Dissipative phase separation boundaries

V. A. Berezin, V. A. Kuz'min, and I. I. Tkachev

Institute for Nuclear Studies, Academy of Sciences of the USSR
(Submitted 4 July 1983)
Zh. Eksp. Teor. Fiz. **86**, 785–795 (March 1984)

Phase separation boundaries with nonzero surface energy-momentum tensor (singular shells) are investigated. It is shown that a shell that leads to an increase in entropy (to creation of particles) in necessarily a singular shell. The equations of motion of such shells are obtained within the framework of special relativity. The most interesting application of these equations is to shells describing the process of combustion or detonation. In particular, it is shown that, under certain definite conditions, the transformation of the vacuum-like metastable state (that arises in cosmological phase transitions) to the stable state is analogous to combustion.

I. INTRODUCTION

We shall be interested in the properties of the separation boundary between two phases[1]—in the first instance, the separation boundary between phases that appear under cosmological conditions.[2]

The motion of a thin separation boundary, for example, the growth of a liquid drop in a supercooled vapor (condensation discontinuity), is usually described by the detonation wave equations.[1] The range of application of these equations is limited, mainly because their derivation is based on the assumption that the phase separation boundary is nonsingular (we recall that a phase separation boundary is called singular if the surface energy-momentum density tensor on the shell is nonzero).

There are, however, many cases in which the phase separation boundary cannot be considered to be nonsingular. Thus, in particular, the shell separating two phases with pure vacuum equations of state is a singular shell. Such shells usually arise in the study of cosmological phase transitions. A planar pure-vacuum phase separation boundary ("domain wall") was considered in Ref. 3, and a spherically-symmetric separation boundary ("new-vacuum bubble") was discussed in Ref. 4.

In this paper, we shall derive the equations describing the motion of a singular shell. In special cases, these equations describe both the growth of a pure-vacuum bubble[4] and the propagation of a detonation wave.[1]

When a pure-vacuum bubble expands, the liberated energy of the metastable vacuum is completely converted into the kinetic energy of the shell. The latent heat of the phase transition is then liberated only after collisions between walls of different bubbles. We shall find that the equations of motion admit of another, and completely different, shell expansion regime in which the entire vacuum energy is converted directly into the thermal energy of the internal medium. The characteristic feature of this regime is that the very occurence of particle creation from vacuum necessarily leads to a nonzero shell tension tensor. In other words, a shell that leads to an increase in entropy is necessarily a singular shell. We shall see that particle creation processes contribute to the components of the shell energy-momentum tensor that are due to its tension (but not with its intrinsic mass), so that in the limit of a plane front, effects due to the singularity of the shell become negligible during its propagation through the medium. In the case of vacuum "combustion,"[5] when the shell velocity tends asymptotically to the velocity of light, the singularity of the shell has an essential effect on its equations of motion for any radius of curvature of the separation boundaries.

The starting point of our analysis is the equation of continuity for the energy-momentum tensor $T_{\mu\nu}$ which can have delta-function type discontinuities on a three-dimensional hypersurface. By integrating the continuity equation over the thickness of the hypersurface, we obtain the required equations of motion of the shell. These equations can also be obtained from the well-known continuity equations for general theory of relativity GTR metrics in the limit as $M_{Pl}\to\infty$ (i.e., when gravitational effects can be neglected).[6]

II. EQUATIONS OF MOTION OF A SINGULAR SHELL

The motion of the phase separation boundary specifies a certain (three-dimensional) hypersurface Σ in four-dimensional spacetime. Let us introduce the coordinates x^i on this hypersurface, so that

$$dl^2={}^3g_{ij}dx^idx^j \tag{1}$$

is an interval on the hypersurface Σ. At each point on Σ we erect the outward normal and introduce the coordinate n measured in the direction of this normal. In the immediate neighborhood of Σ, the 4-interval then assumes the form

$$ds^2=-dn^2+{}^3g_{ij}dx^idx^j. \tag{2}$$

The coordinates for which the metric is given by (2) are called Gaussian. In addition to the Gaussian coordinates, let us construct an arbitrary coordinate system $\{y^\mu\}$:

$$ds^2=g_{\mu\nu}dy^\mu dx^\nu.$$

Here and henceforth Latin indices $i, j, \ldots$ run through these values and label the components of tensors on the hypersurface, whereas Greek indices run through four values and label the components of tensors in spacetime.

