Universal non-equilibrium photon distributions and their formation dynamics

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We consider the kinetic equation for photons interacting with free electrons. We find stationary photon energy contributions taking into account photon sources which are localized in energy space as well as power-law photon sources. We show that in the region between the source and the sink a universal distribution is formed which depends on the integral characteristics of the source.

It is well known that real physical systems of particles and quasi-particles are often in states which are far from equilibrium. The occurrence of disequilibrium is usually connected with the presence of sources and sinks which produce a stationary state of the system. The most striking examples of such possibilities are provided by the theory of weak and strong turbulence. The presence of sources and sinks may lead to the formation of a non-equilibrium situation which is so strong that the approximation of local equilibrium turns out to be insufficient. For instance, in the theory of weak turbulence[1] the presence of a source and a sink leads in the region between them to the appearance of universal turbulence spectra which depend only on the integral characteristics of the source and the sink.

The existence of particle energy distributions similar to turbulence spectra was pointed out for the first time in Refs. 2 and 3. However, because in the situations considered the kinetic equations were integral equations, many problems of principal interest remained unexplained. For instance, first and foremost the universality of the power-law spectra which were obtained in those papers was not proved. It is thus of interest to study examples in which one can explicitly trace the effect of the source on the non-equilibrium distributions and prove their universality. In the present paper we achieve this program for the Kompaneets equation which describes the interaction of radiation with free electrons.

In the present paper we find the stationary photon distribution function in the whole energy range (under non-equilibrium conditions) using the WKB method in the small parameter $T_e/m_e c^2$ (T_e is the electron temperature and m_e the electron mass). We succeed in rather clearly elucidating the effect of the structure of the source on the nature of the photon distribution function. We analyze the possible power-law asymptotic behavior of the distribution function and the regions where it exists depending on the magnitude of the power of the source. The solution obtained in the WKB approximation enables us to give the explicit form of the distribution function for a number of sources (delta-function and power-law sources). The analysis of these distributions shows that between the source and the sink is formed a universal power-law asymptotic form of the distribution function, which depends solely on the integral characteristics of the structure source (its power). At the same time in the range beyond the source (for energies larger than the characteristic energies of the source) a distribution is formed which depends in an essential way on its form. In particular, in the case of a power-law source a distribution is formed with a power-law exponent which is determined by the source. In the case of several concentrated sources we also indicate ranges where there exist universal and non-universal sections of photon distributions. The presence of the above indicated non-self-similar, non-universal sections of the photon spectrum indicate that non-equilibrium particle distributions may serve as powerful diagnostic agents for non-equilibrium sources.

We consider the kinetic equation for the distribution function of photons interacting with free electrons which have an equilibrium energy distribution with temperature T_e:[4]

$$\frac{\partial f}{\partial t}=\frac{1}{x^2}\frac{\partial}{\partial x}x^4\left[T\frac{\partial f}{\partial x}+f+f^2\right]+\frac{Q(x)}{x^2}. \tag{1}$$

This equation is already in terms of the dimensionless variables $t=\sigma n_e c\tau$ where τ is the time, $x=\hbar\omega/m_e c^2$, $T=T_e/m_e c^2$. We have taken into account in this equation the existence of photon sources with intensity $Q(x)$.

From the time when Kompaneets's paper first appeared, many papers and reviews[5–9] have been devoted to the study of the interaction of radiation with free electrons and also to radiation relaxation processes. The interest in those processes is connected with the fact that such an interaction plays an important role in many astrophysical problems involving compact powerful radiation sources. In most papers processes of relaxation to an equilibrium state were studied. However, the operation of the sources may lead to a change in the stationary distribution function. Undoubtedly a stationary solution is possible only in a system where there is, apart from the source, also a sink of like intensity. We shall not give the concrete form of the sink, assuming that the sink is concentrated in the region $x<x_s$ and absorbs just as many particles as reach it from the source. Such a sink can be specified either by fixing the number of particles in the distribution or by using well defined boundary conditions at the sink, for instance, $f(x_s)=0$.

