Effect of Fermi-liquid interaction on a phase transition of order 2½

M. I. Kaganov

Institute of Physics Problems, USSR Academy of Sciences

A. Möbius

Central Institute of Solid State Physics and Materials Research, GDR Academy of Sciences, Dresden

(Submitted 4 July 1983)
Zh. Eksp. Teor. Fiz. **86**, 691–699 (February 1984)

The phase transition of order 2½ [I. M. Lifshitz, Sov. Phys. JETP **11**, 1130 (1959)] is considered with account taken of the Fermi-liquid interaction of the conduction electrons. It is found that although the character of the transition does not differ from that described by Lifshitz, singular parts of the thermodynamic quantities must be renormalized. The electron-system stability, which is ensured by the Pomeranchuk inequalities [I. Ya. Pomeranchuk, Sov. Phys. JETP **8**, 524 (1959)], is not violated by a transition of order 2½.

INTRODUCTION

The phase transition of order 2½ (PT-2½) due to the change of the topology (connectivity) of the Fermi surface (FS) has been attracting interest ever since it was predicted by I. M. Lifshitz in 1960.[1] The experimental discovery of electron-topological phase transitions of metals in elastic strains has apparently stimulated additional developments in PT-2½ theory. Studies devoted to further development of the theory are listed in Ref. 2. We note in addition, two relatively recent papers. Vaks, Trefilov, and Fomichev[3] investigated the behavior of electric resistance and of thermopower in PT-2½, and Lifshitz, Rzhevskiĭ, and Tribel'skiĭ[4] shed light on the character of propagation of nonlinear elastic waves near the transition.

All the papers devoted to PT-2½ theory and to the concomitant phenomena always started out from the "gas" model, assuming that the band structure (the electron dispersion law) does not depend on the electron distribution among the states. All that was taken into account was the shift of the chemical potential ζ, due to the change of the topology (connectivity) of the Fermi surface. The PT-2½ is the result of an external action, say, application of external pressure, that changes the volume of the crystal, and consequently the size of the unit cell. The volume change, however, leads not only to a change of the energy of each state, but to a self-consistent redistribution of the electrons among the states, and this in turn, according to the premises of Fermi-liquid theory, alters the energy of each state. The present paper is devoted to an elucidation of how this circumstance influences the PT-2½. We shall show that theFermi-liquid interaction does not alter the character of the phase transition, but particular parts of the thermodynamic quantities become renormalized. The requirement that the electron subsystem be stable imposes definite conditions on the Landau matrix that describes the Fermi-liquid interaction (Pomeranchuk[5]). If it is assumed that these conditions are satisfied for most electrons, the change of the connectivity of the FS cannot lead to violation of these conditions. The electron redistribution, whereby part of the electrons leaves the critical region (or enters it) does not disturb the stability, since the Fermi-gas term (which must be positive!) increases more rapidly than the Fermi-liquid term. This means that the Fermi-liquid interaction does not "disrupt" the PT-2½. There is no apparent reason why PT-2½ must of necessity be preceded by a first-order transition.

1. MODEL. INITIAL EQUATIONS

Consider a system of N electrons that interact with one another and are located in a field of N_i ion periodically arranged in space. On the whole, the system is of course electrically neutral since the sum of the charges of the electrons and ions is zero ($N = z_i N_i$, where z_i is the ion charge).

A change ΔV of the volume V of the crystal changes the volume of the reciprocal cell (of the first Brillouin zone) and as a consequence changes the quasimomentum of each state (the number of states in the zone, which is determined by the number of crystal cells, is of course preserved). If it is assumed that the relative change of the crystal edges is the same (isotropy), the change of the quasimomentum $\mathbf{p_m}$ of the state $\mathbf{m}$ is

$$\Delta \mathbf{p_m} = -\frac{1}{3}\mathbf{p_m}\frac{\Delta V}{V}, \tag{1}$$

and the change of the energy $\varepsilon_\mathbf{m}$ of this state is

$$\Delta \varepsilon_\mathbf{m} = g_\mathbf{m}\frac{\Delta V}{V} + \frac{1}{N_B}\sum_{\mathbf{m}'} f_{\mathbf{m},\mathbf{m}'}\,\Delta n_{\mathbf{m}'}. \tag{2}$$

The vector $\mathbf{m}$ numbers here the states:

$$p_x = (2\pi\hbar/L_x)\,m_x, \quad p_y = (2\pi\hbar/L_y)\,m_y,$$
$$p_z = (2\pi\hbar/L_z)\,m_z,$$

