phi}}\right)^2\times\left(\frac{Q}{1-iQ/\omega\tau_{phi}}-1\right)\right], \tag{88}$$

$$\alpha_{ph}=\frac{3}{2}\frac{\rho_{n4}}{\rho_4}\frac{\omega}{c}\left[\frac{1}{3}\frac{\lambda^2}{\omega\tau_{phi}}-\mathrm{Im}\left(u+1+\frac{i\lambda}{\omega\tau_{phi}}\right)^2\times\left(\frac{Q}{1-iQ/\omega\tau_{phi}}-1\right)\right]. \quad (89)$$

Equations (88) and (89) become much simpler in the high-frequency limit $\omega\tau_{phi}\gg 1$, where

$$\left(\frac{\Delta c}{c}\right)_{ph}=\frac{3}{2}\frac{\rho_{n4}}{\rho_4}\left\{\frac{u_2}{4}-\frac{1}{3}+\frac{(u+1)^2}{2}\times\left[\ln\frac{2\omega\tau_{phi}}{(1+\langle\delta\rangle^2\omega^2\tau_{phi}^2)^{1/2}}-2\right]\right\}, \quad (90)$$

$$\alpha_{ph}=\frac{3}{4}\frac{\rho_{n4}}{\rho_4}\frac{\omega}{c}(u+1)^2[\mathrm{arctg}\,2\omega\tau_{phi}+\mathrm{arctg}\langle\delta\rangle\omega\tau_{phi}]. \quad (91)$$

The result (92) differs from the analogous result for pure HeII (Ref. 18) in that our procedure yields explicit expressions for the parameters τ_{phi} and $\langle\delta\rangle$.

The problem is simply solved in quadratures at arbitrary $\omega\tau_{\parallel}$ if $\omega\tau_{phi}\gg 1$. In this region we can replace I_{43} in (69) by $-1/t_{phi}$. As a result we get for the matrix element (68)

$$M=\left\langle \psi(y)\left|\left(1-\langle\mathscr{R}_4\rangle\frac{i}{\tau_{\parallel}}\right)^{-1}\omega\langle\mathscr{R}_4\rangle\right|\psi(y)\right\rangle_y \quad (92)$$

$$\mathscr{R}_4=\left(\omega-\mathbf{k}\mathbf{v}_4+\frac{i}{t_{phi}}+\frac{i}{\tau_{\parallel}}\right)^{-1}.$$

Substituting (92) in (55) and (56), we obtain the renormalization and the sound absorption coefficient at arbitary $\omega\tau_{\parallel}$ if $\omega\tau_{phi}\gg 1$.

It follows from (88)–(92) that absorption and renormalization of the speed of sound at $\langle\delta\rangle\omega\tau_{phi}>1$ depends substantially on $\langle\delta\rangle$. In this respect it would be of interest to measure these quantities for solutions with different concentrations. Such experiments would yield information on the dependence of $\langle\delta\rangle$ on x.[19]

Figure 1 shows for $T<0.6$ K the data of Ref. 9 and plots of the equations obtained above. The roton contribution must be taken into account at $T>0.6$ K. The dispersion law was chosen in the form

$$\varepsilon_4=cp_4(1-\gamma_1p_4^2-\gamma_2p_4^4)$$

with parameters $\gamma_1=-1.3\cdot10^{38}$ sec^2/g$^2\cdot$cm^2 and $\gamma_2=9.77\cdot10^{76}$ sec^4/g$^4\cdot$cm^4. We note that calculations with the times from Refs. 2–4 yield values close to those of pure He^4 and lead to additivity of the phonon and impurity contributions, thus contradicting the results of Ref. 9.

CONCLUSION

Our calculations show that the kinetics of a phonon + impurity system are determined essentially by small-angle phonon-phonon scattering with a characteristic time $\tau_{\parallel}$. In the upshot the times τ_l (77) that determine the dissipative coefficients have complicated dependences on the concentration and on the temperature. In the limiting cases $\tau_{\parallel}\to 0$ and $\tau_{\parallel}\to\infty$ the times τ_l differ by more than an order of magnitude, as was observed in experiment.

The procedure employed in this paper yields the renormalization and sound absorption in a wide range of frequencies, temperatures, and concentrations. The speed-renormalization and the absorption of sound are determined by the impurities [Eqs. (48) and (39), by the phonons, and by the phonon-impurity interaction. The phonon-impurity part in (55) and (56) is combined with the phonon part. For low-concentration solutions, the contribution of the impurity part is small. The differences between the renormalization of the speed of sound and the absorption from their values in pure helium is therefore completely determined by the phonon-impurity part, as was observed in the experiments of Ref. 9.

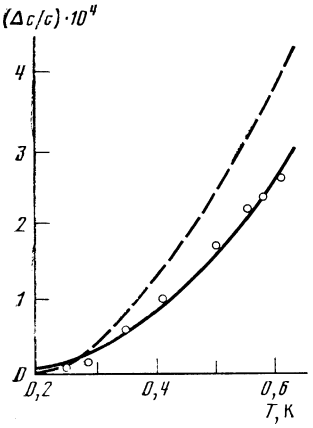

FIG. 1 Temperature dependence of the renormalization of the speed of sound at a frequency $9.93\cdot10^6$ Hz. Dashed line—data of Ref. 9 for pure He II, points—solution with $x=5\cdot10^{-4}$. Solid curve—calculation of the present paper.

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