We shall in what follows study stationary non-equilibrium photon distributions which, as one sees easily from (1), are given by the equation

$$T\frac{\partial f}{\partial x}+f+f^2=\frac{1}{x^4}\int_x^\infty Q(x)\,dx, \tag{2}$$

which is obtained after integrating over x from x to ∞ and using the condition

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$$\left\{x^4\left[T\frac{\partial f}{\partial x}+f+f^2\right]\right\}_{x\to\infty}\to 0,$$

i.e., the photon flux at infinity vanishes and the effect of the source at large x is insignificant (for this it is necessary that the distribution function decrease at infinity faster than x^{-4}).

One can reduce the Riccati Eq. (2) through the substitution $f = Tv'/v - \frac{1}{2}$ to the linear equation

$$v''-\frac{1}{4T^2}q(x)v=0,\qquad q(x)=1+\frac{4}{x^4}\int_x^\infty Q(x)\,dx. \tag{3}$$

The presence of the small parameter $2T$ in that equation allows us to use the WKB method. The corresponding solution of Eq. (3) has the form[10]

$$v=c_1\frac{1}{q(x)^{1/4}}\exp\left(-\frac{\gamma(x)}{2T}\right)+c_2\frac{1}{q(x)^{1/4}}\exp\left(\frac{\gamma(x)}{2T}\right),$$
$$\gamma(x)=\int_{x_s}^{x} q^{1/2}(x)\,dx, \tag{4}$$

and the distribution function is then equal to

$$f(x)=-q^{1/2}(x)\left[c\exp\left(\frac{\gamma(x)}{T}\right)+1\right]^{-1}$$
$$+\frac{1}{2}(q^{1/2}(x)-1)+\frac{T}{x}\left[1+\frac{x^4}{4}\left(\int_x^\infty Q\,dx\right)^{-1}\right]^{-1}. \tag{5}$$

The constant c in the distribution must somehow be determined from the physical conditions and its value depends on the power of the source. We shall in what follows assume that there acts at the point x_s a very powerful sink which absorbs all particles. We shall describe such a sink by the boundary condition $f(x_s) = 0$. Notwithstanding the fact that one of the solutions (4) decreases exponentially, these solutions are of the same order of magnitude as the growing solutions at the point x_s. This fact renders correct the procedure to determine c from the boundary conditions given at the point x_s (see Ref. 10).

From (5) and the condition $f(x_s) = 0$ we get for $x_s \ll T$

$$c=-\frac{1-P^{1/2}/Tx_s}{1+P^{1/2}/Tx_s};\qquad P=\int_{x_s}^\infty Q(x)\,dx. \tag{6}$$

It is clear from (6) that when the power increases the constant c changes from -1 when $P\simeq 0$ to $+1$ as $P\to\infty$, becoming zero when $P = T^2x_s^2$. If we introduce instead the constant c the constant $\mu = -T\ln c$ which plays the role of a chemical potential for the photon distribution, we can write the distribution function (5) for $P\ll T^2x_s^2$ in the form

$$f(x)=\frac{1}{2}q^{1/2}(x)\,\mathrm{cth}\left[\frac{\gamma-\mu}{2T}\right]+\frac{T}{x}-\frac{1}{2},$$
$$\mu=2T\,\mathrm{Arcth}\frac{Tx_s}{P^{1/2}}, \tag{7}$$

and for large $P > T^2x_s^2$ the distribution function takes the form

$$f(x)=\frac{1}{2}q^{1/2}(x)\,\mathrm{th}\left[\frac{\gamma-\mu}{2T}\right]+\frac{T}{x}-\frac{1}{2};\qquad \mu=2T\,\mathrm{Arth}\frac{Tx_s}{P^{1/2}}. \tag{8}$$

We note that the form of the distribution function changes qualitatively when the power of the source changes from $P<T^2x_s^2$ to $P>T^2x_s^2$. Moreover, it is important that the chemical potential μ is a universal function of the source, i.e., it is determined solely by its integral characteristic P.

From Eqs. (7) and (8) which we gave above it is difficult to discern directly the power-law asymptotic behavior of the distribution function and its universality. In what follows we consider a few concrete types of sources which enable us easily to analyze the structure of the stationary distribution functions. We start with a study of a photon source which is localized in energy space:

$$Q(x)=P\delta(x-x_i);\qquad P=J\hbar^3/m^3c^7\sigma n_e, \tag{9}$$

where J is the number of photons appearing per 1 cm^3 in 1 s. It will become clear in what follows that notwithstanding its simplicity this model contains all basic information about possible universal asymptotic behavior in our problem.