$\mathbf{m}_{x,y,z}$ are integers, and $L_{x,y,z}$ are the linear dimensions of the crystal. The first term describes the change of the energy of the state $\mathbf{m}$ as a result of the change of the quasimomentum and potential of the lattice (the lattice potential is changed because the volume change is accompanied by a change in the distances between the ions); the remaining terms (sum over $\mathbf{m}'$) are due to the redistribution of the electrons among the states and are described by the Fermi-liquid effect; $f_{\mathbf{m},\mathbf{m}'}$ is the symmetric Landau correlation function (see Ref. 6),

$\Delta n_{\mathbf{m}}$ is the change of the average occupation numbers, and N_B is the total number of states in the zone. Equation (2) contains no small parameters (other than $\Delta V/V$), and the quantities $g_{\mathbf{m}}$ and $f_{\mathbf{m},\mathbf{m}'}$ in it are of the order of the electron energy on the Fermi surface. Expression (2) formulates the model assumed here for the analysis. The model character of the approach consists in failure to establish the connection between the functions $\varepsilon_{\mathbf{m}}$, $g_{\mathbf{m}}$, and $f_{\mathbf{m},\mathbf{m}'}$, a connection that must exist, inasmuch as for any choice of V the metal is described by a definite dispersion law $\varepsilon_{\mathbf{m}}$ and by a definite Landau matrix $f_{\mathbf{m},\mathbf{m}'}$ (see § 4). The connection can be obtained in principle by considering the deformational interaction between the electrons and the lattice, but this question is not raised here.

The change of the occupation number is determined by the changes of the energy, of the dynamic potential ζ, and of the temperature T:

$$\Delta n_{\mathbf{m}}=\frac{\partial n}{\partial \varepsilon}\left[\Delta\varepsilon_{\mathbf{m}}-\Delta\zeta-(\varepsilon_{\mathbf{m}}-\zeta)\frac{\Delta T}{T}\right]. \tag{3}$$

The electroneutrality conditions

$$\Delta N\equiv\sum_{\mathbf{m}}\Delta n_{\mathbf{m}}=0 \tag{4}$$

relate ΔV, $\Delta\zeta$, and ΔT and makes it possible to calculate $(\partial\zeta/\partial V)_T$ and $(\partial\zeta/\partial T)_V$ from (2) and (3).

We shall need several thermodynamic relations[7]:

$$\frac{\partial P}{\partial V}=\rho\frac{\partial\zeta}{\partial V},\quad \rho=\frac{N}{V}; \tag{5}$$

$$\frac{\partial P}{\partial T}=\rho\left(\frac{\partial\zeta}{\partial T}+s\right),\quad s=\frac{S}{V}; \tag{6}$$

$$C_V/T=\partial S/\partial T. \tag{7}$$

Here P is the pressure while S and $C_V=Nc_V$ are the entropy and heat capacity of the metal electrons:

$$S={}^1/_3\pi^2\nu(\varepsilon_F)T, \tag{8}$$

where $\nu(\varepsilon_F)$ is the density of the electronic states on the Fermi surface. According to the PT-$2\frac{1}{2}$ theory,[1] when the connectivity of the FS is changed the density of states $\nu(\varepsilon_F)$ acquires a nonanalytic increment $\Delta\nu(z)$ that differs from zero from that side of the transition where the number of FS openings is larger. In the general case

$$\Delta\nu(z)=\alpha|z|^{1/2}. \tag{9}$$

The parameter $z=\zeta-\varepsilon_{\text{cr}}$, the distance from the chemical potential ζ to that energy ε_{cr} at which the topology of the equal-energy surfaces changes, describes the proximity to the PT-$2\frac{1}{2}$:

$$\alpha=(2^{1/2}V/\pi^2\hbar^3)\,|m_1^*m_2^*m_3^*|^{1/2},$$

where m_i^* are the effective masses, $i=1,2,3$ (when a new opening is formed, all the effective masses are of like sign, and when a neck is broken the sign of one effective mass differs from that of the two others.) In the "gas" theory[1] of the PT-$2\frac{1}{2}$, the equations that describe the anomalies of the thermodynamic characteristic contain a factor α and a phenomenological parameter γ that connects the change of z with the change of the volume

$$z=\gamma(V-V_{\text{cr}})/V_{\text{cr}}.$$

The approach employed here determines the parameter γ.