Suppose that the equation of the hypersurface Σ in terms of the coordinates $\{y^\mu\}$ has the form $\varphi(y^\mu)=0$. Let us define the coordinate n as follows:

$$n=\varphi(y^\mu)/(g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})^{1/2}, \tag{3}$$

where $\varphi_{,\mu}\equiv\partial\varphi/\partial y^\mu$. It is obvious that, on the hypersurface $n=0$ that we are considering, the components of the unit

 0038-5646/84/030459-06$04.00 © 1984 American Institute of Physics

normal to the hypersurface Σ are

$$N_\mu = n_{,\mu}|_{y^\mu \in \Sigma}. \tag{4}$$

Let T_α^β be the components of the energy-momentum tensor in terms of the Gaussian coordinates, and let the tensor S_α^β be defined by

$$S_\alpha{}^\beta = \lim_{\delta\to 0} \int_{-\delta}^{\delta} T_\alpha{}^\beta \, dn. \tag{5}$$

A shell is said to be nonsingular if all the components of the tensor S_α^β are zero. In the case of a singular shell, the three-dimensional tensor S_i^j is nonzero, and the components S_n^n and S_i^n of the tensor S_α^β are always zero.

Let us integrate the equations of conservation of the energy-momentum tensor[1)]

$$T_\alpha{}^\beta{}_{;\beta} = T_\alpha{}^\beta{}_{,\beta} + T_\alpha{}^\gamma \Gamma_{\gamma\beta}^\beta - T_\gamma{}^\beta \Gamma_{\alpha\beta}^\gamma = 0 \tag{6}$$

with respect to n from $-\delta$ to $+\delta$, and then let δ tend to zero, remembering that only the components T_i^j of the energy-momentum tensor can contain singularities.

The first three equations $T_{i;\beta}^\beta = 0$ can be integrated directly, and we obtain

$$S_i{}^j{}_{,j} + S_i{}^j \Gamma_{jk}^k - S_j{}^k \Gamma_{ik}^j + [T_i{}^n] = 0, \tag{7}$$

or

$$S_i{}^j{}_{|j} = -[T_i{}^n], \tag{8}$$

where the vertical bar in the subscript signifies covariant differentiation in the metric ${}^3g_{ij}$, and $[T_i^n] = T_i^n(n+0) - T_i^n(n-0)$ is the discontinuity in the component T_i^n of the energy-momentum tensor on the hypersurface Σ.

Integrating the equation $T_{n;\beta}^\beta = 0$, we obtain in the limit as $\delta \to 0$

$$[T_n{}^n] - S_i{}^j \Gamma_{nj}^i = 0. \tag{9}$$

The Christoffel symbols in (9) are conveniently expressed in terms of the outward curvature tensor of the hypersurface Σ:

$$K_j{}^i \equiv -N^i{}_{;j} = -\Gamma_{nj}^i, \tag{10}$$

where N^a are the components of the outward normal in Gaussian coordinates $N^n = 1, N^i = 0$, and the first equation in (10) is the definition of the outward curvature tensor of the hypersurface. Thus, equation (9) can be rewritten as follows:

$$S_i{}^j K_j{}^i = -[T_n{}^n]. \tag{11}$$

When the outward normal N^μ is defined in an arbitrary coordinate system, as in (4), the outward curvature tensor is given by

$$K_{ij} = -\frac{\partial y^\mu}{\partial x^i}\frac{\partial y^\nu}{\partial x^j} N_{\mu;\nu}. \tag{12}$$

Equations (8) and (11) determine the form of the hypersurface Σ and, consequently, the evolution of the two-dimensional phase separation boundary of arbitrary shape for arbitrarily specified energy-momentum tensors of both the internal and external phases. Moreover, these equations are valid when the ambient 4-spacetime is curved, provided only we may neglect the reaction of the energy-momentum tensor of the shell to the ambient geometry. Equation (8) is determined by the internal geometry of the hypersurface alone. In this sense, it is simpler than (11) which determines the inscription of the hypersurface into the spacetime surrounding the shell.

1. Spherically symmetric shell

Spherically symmetric phase separation boundaries are of particular interest. The element of length on this type of hypersurface is given by

$$dl^2 = d\tau^2 - \rho^2(\tau)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{13}$$

where τ is the time measured by an observer located on the shell, $\rho(\tau)$ is the shell radius, and θ and φ are the angular coordinates. The covarient derivative in (8) is evaluated directly with the metric (13).