In this model

$$q(x)=1+4P/x^4, \tag{10}$$

and the integral $\gamma(x)$ occurring in the WKB solution can be expressed in terms of hypergeometric functions: $\gamma=\gamma_4(x_s, P, x)$. [See the Appendix for an expression for the functions $\gamma_\delta(x_s, P, x)$.]

Above we used a WKB solution which is valid when the condition

$$\left|\frac{d}{dx}\left(\frac{2T}{q(x)^{1/2}}\right)\right|\ll 1 \tag{11}$$

is satisfied, i.e., when $(4P)^{1/4}\gg T$. Satisfying this inequality guarantees that the solution is valid in the whole range over which x changes. When the power of the source decreases the WKB approximation breaks down near the point $x = P^{1/4}$. The turning point lies in the complex x-plane and when P decreases it approaches the real axis. When the relative position of the source and the sink changes, i.e., when $x_s > x_i$, Eq. (3) remains the same except that the sign in front of the term $4P/x^4$ changes. This leads to the fact that the turning point turns up on the real axis and in the range $x<(4P)^{1/4}$ the distribution function shows fast oscillations. The non-physical behavior of the distribution function indicates that there is no stationary solution for such a relative position of the source and the sink. For small powers of the source, $P\ll T^{1/4}$, one can see from (6) that the constant $c = c_1/c_2 < 0$. However, because for such powers there is a region where the WKB approximation breaks down, the solution given by (5) is valid only when $x\ll P^{1/4}$. To find the solution in the region $x\ll P^{1/4}$ we must determine the constant in the solution (5) bypassing the region of the breakdown. When we accomplish this bypassing using a transfer matrix (see Ref. 10) we check easily that the ratio of the constants c_1/c_2 is unchanged. Therefore, in this case the stationary photon distribution is also given by Eq. (7) or (8), depending on the value of P. When $P < T^2x_s^2$ we can, by using the asymptotic behavior of the funciton $\gamma(x)$, write the distribution function in the form of two asymtotic expressions:

$$f(x)=\frac{T}{x}\left[1-\frac{P^{1/2}}{Tx}\operatorname{cth}\left[\frac{P^{1/2}}{Tx}+\frac{\mu}{2T}-\frac{P^{1/2}}{x_sT}\right]\right],\quad x\ll P^{1/4}, \tag{12}$$

$$f(x)=\left\{\exp\left[\frac{1}{T}\left(x-\mu+\frac{2P^{1/2}}{x_s}\right)\right]-1\right\}^{-1}+\frac{P}{x^4};\quad x\gg P^{1/4}. \tag{13}$$

As the power of the source $P\rightarrow 0$ the asymptotic form (12) cannot be formed and the distribution function is given by Eq. (13) in the whole of the region between the source and the sink. The distribution goes in that case for $P=0$ over into the equilibrium distribution.

We note that the solution (12) is formally the same as the photon distribution found in Ref. 11, but in contrast to Ref. 11 it is clear that this solution is valid only when $P^{1/2}<Tx_s$. For large powers of the source one must use the other branch of the solution (8). One notes easily that this solution has a simple power-law asymptotic form (see also Refs. 11, 12):

$$f(x)\approx\frac{T}{x}+\frac{P^{1/2}}{x^2},\quad x_s<x\ll P^{1/4},$$

$$f(x)\approx\frac{P}{x^4},\quad P^{1/4}\ll x<x_i. \tag{14}$$

When the intensity of the source increases the region occupied by the asymptotic of $P^{1/2}/x^2$ broadens and when the inequality $(4P)^{1/4}\gg x_i$ is satisfied it occupies nearly the whole interval between the source and the sink and the second asymptotic form vanishes.

In the region $x\gg x_i$, by virtue of the strict localization of the source, the stationary distribution function is given by Eq. (2) with $Q=0$. In other words, in the region of large energies the distribution function turns out to be a quasiequilibrium one:

$$f(x)=\left[\exp\left(\frac{x+\mu}{T}\right)-1\right]^{-1}. \tag{15}$$

The chemical potential μ is determined from the condition that the distribution function by continuous at the source and equal to

$$\mu=T\ln(1+1/f(x_i))-x_i,$$

where $f(x_i)$ is determined by the distribution function found above [e.g., Eq. (14)].