To transform in (2) and (4) from summation to integration it is convenient to make the quasimomentum dimensionless ($\mathbf{p}\to\tilde{\mathbf{p}}$, where $\tilde{p}_x=p_xa_x/2\pi\hbar$ etc); the volume of the first Brillouin zone is then unity, the electron state is uniquely determined by the quasimomentum $\tilde{\mathbf{p}}$, and $\mathbf{p}$ is independent of volume. Leaving out the tilde over $\mathbf{p}$, we have from (2)–(4)

$$\Delta\varepsilon(\mathbf{p})=g(\mathbf{p})\frac{\Delta V}{V}+\int d^3p'f(\mathbf{p},\mathbf{p}')\Delta n(\mathbf{p}'), \tag{10}$$

$$\Delta n(\mathbf{p})=\frac{\partial n}{\partial\varepsilon}\left[\Delta\varepsilon(\mathbf{p})-\Delta\zeta-(\varepsilon(\mathbf{p})-\zeta)\frac{\Delta T}{T}\right], \tag{11}$$

$$\Delta N\equiv N_B\int\Delta n(\mathbf{p})\,d^3p=0. \tag{12}$$

The integration is over the first Brillouin zone. We note that the elements of the symmetric matrix $f(\mathbf{p},\mathbf{p}')$ in (10), just as in (2), have the dimension of energy, so that they differ from the customary ones by a dimensional factor (cf. Ref. 6).

2. ABSOLUTE ZERO TEMPERATURE ($T=0$)

Strictly speaking, the change of the FS topology leads to a PT-$2\frac{1}{2}$ only at absolute zero temperature ($T=0$). At a finite temperature all the thermodynamic quantities are continuous, although they do have abrupt anomalies.[1] It is therefore natural to investigate in detail the case of zero temperature.

According to (5), to determine a singularity of the compressibility it is necessary to calculate with the aid of (10)–(12) the singularity of $\partial\zeta/\partial V$. We carry out the calculations first with the Fermi-liquid interaction neglected, i.e., we repeat the results of Ref. 1 within the framework of the model (10)–(12). For the sake of clarity we confine ourselves to the case when a new opening is produced at $V=V_{\text{cr}}$. Thus, at $V\neq V_{\text{cr}}$ the FS has besides the main opening also a small (anomalous) one, in the general case an ellipsoid whose equation is

$$\zeta(V)-\varepsilon_{\text{cr}}(V)=\sum_{i=1}^{3}\frac{p_i^2}{2m_i(V)}. \tag{13}$$

The quasimomentum is reckoned here from the point $\mathbf{p}=\mathbf{p}_{\text{cr}}$, at which a new opening is generated. The critical value V_{cr} of the volume is given by

$$\zeta(V_{\text{cr}})=\varepsilon_{\text{cr}}(V_{\text{cr}}). \tag{14}$$

It is convenient to reformulate the electroneutrality condition (12) as the condition that the volume of the total FS be independent of V:

$$\Omega_{\text{main}}(V)+\Omega_{\text{an}}(V)=\Omega_{\text{main}}(V_{\text{cr}}); \tag{15}$$

where $\Omega_{\text{main}}(V)$ is the volume of the main opening of the FS, and Ω_{an} that of the produced anomalous one. Taking (11) and (12) into account, we can expand $\Omega_{\text{main}}(V)$ in powers of ΔV and, using (13), obtain an equation from which we can determine $\partial\zeta/\partial V$ by successive approximations with allowance for the anomalous term:

$$0=\oint\limits_{(\mathrm{main})}\frac{dS}{v}\Delta\zeta-\oint\limits_{(\mathrm{main})}g(\mathbf{p})\frac{dS}{v}\frac{\Delta V}{V}+\frac{4\pi}{3}[2m^*(V)]^{3/2}[\zeta(V)-\varepsilon_{\mathrm{cr}}(V)]^{3/2},$$

$$m^*=(m_1^*m_2^*m_3^*)^{1/3}. \tag{15'}$$

The integration is over the main opening of the FS at $V=V_{\mathrm{cr}}$, dS is the surface area element, and $\mathbf{v}=\partial\varepsilon/\partial\mathbf{p}$. Thus,

$$\frac{\partial\zeta}{\partial V}=\left(\frac{\partial\zeta}{\partial V}\right)_{\mathrm{cr}}+\Delta\frac{\partial\zeta}{\partial V}, \tag{16}$$

$$\left(\frac{\partial\zeta}{\partial V}\right)_{\mathrm{cr}}=\frac{\langle g\rangle}{V_{\mathrm{cr}}},\quad \langle g\rangle=\oint\limits_{(\mathrm{main})}g(\mathbf{p})\frac{dS}{v}\Big/\oint\limits_{(\mathrm{main})}\frac{dS}{v}, \tag{16'}$$

$$\Delta\frac{\partial\zeta}{\partial V}=-\frac{2\pi[2m^*(V_{\mathrm{cr}})]^{3/2}}{V_{\mathrm{cr}}\oint\limits_{(\mathrm{main})}(dS/v)}\gamma^{3/2}\left(\frac{\Delta V}{V_{\mathrm{cr}}}\right)^{1/2},$$

$$\gamma=\langle g\rangle-g_{\mathrm{cr}},\quad g_{\mathrm{cr}}=V_{\mathrm{cr}}\left[\frac{\partial\varepsilon_{\mathrm{cr}}(V)}{\partial V}\right]_{V=V_{\mathrm{cr}}}. \tag{16''}$$