By virture of spherical symmetry, $S_2^2 = S_3^3$ and $S_i^j = 0$ if $i \neq j$; $S_{2|j}^j = S_{3|j}^j \equiv 0$, and only one equation remains in (8), namely,

$$S_0{}^j{}_{|j} = S_0{}^0{}_{,0} + \frac{2\dot\rho}{\rho}(S_0{}^0 - S_2{}^2) = -[T_0{}^n], \tag{14}$$

where

$$\dot\rho \equiv d\rho/d\tau.$$

We shall now consider the evolution of spherically-symmetric bubbles in flat spacetime. The most convenient coordinates y^μ are therefore the usual spherical polar coordinates, in which the 4-interval takes the form

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{15}$$

These coordinates are used by an observer at rest at the center of a bubble. The equation of the surface Σ assumes the form

$$r - R(t) = 0,$$

where $R(t)$ is an unknown function.

In accordance with (3), we define the coordinate n by

$$n = (r - R(t))/(1 - (dR/dt)^2)^{1/2}. \tag{16}$$

To determine the outward curvature tensor, we need, in addition, the connection between the coordinates (13) and (15) on the shell. We have

$$\begin{aligned} R(t) &= \rho(\tau), \\ dt/d\tau &= (1+\dot\rho^2)^{1/2}, \\ R' \equiv dR/dt &= \dot\rho/(1+\dot\rho^2)^{1/2}. \end{aligned} \tag{17}$$

Using (12), (4), and (16), we obtain

$$\begin{aligned} K_2{}^2 &= -(1+\dot\rho^2)^{1/2}/\rho, \\ K_0{}^0 &= -\ddot\rho/(1+\dot\rho^2)^{1/2}. \end{aligned} \tag{18}$$

We must now specify the right-hand sides of (8) and (11). Here we sacrifice generality and suppose that, both outside and inside the bubble, T_μ^ν is the energy-momentum tensor of an ideal liquid:

$$T_\mu{}^\nu = (\varepsilon + p) u_\mu u^\nu - p\delta_\mu{}^\nu, \tag{19}$$

where u_μ is the 4-velocity of an element of the medium and ε and p are, respectively, the energy density and momentum measured in the frame in which the medium is at rest. Any possible generation of entropy (and particle creation) is thus associated with the transition layer and is ascribed to the shell.

We shall suppose that the ambient medium is at rest relative to the bubble center (i.e., apart from the possible motions of the ambient medium, we shall also neglect reflection of particles back into this medium). The velocity of the external medium relative to the shell is then R'. It is natural to suppose that the medium in the interior of the bubble is spherically symmetric. Hence the modulus of the 3-velocity of the internal medium relative to the shell is $u = -u^1/u^0$. Under these assumptions, Eqs. (8) and (11) assume the following form for a spherically symmetric shell:

$$\frac{dS_0^{\;0}}{d\tau}+\frac{2\dot\rho}{\rho}(S_0^{\;0}-S_2^{\;2})=(\varepsilon+p)_{out}\frac{R'}{1-(R')^2}-(\varepsilon+p)_{in}\frac{u}{1-u^2}, \tag{20a}$$

$$S_0^{\;0}\frac{\ddot\rho}{(1+\dot\rho^2)^{1/2}}+\frac{2S_2^{\;2}(1+\dot\rho^2)^{1/2}}{\rho}=p_{in}+(\varepsilon+p)_{in}\frac{u^2}{1-u^2}-p_{out}-\frac{(\varepsilon+p)_{out}(R')^2}{1-(R')^2}. \tag{20b}$$

Using the Einstein formula for the addition of velocities, we can rewrite (20) in terms of the 4-velocities v^μ of the internal medium relative to the center of the bubble:

$$\frac{dS_0^{\;0}}{d\tau}+\frac{2\dot\rho}{\rho}(S_0^{\;0}-S_2^{\;2})=(\varepsilon+p)_{out}\dot\rho(1+\dot\rho^2)^{1/2}-(\varepsilon+p)_{in}\times[\dot\rho(1+\dot\rho^2)^{1/2}((v^0)^2+(v^1)^2)-v^0v^1(2\dot\rho^2+1)], \tag{21a}$$

$$S_0^{\;0}\ddot\rho/(1+\dot\rho^2)^{1/2}+2S_2^{\;2}(1+\dot\rho^2)^{1/2}/\rho=\varepsilon_{out}-\varepsilon_{in}+(\varepsilon+p)_{in}[v^0(1+\dot\rho^2)^{1/2}-v^1\dot\rho^2]^2-(\varepsilon+p)_{out}(1+\dot\rho^2). \tag{21b}$$

In the ensuing analysis it will be useful to have the two forms of the equations of motion of shells, given by (20) and (21).

Equations (20) become identical with the well-known equations for detonation waves[1] in the case of a nonsingular shell $S_i^{\,j}=0$, and also when $\dot\rho\to$const and $S_i^{\,j}\to$const in the limit as $\rho\to\infty$, i.e., in the limit of a plane shell whose velocity tends to the velocity of light. Equations (20) are suitable for a wide class of shells, including the special case of the propagation of detonation waves, condensation discontinuities, and so on.[2)]

The energy-momentum surface density tensor on the shell must be known before specific cases can be investigated.