It will become clear from what follows that the results obtained for a strictly localized source and precisely the power-law asymptotic behavior of the distribution function between the source and the sink turn out to be practically unchanged also for non-localized sources under rather weak conditions.

Generally speaking, the Kompaneets equation is valid only for sufficiently smeared out distribution functions (and hence also smeared out sources) with a width $\Delta\omega\gg\Delta\omega_D$ where $\Delta\omega_D\sim\omega T^{1/2}$.[13] The preceding case with a delta-function source can thus be considered only as a model case. A source which does not go beyond the limits of applicability of the Kompaneets equation must necessarily have a reasonable width. However, the justification of the delta-function model is the following: the use instead of the δ-function of some approximation with a finite width does not lead to any change in the asymptotic behavior of the distribution function (14) at distances from the point of maximum intensity larger than the width of the source. Below we show even more—the asymptotic behavior (14) obtained in the delta-function model is valid also in the case of non-localized power-law sources which occur in many astrophysical problems.[14] We give a source of the form

$$Q(x)=\frac{Q}{x_i}\left[\left(\frac{x}{x_i}\right)^{\alpha}\theta(x_i-x)+\left(\frac{x}{x_i}\right)^{-1-\beta}\theta(x-x_i)\right]. \tag{16}$$

For such a source the potential function has the form

$$q(x)=1+\frac{4P}{x^4}\left[1-\frac{\beta}{\alpha+\beta+1}\left(\frac{x}{x_i}\right)^{\alpha+1}\right]\theta(x-x_i)$$
$$+\frac{4P}{x^{\beta+4}}\frac{\alpha+1}{\alpha+\beta+1}x_i^{\beta}\theta(x_i-x),\quad P=Q\frac{\alpha+\beta+1}{\beta(\alpha+1)}. \tag{17}$$

From this it is clear that the second term within the square brackets is always small compared to the first one when $x<x_i$. Neglecting these terms we find easily the following expression for the phase integral:

$$\gamma(x)=\theta(x-x_i)\gamma_4(x_s,P,x)+\theta(x_i-x)\left[\gamma_4(x_s,P,x_i)\right.$$
$$\left.+\gamma_{\beta+4}\left(x_i,P\frac{\alpha+1}{\alpha+\beta+1}x_i^{\beta},x\right)\right]. \tag{18}$$

[See the Appendix for an expression for the functions $\gamma_\delta(x)$.]

It is clear from (17) and (18) that $\gamma(x)$ and $q(x)$ in the region $x\ll x_i$ are the same as for a concentrated source. In the region between the source and the sink the distribution function has a universal form, i.e., it is insensitive to the actual form of the source (it is merely necessary that the intensity drops towards the sink) and depends on its integral intensity.

The universality of the spectrum between the source and the sink suggests that the distribution $N=x^2f(x)\sim P^{1/2}$ can be found from purely Kolmogorov-like considerations. Indeed, far from the source and the sink one can write down an equation of continuity for the number of particles:

$$\frac{\partial N}{\partial t}+\frac{\partial P}{\partial k}=0.$$

Hence it follows that $N\propto\tau P/k$, where τ is a characteristic interaction time given according to the Kompaneets equation by $\tau^{-1}\sim N/k$. As a result we get a distribution $N\propto P^{1/2}$ which is thus Kolmogorov-like with a constant flux of photons along the spectrum.

We now examine what is the shape of the distribution function for $x>x_i$. If the source is at such high energies that $4P/x_i^4\ll 1$, we can use the asymptotic forms of the functions $q^{1/2}(x)$ and $\gamma(x)$:

$$q^{1/2}(x)\approx 1+2Q/\beta x^{\beta+4},$$
$$\gamma(x)\approx 2P^{1/2}/x_s \tag{19}$$

and from Eq. (5) we find that

$$f(x)=\left\{\exp\left[\gamma(x)-\frac{\mu}{T}\right]\right\}^{-1}+\frac{Q}{\beta x^{\beta+4}}, \tag{20}$$

where the chemical potential $\mu=-T\ln|c|$ is determined from the condition that the function be continuous in the point $x = x_i$ and is equal to

$$\mu = T\gamma(x_i) - T\ln\left(1 + \frac{\beta(\alpha+1)}{\alpha+1+\beta}\frac{x_i^4}{Q}\right).$$

The power-law asymptotic behavior of the distribution function obtained here is, in contrast to (14), not universal and determined by the source. The existence in the region $x > x_i$ of a non-universal asymptotic form enables us to use effectively the non-equilibrium distributions for diagnostics and non-equilibrium sources (enabling us to determine their shape, power and localization region). For the purpose of diagnostics of non-equilibrium states it is of interest to elucidate the features of the distribution function also in the more general case. For instance, it is of interest to consider the case of several power-law sources as such a situation is relatively often met with in actual problems.