As already mentioned, the employed model (2) determines the parameter and refines thereby the condition for the PT-$2\frac{1}{2}$, viz., the transition takes place at $\gamma\Delta V=(\langle g\rangle-g_{\mathrm{cr}})\Delta V>0$. The singular part of $\Delta\partial\zeta/\partial V$ can naturally be related to the singular part of the density of states (9):

$$V_{\mathrm{cr}}\Delta\frac{\partial\zeta}{\partial V}=-(\langle g\rangle-g_{\mathrm{cr}})\frac{\Delta\nu}{\nu_{\mathrm{cr}}},$$

$$\nu_{\mathrm{cr}}=\frac{2V_{\mathrm{cr}}}{(2\pi\hbar)^3}\oint\limits_{(\mathrm{main})}\frac{dS}{v}, \tag{17}$$

$$\Delta\nu=\frac{V_{\mathrm{cr}}[2m^*(V_{\mathrm{cr}})]^{3/2}}{2\pi^2\hbar^3}[\zeta(V)-\varepsilon_{\mathrm{cr}}(V)]^{1/2}.$$

Let us stress a number of circumstances: 1) Eqs. (16) and (17) contain the difference $\langle g\rangle-g_{\mathrm{cr}}$; this means that the point from which the electron energy is reckoned is immaterial; 2) although the effective masses in (13) depend on V, only the volume dependence of the critical energy $\varepsilon_{\mathrm{cr}}$ is substantial in the approximation employed; 3) finally, if the PT-$2\frac{1}{2}$ is caused by a break of the neck, Eq. (16) remains in force (when determining m^* it must be recognized that sgn $m_1^*m_2^*m_3^*<0$, i.e.,

$$m^*=|m_1^*m_2^*m_3^*|^{1/3}).$$

The purpose of the present section is to show how the coefficients in (16) and (17) are renormalized on account of the Fermi-liquid interaction (the dependence on $\Delta V/V$ is preserved). To this end we analyze anew the consequences of (16), taking into account relations (10) and (12) which (we emphasize this) are defined at all points of **p**-space (and particularly near the place where the new opening is formed). The main technical complication that arises here is that the term of (15) which is due to the main FS opening also contains an anomalous part (the matrix $f(\mathbf{p},\mathbf{p}')$ leads to a connection between different points of **p**-space, the usual ones and those close to $\mathbf{p}_{\mathrm{cr}}$). As is customary in Fermi-liquid theory, it is convenient to work with infinitely small changes (with the differential δV and $\delta\zeta$) rather than with finite increments ($\Delta V,\Delta\zeta$). According to (15),

$$\delta\Omega_{\mathrm{main}}+\delta\Omega_{\mathrm{an}}=0. \tag{15''}$$

It is convenient to introduce the function $\delta\tilde{\xi}=\delta\tilde{\xi}(\hat{\mathbf{p}},\zeta)$, specified by the equality

$$\delta n(\hat{\mathbf{p}},\zeta)=-\frac{\partial n}{\partial\varepsilon}\delta\tilde{\xi}(\hat{\mathbf{p}},\zeta), \tag{18}$$

which describes the change of the FS with change of volume ($\hat{\mathbf{p}}$ is a two-dimensional coordinate on the FS). The change of variables $\hat{\mathbf{p}}$, $\zeta\to\mathbf{p}$ introduce a new infinitely small function $\delta\xi(\mathbf{p})=\delta\tilde{\xi}(\hat{\mathbf{p}},\zeta)$ that defines at each point of **p**-space a shift equal to $\delta\xi(\mathbf{p})\mathbf{v}/v^2$. The shift has a physical meaning only when the point **p** is on the FS. From (10), (11), and (18) we obtain the integral equation

$$\delta\xi(\mathbf{p})+\oint\frac{dS'}{v'}f(\mathbf{p},\mathbf{p}')\delta\xi(\mathbf{p}')=-g(\mathbf{p})\frac{\delta V}{V}+\delta\zeta, \tag{19}$$

which is conveniently solved by introducing two functions $u(\mathbf{p})$ and $w(\mathbf{p})$ that satisfy the following equations:

$$u(\mathbf{p})+\oint\frac{dS'}{v'}f(\mathbf{p},\mathbf{p}')u(\mathbf{p}')=-g(\mathbf{p}),$$
$$w(\mathbf{p})+\oint\frac{dS'}{v'}f(\mathbf{p},\mathbf{p}')w(\mathbf{p}')=1. \tag{20}$$