III. ENERGY-MOMENTUM SURFACE DENSITY TENSOR

We shall now consider the tensor $S_i^{\,j}$ for shells that arise in cosmological phase transitions. The order parameter in such transitions is the average $\bar\varphi\equiv\langle\varphi\rangle$ of the operator for the scalar field φ. The evolution of the field is described by a Lagrange density that is invariant under a gauge group G:

$$\mathscr{L}=(D_\mu\varphi)^*(D^\mu\varphi)-V(\varphi)-{}^1/_4F_{\mu\nu}^{\;a}F^{a\mu\nu},$$

$$D_\mu\equiv\partial_\mu-igA_\mu^{\;a}T^a, \tag{22}$$

$$F_{\mu\nu}^{\;a}\equiv\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+gf^{abc}A_\mu^aA_\nu^c,$$

where g is the gauge coupling constant of the group, T^a are the generators in the representation of the fields φ, f^{abc} are the structure constants of the group, and $V(\varphi)$ is a G-invariant polynomial of degree not higher than 4 in the fields φ. Using the formula

$$T_{\mu\nu}=\frac{2}{(-g)^{1/2}}\left(\frac{\partial(-g)^{1/2}\mathscr{L}}{\partial g^{\mu\nu}}\right)-\frac{2}{(-g)^{1/2}}\left(\frac{\partial(-g)^{1/2}\mathscr{L}}{\partial(\partial g^{\mu\nu}/\partial x^\alpha)}\right)_{;\alpha}, \tag{23}$$

where $g=\det g_{\mu\nu}$, $\partial(-g)^{1/2}/\partial g^{\mu\nu}=-\frac{1}{2}(-g)^{1/2}g_{\mu\nu}$, we obtain the following expression for the energy-momentum tensor

$$T_{\mu\nu}=(D_\mu\varphi)^*(D_\nu\varphi)-{}^1/_2F_{\mu\varkappa}^{\;a}F_\nu^{\;a\varkappa}+{}^1/_2g_{\mu\nu}T+\text{H.c.},$$

where

$$T\equiv V(\varphi)-(D^\alpha\varphi)^*(D_\alpha\varphi)+{}^1/_4F_{\lambda\varkappa}^{a}F^{a\lambda\varkappa}. \tag{24}$$

It is clear that prior to the transition to the limit of zero thickness of the region separating two phases with different values of $\bar\varphi$, we must take the normal Gaussian coordinate system to be the system in which surfaces of constant n coincide with surfaces of constant $\bar\varphi$. Let x^i be the coordinates on the $n=$ const surfaces; we then have $\partial\bar\varphi/\partial x^i=0$. We now define the field Φ as follows: $\varphi=\bar\varphi+\phi$, i.e., $\langle\phi\rangle=0$. Substituting $\varphi=\bar\varphi+\phi$ in (24), we obtain

$$T_{ij}=(D_i\phi)^*(D_j\phi)-{}^1/_2F_{i\varkappa}^{\;a}F_j^{\;a\varkappa}+{}^1/_2g_{ij}T(\bar\varphi,\phi)+\text{H.c.} \tag{25}$$

Finally, averaging this relation and integrating with respect to n between $-\delta$ and δ, we find that in the limit as $\delta\to0$:

$$S_2^{\;2}=S_0^{\;0}+\tilde S_2^{\;2},$$

where

$$\tilde S_{22}=\lim_{\delta\to0}\int dn\{2\langle(D_2\phi)^*(D_2\phi)\rangle+\langle F_{0\varkappa}^{\;a}F_0^{\;a\varkappa}\rangle-(0\to2)\}. \tag{26}$$

1. Vacuum case

If we neglect field fluctuations in (26), we find that $\tilde S_{22}=0$ and $S_i^{\,j}=S\delta_i^j$. We shall call this vacuum shell. For a shell separating two phases with vacuum equations of state $(\varepsilon+p)_{in}=(\varepsilon+p)_{out}=0)$, it follows from (20a) that $dS_0^0/d\tau=0$, and the surface energy density of the expanding bubble is independent of its radius. We can therefore determine S_0^0 by evaluating S_0^0 for flat wall:[3,4]

$$S_0^{\;0}=\int dn\left(\frac{1}{2}(\partial_n\bar\varphi)^2+V(\bar\varphi)\right)=\int_{\bar\varphi_1}^{\bar\varphi_2}d\bar\varphi(2V(\bar\varphi))^{1/2}, \tag{27a}$$

where $\bar\varphi_1$ and $\bar\varphi_2$ are the values of the field $\bar\varphi$ in the corresponding phases. In particular, for the potential

$$V={}^1/_2\lambda(\bar\varphi^2-\bar\varphi_0^{\;2})^2$$

we obtain

$$S_0^{\;0}=\int_{-\bar\varphi_0}^{\bar\varphi_0}d\bar\varphi(2V(\bar\varphi))^{1/2}=\frac{4}{3}\lambda^{1/2}\bar\varphi_0^{\;3}. \tag{27b}$$