Let the source have the form

$$Q(x) = \frac{Q_1}{x_1}\left[\left(\frac{x}{x_1}\right)^{\alpha_1}\theta(x_1-x) + \left(\frac{x}{x_1}\right)^{-\beta_1-1}\theta(x-x_1)\right] + \frac{Q_2}{x_2}\left[\left(\frac{x}{x_2}\right)^{\alpha_2}\theta(x_2-x) + \left(\frac{x}{x_2}\right)^{-\beta_2-1}\theta(x-x_2)\right], \quad (21)$$

$$x_2 \gg x_1.$$

The solution of the kinetic equation can then as before be found from the general Eq. (5). However, we shall not write down the solution as it is unwieldy and merely restrict ourselves to its analysis. It is clear that as earlier when $x < x_1$ a universal distribution of the shape (14) will be formed with a power of the source

$$P = \frac{Q_1 x_1(\alpha_1+\beta_1)}{(\beta_1-1)(\alpha_1+1)} + \frac{Q_2 x_2(\alpha_2+\beta_2)}{(\alpha_2+1)(\beta_2-1)}.$$

In the region $x \gg x_2$ a distribution of the shape (20) is formed with a non-universal power-law contribution $Q_2/\beta_2 x^{4+\beta_2}$. As far as the section between x_1 and x_2 is concerned there can be several possibilities. There will in that interval be in the expression for $q(x)$ competition between the non-universal part from the source at the point $x = x_1$ and the universal part from the source concentrated in the point x_2:

$$q(x) \approx 1 + \frac{4Q_2 x_2(\alpha_2+\beta_2)}{(\alpha_2+1)(\beta_2-1)x^4} + \frac{4Q_1}{(\beta_1-1)x_1^3}\left(\frac{x_1}{x}\right)^{\beta_1+3}, \quad (22)$$

$$x_1 \ll x \ll x_2.$$

Depending on the ratio of the two last terms there will be formed either a universal asymptotic form determined by the source Q_2 or a non-universal part from the source Q_1.

The above-considered power-law photon sources are, as we have already noted, widespread. For instance, it is known[14] that for quasars and the nuclei of Seyfert galaxies 90% of all energy emitted by them lies in the band $\omega_i \sim 10^{13}\mathrm{s}^{-1}$ ($x_i \approx 10^{-8}$) and the sources have spectra of the form (16) with $\alpha = 3.5$; $\beta = 2.5$. Using the data given in Ref. 14 we estimate for these sources the parameter $P/x_i^4 \sim 10^4$. Under such conditions the asymptotic behavior (14) of the photon distribution function must appear and the formation of it can lead to an appreciable heating of the electrons in the plasma surrounding the source, as the energy of the photon gas goes to the heating of the electron subsystem which according to Ref. 15 always has a Maxwellian distribution.

The stationary electron temperature can be expressed in terms of the photon distribution function as follows (see, e.g., Ref. 9):

$$T_{st} = \frac{1}{4}\int_0^\infty (N^2 + x^2 N)\,dx\left(\int_0^\infty xN\,dx\right)^{-1}. \quad (23)$$

For the asymptotic form of the distribution function $N = x^2 f(x) \approx P^{1/2}(x_s < x < x_i)$, whence we get the following temperature:

$$T_{st} \approx P^{1/2}/2x_i + x_i/6 \approx P^{1/2}/2x_i \quad (P \gg x_i^4).$$

Conclusion

We give a summary of the study just performed. Let the source be localized near the energy x_i and the sink near the energy x_s. In that case:

1. The form of the distribution function depends significantly on the relation between x_s and x_i. If $x_s > x_i$ a stationary photon distribution is possible only under the condition $4P/x_i^4 < 1$. If $x_i > x_s$ a stationary distribution is possible for any finite value of P. In what follows we shall assume $x_i > x_s$, $x_s < T$.