Then

$$\delta\xi(\mathbf{p})=u(\mathbf{p})\delta V/V+w(\mathbf{p})\delta\zeta. \tag{21}$$

When integrating in (19) and (20) we must take into account the formation of a FS opening whose dimensions depend on $\delta V/V_{\mathrm{cr}}$. As a result of this the functions $u(\mathbf{p})$ and $w(\mathbf{p})$ contain nonanalytic terms

$$u_1(\mathbf{p}),\ w_1(\mathbf{p})\propto(\delta V/V_{\mathrm{cr}})^{1/2},$$

even if **p** belongs to the surface of the main zone

$$u(\mathbf{p})=u_0(\mathbf{p})+u_1(\mathbf{p}),\quad w(\mathbf{p})=w_0(\mathbf{p})+w_1(\mathbf{p}). \tag{22}$$

Actually, separating in (20) the interaction over the surface of the newly produced zone, we have

$$\left|\begin{matrix}-g(\mathbf{p})\\1\end{matrix}\right|=\left|\begin{matrix}u(\mathbf{p})\\w(\mathbf{p})\end{matrix}\right|+\oint\limits_{(\mathrm{main})}f(\mathbf{p},\mathbf{p}')\left|\begin{matrix}u(\mathbf{p}')\\w(\mathbf{p}')\end{matrix}\right|\frac{dS'}{v'}+\int\limits_{(\mathrm{an})}f(\mathbf{p},\mathbf{p}')\left|\begin{matrix}u(\mathbf{p}')\\w(\mathbf{p}')\end{matrix}\right|\frac{dS'}{v'}. \tag{23}$$

Hence

$$\left|\begin{matrix}u_0(\mathbf{p})\\w_0(\mathbf{p})\end{matrix}\right|+\oint\limits_{(\mathrm{main})}\frac{dS'}{v'}f(\mathbf{p},\mathbf{p}')\left|\begin{matrix}u_0(\mathbf{p}')\\w_0(\mathbf{p}')\end{matrix}\right|=\left|\begin{matrix}-g(\mathbf{p})\\1\end{matrix}\right|, \tag{24}$$

and

$$\left|\begin{matrix}u_1(\mathbf{p})\\w_1(\mathbf{p})\end{matrix}\right|+\oint\limits_{(\mathrm{main})}\frac{dS'}{v'}f(\mathbf{p},\mathbf{p}')\left|\begin{matrix}u_1(\mathbf{p}')\\w_1(\mathbf{p}')\end{matrix}\right|$$
$$=-f(\mathbf{p},\mathbf{p}_{\mathrm{cr}})2\pi[2m^*(V_{\mathrm{cr}})]^{3/2}[\zeta(V)-\varepsilon_{\mathrm{cr}}(V)]^{1/2}\left|\begin{matrix}u_0(\mathbf{p}_{\mathrm{cr}})\\w_0(\mathbf{p}_{\mathrm{cr}})\end{matrix}\right|. \tag{25}$$

The matrix element $f(\mathbf{p},\mathbf{p}_{\mathrm{cr}})$ contained in (25) connects **p**-

space points belonging to different openings and has in general no special small quantity.

We write the formal solution of Eqs. (24) and (25) by introducing an operator $\widetilde{G}$ defined with the aid of the main opening of the FS

$$u_0(\mathbf{p})=-\hat{G}\{g(\mathbf{p})\}, \quad w_0(\mathbf{p})=\hat{G}\{1\} \tag{26}$$

and

$$\begin{vmatrix} u_1(\mathbf{p}) \\ w_1(\mathbf{p}) \end{vmatrix} = - \begin{vmatrix} u_0(\mathbf{p}_{\rm cr}) \\ w_0(\mathbf{p}_{\rm cr}) \end{vmatrix} 2\pi[2m^*(V_{\rm cr})]^{3/2} \times[\zeta(V)-\varepsilon_{\rm cr}(V)]^{1/2}\hat{G}\{f(\mathbf{p},\mathbf{p}_{\rm cr})\}. \tag{27}$$

The factors $u_0(\mathbf{p}_{\rm cr})$ and $w_0(\mathbf{p}_{\rm cr})$ are determined directly from (24) and (26) at $\mathbf{p}=\mathbf{p}_{\rm cr}$.