By integrating equation (21b), we obtain the equation of motion of the shell of a vacuum bubble:

$$\dot\rho^2=\left(\frac{\varepsilon_{out}-\varepsilon_{in}}{3S_0^{\;0}}\right)^2\rho^2-1+C\rho^{-3}, \tag{28}$$

where C is an arbitrary constant of integration. For $C\neq0$, equation (28) becomes identical with the equation of motion of a bubble with nonzero Schwarzschild mass.[7] For spontaneously appearing bubble of new phase, the external mass must be zero, and the integral of (28) for $C = 0$ in terms of the coordinates of an observer at rest at the centre of the bubble assumes the form

$$r^2 - t^2 = \left(\frac{3S}{\varepsilon_{out} - \varepsilon_{in}} \right)^2 . \tag{29}$$

This is the well-known[4] equation of motion of a pure-vacuum shell.

2. Nonvacuum shell

Of the greatest interest, however, are nonvacuum shells. In fact, even when the shell is initially a pure vacuum shell, its expansion is accompanied by particle creation. We then have the key question: will this type of bubble remain "empty"? The process of particle creation by a vacuum shell has recently attracted a number of papers.[8] This process is usually looked upon as a small perturbation of the vacuum solution (29), but a complete analysis of this problem (within the range of validity of the thin-wall approximation) can be based on solutions of (20). The structure of the tensor S_i^j must then be additionally specified and, in its turn, can be found (in principle) from quantum field theory.

In this paper, we confine our attention to some characteristic properties of (20) that do not depend in an essential way on the structure of S_i^j.

We begin by considering the case where the external medium is a pure vacuum $(\varepsilon + p)_{out} = 0$ (metastable vacuum, $\varepsilon_{out} > 0$), the vacuum energy density in the interior of the bubble is zero, and ε and p are related by some (nonequilibrium) equation of state (for example, $\varepsilon = 3p$). The equations given by (20) then admit of the existence of a shell with $S_0^0 = 0$. However, we cannot then consider that $S_2^2 = 0$. Actually, our shell is a source of particles, and this necessarily leads to a nonzero value of the component S_2^2 of the shell tension tensor. It is clear from (20) that $S_2^2 > 0$ for this shell. If, on the other hand, the internal medium is a vacuum (residue of old phase), and the shell collapses, creating a particle on the outside, equation (20) has a solution with $S_0^0 = 0$ only for $S_2^2 < 0$. Next, we may suppose that the internal medium is in thermodynamic equilibrium (in which case we consider that the entire nonequilibrium transition layer is thin, and is ascribed to the shell). To be specific, let us suppose that the chemical potential of the internal medium is zero, so that $(\varepsilon + p)_{in} = Ts$. The internal medium has a nonzero entropy density s, and the entropy density of the external vacuum medium is zero. Consequently, the shell contains a nonzero entropy source:

$$\frac{su}{(1-u^2)^{1/2}} = \sigma. \tag{30}$$

When $S_0^0 = 0$, we have $S_2^2 = \tilde{S}_2^2$ and (20a) yields the following relation between the entropy source and $\tilde{S}_2^2$:

$$\tilde{S}_2{}^2 = \frac{\rho T}{2\dot{\rho}(1-u^2)^{1/2}} \sigma. \tag{31}$$

The shell with $S_0^0 \approx 0$ and $\tilde{S}_2^2 \neq 0$, is evidently a good approximation when the process of chemical burning is examined. We have seen that (20) admits of the existence of a shell with this structure even in the case of an external medium with a pure vacuum equation of state. Thus, in principle, vacuum burning becomes possible,[5] and corresponds to the following structure of S_i^j:$S_0^0 = 0$, $S_2^2 = \tilde{S}_2^2 \neq 0$. Let us examine this effect in greater detail.