2. If the power of the source satisfies the condition $4P \gg T^4$ in the whole region between the source and the sink the distribution function has the form (8) with the simple power-law asymptotic forms (14):

$$f(x) = P^{1/2}/x^2 + T/x, \quad x_s < x \ll P^{1/4},$$
$$f(x) \approx P/x^4, \quad P^{1/4} \ll x < x_i.$$

3. If the power is so small that $4P < T^4$ the distribution (5) is valid in the regions $x \ll P^{1/4}$ and $x \gg P^{1/4}$, and if $x_s^2 < P/T^2 < T^2$ the distribution has the form (8) with asymptotic behavior (14).

If, however, $P/T^2 < x_s^2$ the distribution has the form (7) with the asymptotic forms

$$f(x) \approx \frac{T}{x}\left[1 - \frac{P^{1/2}}{x^2}\operatorname{cth}\left[\frac{P^{1/2}}{T}\left(\frac{1}{x} - \frac{1}{x_s}\right) + \frac{\mu}{2T}\right]\right], \quad x \ll P^{1/4},$$

$$f(x) \approx \left[\exp\left[\left(x - \mu + \frac{2P^{1/2}}{x_s}\right)\frac{1}{T}\right] - 1\right]^{-1} + \frac{P}{x^4}, \quad x \gg P^{1/4},$$

$$\mu = 2T\,\operatorname{Arcth}\frac{Tx_s}{P^{1/2}}.$$

4. The results given above remain practically unchanged in the case of unlocalized sources. All formulae simply have an asymptotic meaning when $x \ll x_i$. This shows the universality of the distribution between the source and the sink.

5. In the energy range $x > x_i$ two cases may occur:

a) if the source is a δ-function source the distribution in the range $x > x_i$ is a quasi-equilibrium one [see (15)];

b) If the source in non-localized the non-universal distribution (20) is formed in the region $x > x_i$. By virtue of this the non-equilibrium photon distributions can be used for the diagnostics of sources enabling us to determine their arrangement, power, and shape.

One can generalize the results obtained to sources of a more general form (for instance, to the case of several power-law sources). We note that considering the evolution in time of deviations from the obtained stationary distributions and linearizing the kinetic equation one may prove the stability of the solutions obtained.

Appendix

For the WKB analysis of Eq. (3) it is necessary to evaluate the phase integral $\gamma(x)$ and know its asymptotic behavior. All integrals $\gamma(x)$ encountered in the paper have the form

$$\gamma_\delta(x_s, P, x)=\int_{x_s}^{x} dx\,(1+4Px^{-\delta})^{1/2}.$$

Such integrals can be expressed in the general case in terms of hypergeometric functions. We give here an expression for the function $\gamma(x)$:

$$\gamma_\delta(x_s, P, x)=(4P)^{1/\delta}\left\{\frac{2}{2+\delta}\left[\left(\frac{4P}{x^\delta}\right)^{-(\delta+2)/2\delta}\left(1+\frac{x^\delta}{4P}\right)^{-1/2}\right.\right.$$
$$\times F\left(\frac{1}{2},1;\frac{3\delta+2}{2\delta};\left(1+\frac{4P}{x^\delta}\right)^{-1}\right)$$
$$-\left(\frac{x_s{}^\delta}{4P}\right)^{(\delta+2)/2\delta}\left(1+\frac{x_s{}^\delta}{4P}\right)^{-1/2}$$
$$\left.\times F\left(\frac{1}{2},1;\frac{3\delta+2}{2\delta};\left(1+\frac{4P}{x_s{}^\delta}\right)^{-1}\right)\right]$$
$$+\frac{1}{\delta-1}\left[\left(\frac{x_s{}^\delta}{4P}\right)^{(2-\delta)2\delta}\left(1+\frac{x_s{}^\delta}{4P}\right)^{-1/2}\right.$$
$$\times F\left(\frac{1}{2},1;\frac{2\delta-1}{\delta};\left(1+\frac{x_s{}^\delta}{4P}\right)^{-1}\right)$$
$$-\left(\frac{x^\delta}{4P}\right)^{(2-\delta)/2\delta}\left(1+\frac{x^\delta}{4P}\right)^{-1/2}$$
$$\left.\left.\times F\left(\frac{1}{2},1;2-\frac{1}{\delta};\left(1+\frac{x^\delta}{4P}\right)^{-1}\right)\right]\right\}.$$

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