We now specify concretely the differential electroneutrality condition (15″):

$$\delta\Omega_{\rm main}=\oint\limits_{(\rm main)}\frac{dS}{v}u_0(\mathbf{p})\frac{\delta V}{V_{\rm cr}}+\oint\limits_{(\rm main)}\frac{dS}{v}w_0(\mathbf{p})\delta\zeta + \oint\limits_{(\rm main)}\frac{dS}{v}u_1(\mathbf{p})\frac{\delta V}{V_{\rm cr}}+\oint\limits_{(\rm main)}\frac{dS}{v}w_1(\mathbf{p})\,d\zeta, \tag{28}$$

$$\delta\Omega_{\rm an}=2\pi[2m^*(V_{\rm cr})]^{3/2}[\zeta(V)-\varepsilon_{\rm cr}(V)]^{1/2}(\delta\zeta-\delta\varepsilon_{\rm cr}). \tag{29}$$

Leaving out the anomalous terms of (28), we obtain from it the Fermi-liquid generalization (16′):

$$\left(\frac{\partial\zeta}{\partial V}\right)_{\rm cr}=-\frac{1}{V_{\rm cr}}\oint\limits_{(\rm main)}\frac{dS}{v}u_0(\mathbf{p})\Big/\oint\limits_{(\rm main)}\frac{dS}{v}w_0(\mathbf{p}) \equiv-\frac{1}{V_{\rm cr}}\frac{\langle u_0\rangle}{\langle w_0\rangle}, \tag{30}$$

which in turn makes it possible to determine the structure of the argument of the anomalous term:

$$\delta\zeta-\delta\varepsilon_{\rm cr}=\gamma_{\rm FL}\frac{\delta V}{V_{\rm cr}}, \quad \gamma_{\rm FL}=u_0(\mathbf{p}_{\rm cr})-w_0(\mathbf{p}_{\rm cr})\frac{\langle u_0\rangle}{\langle w_0\rangle}. \tag{31}$$

If we neglect the Fermi-liquid interaction, we have

$$u_0(\mathbf{p}_{\rm cr})\to-g_{\rm cr}, \quad u_0(\mathbf{p})\to-g(\mathbf{p}),$$
$$w_0(\mathbf{p})\to1, \quad \gamma_{\rm FL}\to\langle g\rangle-g_{\rm cr}$$

(see above). Expression (27) contains the integral

$$I=\oint\limits_{(\rm main)}\frac{dS}{v}\hat{G}\{f(\mathbf{p},\mathbf{p}_{\rm cr})\}.$$

Owing to the symmetry of the matrix $f(\mathbf{p},\mathbf{p}')$, the operator $\hat{G}$ is self-adjoint. Therefore

$$I=\oint\limits_{(\rm main)}\frac{dS}{v}f(\mathbf{p},\mathbf{p}_{\text{кр}})G\{1\} =\oint\limits_{(\rm main)}\frac{dS}{v}f(\mathbf{p},\mathbf{p}_{\rm cr})w_0(\mathbf{p})=1-w_0(\mathbf{p}_{\rm cr}).$$

The last equation was obtained by permuting the arguments of $f(\mathbf{p},\mathbf{p}_{\rm cr})$.

We now gather all the anomalous terms of (15″). With the aid of the preceding formula and (31) they can be written in the form

$$2\pi[2m^*(V_{\rm cr})]^{3/2}\left(\gamma_{\rm FL}\frac{\delta V}{V_{\rm cr}}\right)^{1/2}w_0(\mathbf{p}_{\rm cr})\gamma_{\rm FL}\frac{\delta V}{V_{\rm cr}};$$

whence

$$\Delta\frac{\partial\zeta}{\partial V}=-\frac{2\pi[2m^*(V_{\rm cr})]^{3/2}}{V_{\rm cr}\oint\limits_{(\rm main)}\frac{dS}{v}w_0(\mathbf{p})}\gamma_{\rm FL}^{3/2}w_0(\mathbf{p}_{\rm cr})\left(\frac{\delta V}{V_{\rm cr}}\right)^{1/2}. \tag{32}$$

We see that the Fermi-liquid interaction renormalizes both the parameter γ and the coefficient preceding $\gamma^{3/2}(\delta V/V_{\rm cr})^{1/2}$. An important role in the renormalization is played by the circumstance that the change of the shape of the main FS depends on the parameters of the Fermi-liquid interaction. Since the singular part of $\partial\zeta/\partial V$ depends in essence on the anomaly of the density of states, the result is not changed if the PT-$2\frac{1}{2}$ is produced not via formation of a new zone, but via breaking of a neck (in this case $m^*=|m_1m_2m_3|^{1/3}$).

We have purposely gone into so much detail in the derivation, to show the following: knowledge of the dispersion law at $V=V_{\rm cr}$, of $g(p)$ meaning also of $g_{\rm cr}$, of the Landau matrix, as well as of the solution of Eqs. (24) makes possible calculation of all the expressions in (32) (the values of $u_0(\mathbf{p})$ and $W_0(\mathbf{p})$ are obtained by integrating with over the FS at $V=V_{\rm cr}$).