IV. VACUUM BURNING

Thus, suppose that $(\varepsilon + p)_{out} = 0$, $S_0^0 = 0$, $S_2^2 \neq 0$. Equations (20) for this shell do not contain the second derivative with respect to τ. Let us solve (20) for $\dot{\rho}$ and u:

$$(1+\dot{\rho}^2)^{1/2} = \frac{(\varepsilon_{out} - \varepsilon_{in})(\varepsilon_{out} + p_{in})}{2S_2{}^2[\varepsilon_{out} - \varepsilon_{out}) + (\varepsilon_{out} + p_{in})]} \rho + \frac{2S_2{}^2}{\rho[(\varepsilon_{out} - \varepsilon_{in}) + (\varepsilon_{out} - p_{in})]}, \tag{32}$$

$$u^2 = \frac{\rho^2 (p_{in} + \varepsilon_{out})^2 - 4(S_2{}^2)^2}{\rho^2 (\varepsilon_{out} - \varepsilon_{in})^2 - 4(S_2{}^2)^2} . \tag{33}$$

Let us consider in these equations the limit as $\rho \to \infty$. In this limit, the velocity of the shell tends to the velocity of light ($\rho \to \infty$), and the parameters of the internal medium fit the detonation adiabatic curve[1]

$$\varepsilon_{in} u - p_{in} = \varepsilon_{out}(1+u),$$

if $(S_2^2/\rho) \to 0$. If, on the other hand, $(S_2^2/\rho) \to$const, the shell may move with constant velocity. Finally, when $(S_2^2/\rho) \to \infty$, the velocity of the shell again tends to the velocity of light and $u \to 1$, i.e., in this case, as $\rho \to \infty$, the velocity of the internal medium relative to the shell is not constant, but the velocity of the medium relative to the bubble center is. It is then more convenient to use the equations of motion (21) in the analysis of the shell. In particular, the solution with the internal medium at rest relative to the bubble center is possible in this case.

It also follows from (32) that, in the limit as $\rho \to \infty$, we have $\varepsilon_{in} > 2\varepsilon_{out} + p_{in}$ if $(S_2^2/\rho) \to 0$ and $\varepsilon_{in} < 2\varepsilon_{out} + p_{in}$ if $(S_2^2/\rho) \to \infty$, but always $\varepsilon_{in} \gtrsim \varepsilon_{out}$.

Whatever the magnitude of $\tilde{S}_2^2$, the entire energy of the metastable vacuum converts into the energy of the internal medium. This is, of course a direct consequence of the fact that $S_0^0 = 0$, and the energy released in the course of the phase transition cannot be expended in increasing the kinetic energy of the wall, as in the pure vacuum case.[4] The shell expansion velocity, on the other hand, depends on $\tilde{S}_2^2$ (and thus the rate of conversion of vacuum energy into the energy of the internal medium also depends on this quantity).

The quantity S_0^0 is the amount of energy contained in the transition layer between the two phases. Although, formally, S_0^0 can be zero, if T_0^0 experiences only a discontinuity and does not contain δ-function singularities in the limit of an infinitely thin transition layer, the quantity S_0^0 may be not zero in the case of a vacuum phase transition.

We now turn to shells with $S_0^0 \neq 0$. We shall find the conditions under which we may say that vacuum combustion takes place.

The particles created during the bubble expansion process can form bound states with the shell.[9] In that case, $S_0^0 = S_0^0(\rho)$ and S_0^0 increases with increasing ρ. We shall neglect this effect, and consider that $S_0^0 = \text{const}$. Next, particle creation effects lead to $S_2^2 = S_0^0 + \tilde{S}_2^2$ where $\tilde{S}_2^2$ is connected with the entropy source (30).

Generally speaking, $\tilde{S}_2^2$ is a complicated function of the state of the medium and motion of the shell. $\tilde{S}_2^2$ contains terms proportional to $\rho, \dot{\rho}, \ddot{\rho}$, and so on. However, in any of these cases, the quantity $\tilde{S}_2^2$ increases with ρ, so that the amount of vacuum energy converted into the energy of the internal medium can only increase. Below, we shall investigate the motion of a bubble on the assumption that $\tilde{S}_2^2 = \text{const}$, and hence find the lower bound for the generation of heat from vacuum during the process of expansion of the bubble of new phase.

Equation (20b) is readily integrated on the assumption that all the quantities other than $\rho(\tau)$ are constants:

$$\frac{(1+\dot{\rho}^2)^{1/2}}{\rho} = \frac{\varepsilon_{out} - \varepsilon_{in} + (\varepsilon + p)_{in}/(1-u^2)}{3S_0{}^0 + 2\tilde{S}_2{}^2} + C'\rho^{-(3S_0{}^0 + 2\tilde{S}_2{}^2)/S_0{}^0}, \tag{34}$$

where C' is the constant of integration. A spontaneously arising, pure-vacuum bubble corresponds to $C = 0$ in (28). On the other hand, the radius of the bubble produced in a thermostat (where the bubble can rise with a nonvanishing external mass), is determined by the conditions of equilibrium for the bubble in the medium $\dot{\rho} = 0, \ddot{\rho} = 0$ at the time of its creation. It then follows from (20b) that $\rho_0 = 2S_2^2/(p_{in} - p_{out})$—a well-known formula in thermodynamics. In any case, for sufficiently large ρ, the specific value of C is unimportant. As $\rho \to \infty$, equations (34) and (20a) yield

$$2\tilde{S}_2{}^2(\varepsilon_{out} + p_{in} - u^2(\varepsilon_{out} - \varepsilon_{in})) = (3S_0{}^0 + 2\tilde{S}_2{}^2)\, u\, (\varepsilon + p)_{in}. \tag{35}$$

Hence, it is readily seen that $\varepsilon_{in} \sim \varepsilon_{out}$ for any $\tilde{S}_2^2$, however small, provided $u \sim \tilde{S}_2^2/S_0^0$. However, it is important to note that ε_{in} is not an adequate criterion for estimating whether or not the vacuum is burning. To establish this criterion, we proceed as follows. Let E_{part} be the proportion of the liberated vacuum energy that has been converted into the energy of the internal medium, and let E_{kin} be the kinetic energy of the bubble walls, $E_{\text{kin}} = 4\pi\rho^2 S_0^0(1+\rho^2)^{1/2}$. It is clear that

$$E_{\text{part}}/E_{\text{kin}} = {}^4/_3\rho^3\varepsilon_{out}/E_{\text{kin}} - 1. \tag{36}$$

We shall say that the vacuum is burning if $E_{\text{part}}/E_{\text{kin}} \gtrsim 1$. From (36) and (34) we find that

$$E_{\text{part}}/E_{\text{kin}} = \frac{2\varepsilon_{out}\tilde{S}_2{}^2(u-1) + 3uS_0{}^0\varepsilon_{in}}{3S_0{}^0 u(\varepsilon_{out} - \varepsilon_{in})}. \tag{37}$$

From the relation between ε_{in} and u that follows from (35) under the condition that the equation of state of the internal medium is $p_{in} = v_s{}^2\varepsilon_{in}$, we finally obtain

$$\frac{E_{\text{part}}}{E_{\text{kin}}} = \frac{2\tilde{S}_2{}^2}{3S_0{}^0}\frac{(1-u)(u-v_s{}^2)}{u(1+v_s{}^2)}, \tag{38}$$

where v_s is the velocity of sound. We see that $E_{\text{part}}/E_{\text{kin}}$ vanishes for $u = 1$ and for $u = 1$ and for $u = v_s^2$, and reaches its maximum for $u = v_s$. Thus, for $S_0^0 = \text{const}$ and $\tilde{S}_2^2 = \text{const}$ we find that $E_{\text{part}}/E_{\text{kin}} \sim \tilde{S}_2^2/S_0^0$.

What are the possible values of this ratio?

It may be considered that $\tilde{S}_2^2/S_0^0 \sim 1$ can arise in the realistic field-theory models. Firstly, S_0^0 is determined by the coupling constants of scalar fields, whereas $\tilde{S}_2^2$ is determined by the maximum coupling constant in the model [gauge constant g in Grand Unification Theories (GUT)]. From the point of view of cosmological consequencies, the most interesting models are those that allow considerable supercooling of the metastable phase. On the other hand, this type of supercooling arises in GUT with $\lambda \sim g^4$ (see, for example, Ref. 10). Secondly, in GUT, phase transitions with considerable supercooling occur at temperatures of the order of the reciprocal of the confinement radius in the metastable phase, for which all the coupling constants grow.

Suppose now that the state of the external medium is not a vacuum either. As in the vacuum case, (20) again allows the motion of the shell with $\rho \to \infty$ for $\rho \to \infty$ if

$$v \to \frac{(\varepsilon+p)_{in} - (\varepsilon+p)_{out}}{(\varepsilon+p)_{in} + (\varepsilon+p)_{out}}, \tag{39}$$

where $v = v^1/v^0$ [see (21)].