As seen from a comparison of (32) and (16), the character of the transition remains unchanged. With the aid of (5) we easily obtain the singular part of the compressibility

$$\Delta\frac{\partial P}{\partial V}=\left\{-2\pi\rho[2m^*(V_{\rm cr})\gamma_{\rm FL}]^{3/2}w_0(\mathbf{p}_{\rm cr})/V_{\rm cr}\oint\limits_{(\rm main)}\frac{dS}{v}w_0(\mathbf{p})\right\} \times\left(\frac{\delta V}{V}\right)^{1/2}. \tag{33}$$

Owing to the factors that include $w_0(p)$, a definite sign of $\Delta\partial P/\partial V$ is no longer an inevitable consequence of the positiveness of the anomalous part of the density of states (9).

3. FINITE TEMPERATURE ($T\neq0$)

At a finite temperature the singularities due to the changes of the connectivity of the FS become smeared out, but the existence of a critical chemical potential or proximity to it is felt, naturally also in the temperature dependence of the free energy. At $|z|\gg T$, when the temperature plays the least role, the electrons of the metals constitute the usual Fermi liquid—in the sense that the temperature-induced increment to the free energy from a large number of FS openings is proportional[1)] to $(T/|z|^2)$; this, naturally, can be deduced also from an analysis of Eqs. (11) and (12).

To calculate the temperature dependence of the singular part of a thermodynamic quantity it is necessary to generalize expression (18) and (21) as follows:

$$\delta n(\hat{\mathbf{p}},\zeta)=-\frac{\partial n}{\partial\varepsilon}\sum_q\delta\tilde{\xi}_q(\hat{\mathbf{p}},\zeta)(\varepsilon-\zeta)^q,$$
$$\delta\xi_q=u_q\frac{\delta V}{V}+w_q\delta\zeta+\zeta_q\delta T. \tag{34}$$

Using an expansion of a system of integral equations similar to (20) in powers of T/z, as well as Eqs. (6) and (7), we obtain expressions for $\Delta\ (\partial P/\partial T)$ and ΔC_V. The normalization of the latter, due to the Fermi-liquid interaction, coincides with the renormalization of $\Delta \partial P/\partial V$ (cf. (33) with Eq. (13.10) of Ref. 7). The quantities $(\partial P/\partial T)_V$ and $\Delta C_V/T$ are the most suitable for the analysis of the experimental data.

At those values of z at which the number of openings is smaller, the anomalous part of the temperature increment is exponentially small and the principal role in the temperature dependence of F is assumed by the term due to the main opening of the FS.

At relatively high temperatures ($|z| \ll T$), the most interesting case is $z = 0$, i.e., $\varepsilon_F(V_{\rm cr}) = \varepsilon(V_{\rm cr})$. The existence of a critical point $\mathbf{p}_{\rm cr}$ (a conical point on the main FS or a point at which a new FS opening is created) manifests itself in the fact that the temperature dependence of F contains terms with fractional powers of the temperature. As a result, in particular, the linear term of the expansion of the electronic heat capacity in terms of temperatures should be proportional to $T^{3/2}$ (and hot T^3 as in the usual case). This fact is not connected directly with the Fermi-surface interaction between the electrons. It takes place also in the gas approximation.[8]

4. CONCERNING THE STABILITY OF THE FS NEAR A TRANSITION OF ORDER 2½

In the phenomenological approach used here it is strictly speaking impossible to examine the stability of the FS. Indeed, the entire approach is based on the fact that at each value of the volume V (particularly also at $V = V_{\rm cr}$) there exists a stable structure [having a dispersion $\varepsilon(\mathbf{p})$ and a correlation Landau matrix $f(\mathbf{p},\mathbf{p}')$] that generates a definite FS at a fixed value of the number of electrons. In such an approach, of course, it would be necessary to spell out the conditions (that generalize Pomeranchuk's results5)) imposed on the matrix $f(\mathbf{p},\mathbf{p}')$ at an arbitrary shape of the FS. In the general case, to our knowledge, no such conditions were derived. We repeat, however, that we must assume that these (unknown) conditions hold. There remains still the following question: does not the approach to the PT-2½ favor the violation of the stability conditions? Or more accurately: should not the stability conditions be certainly violated as $V \to V_{\rm cr}$? It can be verified that they should not. Indeed, the stability conditions stem from the requirement that the energy increment δE of the Fermi-particle system be positive when the FS is deformed (we confine ourselves, as in Ref. 5, to $T = 0$):

$$\delta E = \int \varepsilon(\mathbf{p})\,\delta n(\mathbf{p}) \frac{2d^3p}{(2\pi\hbar)^3} + \frac{1}{2} \iint f(\mathbf{p},\mathbf{p}')\,\delta n(\mathbf{p})\,\delta n(\mathbf{p}') \frac{4d^3p\,d^3p'}{(2\pi\hbar)^6}, \qquad (35)$$