We have considered the case $S_0^0 \to \text{const}$, $S_2^2 \to \text{const}$. However, apart from the solution given by (39), the shell may move with constant velocity $\rho \to \text{const}$ as $\rho \to \infty$ when $(\varepsilon + p)_{out} \neq 0$. For sufficiently large ρ, this kind of shell enters the detonation-wave regime, whose properties are well-known (see for example Ref. 1). It was suggested in a recent paper[11] that the growth of bubbles in cosmological phase transitions is analogous to the propagation of spherical detonation waves. We now see that the singular shell can actually expand like a detonation wave, and we can identify the conditions under which this can take place. When $(\varepsilon + p)_{out}$ is small, the velocity of the shell is close to, but not equal to, the velocity of light, and the internal medium is characterized by parameters u, ε_{in}, p_{in} satisfying (33). This regime is remarkable in that it should involve the complete conversion of the vacuum energy into the energy of the internal medium independently of S_0^0 and $\tilde{S}_2^2$ (since $\dot{\rho}$ is constant, the liberated vacuum energy cannot be compensated by an increase in the kinetic energy of the wall for large ρ).

V. CONCLUSION

Using the energy-momentum conservation law $T^{\nu}_{\mu;\nu} = 0$ as our starting point, we have shown that vacuum burning is possible in principle. To calculate the magnitude of the effect in a realistic field-theory model, we must investigate the creation of particles by a classical field with time dependent field gradient (expanding bubble wall). The number of created particles and, consequently, the quantity S_2^2, will depend, in particular, on how the wall moves. S_2^2 in its turn, appears in the equation of motion of the shell given by (20). We can thus obtain a closed set of equations by augmenting (20) with the evaluation of S_i^j from (25) and (26) within the framework of field theory.

We have not implemented this program. Nevertheless, we shall suppose that the expansion of the vacuum bubble in realistic Grand Unification Theories is actually analogous to combustion. We may expect, at least, that if the initial temperature inside the bubble is high enough, this value of the temperature will persist during the expansion process. Vacuum burning can lead to interesting cosmological consequences, some of which were discussed in Ref. 5.

We note here that if the vacuum burning effect does actually take place, the initial state of the Universe (intermediate state in the oscillating Universe[12]) could have been a pure vacuum or near-vacuum state. A detailed study of the specific spectrum of the resulting adiabatic inhomogeneities could have shown that either they are "useful" in the theory of the origin of galaxies (see, for example, Ref. 13) or, if they are too large, they should impose restrictions on the parameters characterizing the phase transition with vacuum burning.

The authors are indebted to A. Yu. Ignat'ev, V. A. Matveev, V. A. Rubakov, A. N. Tavkhelidze, and M. E. Shaposhnikov for useful discussions.

[1)]The necessity for the covariant derivative in flat spacetime is connected with the curvature of the separation hypersurface and the existence of a privileged curvilinear coordinate system.

[2)]The generality of (20) is most particularly limited by the fact that we have neglected the reflection of particles into the external medium—a property that a singular shell may have.

[1]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953 [English transl. by Pergamon]; Ya. B. Zel'dovich and Yu. P. Raĭzer, Fizika udarnykh voln i vysokotemperaturnykh yavlenii (Physics of Shock Waves and High-Temperature Phenomena), Fizmatgiz, 1966 (English transl. by Academic Press).
[2]D. A. Kirzhnits, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 755 (1972) [JETP Lett. **15**, 506 (1972)]; D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972).
[3]Ya. B. Zel'dovich and I. Yu. Kobzarev, Zh. Eksp. Teor. Fiz. **67**, 3 (1974) [Sov. Phys. JETP **40**, 1 (1975)].
[4]M. B. Voloshin, I. Yu. Kobzarev, and L. B. Okun', Yad. Fiz. **20**, 1229 (1974) [Sov. J. Nucl. Phys. **20**, 644 (1975)].
[5]V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Preprint ICTP 83/16; Phys. Lett. 124B, 479 (1983).
[6]W. Israel, Nuovo Cimento, **44B**, 1 (1966); **48B**, 463 (1967).
[7]V. A. Berezin, V. A. Kuz'min, and I. I. Tkachev, Phys. Lett. **120**, 91 (1983).
[8]R. F. Sawyer, Phys. Rev. **D24**, 1581 (1981); H. Aoyama, Preprint CALT 68-925 (1982).
[9]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. **D10**, 4130 (1974).
[10]V. A. Kuz'min, M. E. Shaposhnikov, and I. I. Tkachev, Phys. Lett. **102B**, 379 (1981); Z. Phys. **C12**, 83 (1982).
[11]P. J. Steinhardt, Phys. Rev. **D25**, 2074 (1982).
[12]M. A. Markov, in: Teoretiko-gruppovye metody v fizike (Group-theoretic Methods in Physics), Nauka, Moscow, 1983.
[13]Ya. B. Zel'dovich and I. D. Novikov, Stroenie i evolyutsiya Vselennoĭ (Structure and Evolution of the Universe), Nauka, 1975.

Translated by S. Chomet