where $\delta n(\mathbf{p}) = -1$ at those $\mathbf{p}$-space regions from which the electrons "departed," and $\delta n(\mathbf{p}) = +1$ in those regions in which they "arrived." Assuming the deformation on the FS to be small, the displacement of the FS can be represented in the form $\vartheta\chi(\hat{\mathbf{p}})$, where $\vartheta \ll 1$ and $\chi(\hat{\mathbf{p}})$ is a function specified on the FS and satisfying the neutrality condition

$$\oint \frac{dS}{v} \chi(\hat{\mathbf{p}}) = 0. \qquad (36)$$

The functional (35) can be recast in the form 2)

$$\delta E = \frac{\vartheta^2}{2} \left\{ \oint \frac{dS}{v} \chi^2(\hat{\mathbf{p}}) + \frac{1}{2} \oint\oint \frac{dS}{v} \frac{dS'}{v'} f(\hat{\mathbf{p}},\hat{\mathbf{p}}')\chi(\hat{\mathbf{p}})\chi(\hat{\mathbf{p}}') \right\}. \qquad (37)$$

If $|\chi(\hat{\mathbf{p}})| \sim 1$, the existence of an anomalous zone (or of an anomalously small distance between the FS openings in the case of neck breaking) cannot change the sign of the inequality, since the contribution of the critical region (near $\mathbf{p}_{\rm cr}$) is small ($\propto |z|\,1/2$), and the conditions imposed on $f(\mathbf{p},\mathbf{p}')$ do not contain a special small quantity.

There may be danger in functions $\chi(\hat{\mathbf{p}})$ for which the neutrality condition (34) is satisfied on account of a redistribution of the electrons between the main and anomalous zones, so that $\chi_{\rm an} \propto |z|^{-1/2}$. It can be seen, however, that in this case the first (gas) term in (37), which is certainly positive, diverges like $|z|^{-1/2}$, while the second (Fermi-liquid) term remains finite. Consequently such a perturbation can likewise not upset the stability. Of course, the entire reasoning is valid if the matrix $f(\mathbf{p},\mathbf{p}')$ has no singularities at $V = V_{\rm cr}$ (see above). An analogy can be discerned in this analysis with that of the singularity of the conductivity in a PT-2½ (Ref. 3): the mean free path for scattering by defects contains an irregular term $\Delta\ 1 \infty |z|^{1/2}$ due to the singularity of the density of states (the singularity of the matrix elements of the transition can be disregarded here).

In conclusion, we take the opportunity to thank I. M. Lifshitz, L. Pitaevskiĭ, and H. Eschrig for helpful advice, as well as G. Voigt for constant interest in the work.

In addition, one of us (A.M.) is grateful for hospitality to Academician P. L. Kapitza and to the Institute of Physics Problems, where this work was performed.

1) As always, the denominator of the expansion in temperature contains the distance from ε_F to the nearest critical energy $\varepsilon_{\rm cr}$, i.e., $\varepsilon_F - \varepsilon_{\rm cr} = z$.

2) The Landau function is normalized in a different manner in Eqs. (35) and (37) (see end of §1).

---

[1] I. M. Lifshitz, Zh. Eksp. Teor. Fiz. **38**, 1569 (1960) [Sov. Phys. JETP **11**, 130 (1960)].

[2] M. I. Kaganov and I. M. Lifshitz, Usp. Fiz. Nauk **129**, 487 (1979) [Sov. Phys. Usp. **22**, 904 (1979)].

[3] V. G.Vaks, P. V. Trefilov, and V. S. Fomichev, Zh. Eksp. Teor. Fiz. **80**, 1613 (1981) [Sov. Phys. **53**, 830 (1981)].

[4] I. M.Lifshitz, V. V. Rzhevskiĭ, and M. I. Tribel'skii, *ibid.* **81**, 1528 (1981) [**54**, 810 (1981)].

[5] I. Ya. Pomeranchuk, *ibid.* **35**, 524 (1958) [**8**, 361 (1959)]; see also: Collected Works [in Russian], Nuka, Vol. 1, 1972.

[6] L. D. Landau and E. M. Lifshitz, Statistical Physics, Pergamon, 1980, Vol. 2, Chap. 1. See also L. D. Landau, Collected Works Pergamon, Vol. 2.

[7] I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, Electron Theory of Metals, Cons. Bureau, 1973.

[8] S. S. Nedorezov, Zh. Eksp. Teor. Fiz. **51**, 858, 1575 (1966) [Sov. Phys. JETP **24**, 572, (1967)].

Translated by J. G. Adashko