Contribution to the theory of the quantum Hall effect in a quasi-two-dimensional electron + impurity system

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The principal relations of the quantum Hall effect in a quasi-two-dimensional electron system are studied in the single-electron approximation on the basis of the Kubo formula. A modified Kubo formula and several useful sum rules are derived directly. It is shown that if the density of states of the system has gaps no correction need be made to the conductivity of completely filled Landau levels in any finite order of perturbation theory. This leads to an exact quantization of the Hall conductivity of filled Landau levels. The existence of equidistant plateaus on the plot of the Hall conductivity vs density or magnetic field follows from the presence of localized electronic states on the lower edge of the broadened Landau level and of localized hole state on the upper edge of this level. The feasibility of experimentally isolating multielectron-correlation effects is discussed.

§1. INTRODUCTION

The quantum Hall effect (QHE) is now attracting increasing interest as a new pronounced manifestation of quantum relations on a macroscopic level. Isolated important features of the QHE were investigated in Refs. 2–6 in the single-electron approximation, but no coherent theoretical analysis of the principal experimental facts was presented so far. In the present paper we also confine ourselves to the single-particle approximation, for it provides a satisfactory explanation of the basic relations in the QHE. In addition, it is necessary to study all the consequences of the single-electron approximation in order to be able to distinguish in an experiment the possible effects of interelectron Coulomb interaction. Some of the recently obtained data[7–11] do not seem to fit the single-electron scheme. To interpret them it is necessary to take into account the possibility of formation, in a quasi-two-dimensional electron system, of a Wigner lattice or of a charge-density wave interacting with disordered impurity centers.[12,13] By confining ourselves to the single-electron approximation we also avoid dealing with details connected with spin and valley splitting of the Landau levels.

To understand more clearly the unusual features of the QHE, we compare the experimental behavior of the Hall conductivity component σ_{xy} of a quasi-two-dimensional electron system,[1,7,9] shown schematically in Fig. 1, with the Hall component, shown in the same figure, of an ideal quasi-two-dimensional system, $|\sigma_{xy}^{(0)}| = ecN_e/BS$, where N_e is the number of electrons in the system, B is the magnetic field strength, and S is the sample area. If in an ideal system we fill with electrons an integer number p of Landau levels, so that the electron density is $n_e = pn_0$, where $n_0 = 1/2\pi\lambda^2$ is the electron density on one filled Landau level and $\lambda = (c\hbar/eB)^{1/2}$ is the magnetic length, the Hall conductivity of the ideal system turns out to be a multiple of the universal constant $e^2/2\pi\hbar$:

$$\sigma_{xy}^{(0)}(p) = \frac{ec}{B} p \frac{1}{2\pi\lambda^2} = p\frac{e^2}{2\pi\hbar}. \tag{1}$$

In an ideal system relation (1) is satisfied only at discrete points $n_e = pn_0$. In described experiments, on the contrary, the correction $\Delta\sigma_{xy} = \sigma_{xy} - \sigma_{xy}^{(0)}(p)$ to the conductivity σ_{xy} of a real quasi-two-dimensional system turns out to be extremely small ($|\Delta\sigma_{xy}/\sigma_{xy}^{(0)}(p)| < 10^{-6}$ according to the data of Ref. 14) not only at the points pn_0, but also in the entire region of electron densities near these points. This gives a series of equidistant plateaus on the plot of σ_{xy} vs the density in a constant magnetic field (Fig. 1). The width of each plateau, i.e., the region Δn_e of densities near $n_e = pn_0$ where $\sigma_{xy}(n_e) \approx \sigma_{xy}^{(0)}(p)$, is a steep function of the temperature.

With decreasing temperature the plateaus can broaden on both sides of the point pn_0. This important experimental fact means that for values $n_e \gtrsim pn_0$ the system conductivity is less than the ideal one at the same electron densities, while at $n_e \lesssim pn_0$ the Hall conductivity of a real system is larger than the ideal value. A qualitative explanation of the observed effects can be obtained in the single-electron approximation by recognizing that the random perturbing potential present in the system broadens the degenerate Landau level and localizes some of the electronic states on the tails of these levels. With increasing electron density in the system, the Fermi level lands periodically in the region of the localized

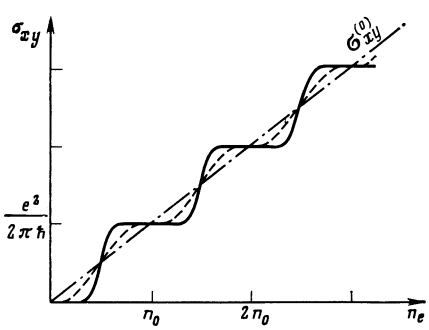

FIG. 1. Qualitative behavior of Hall conductivity σ_{xy} of a quasi-two-dimensional electron system in the quantum Hall effect[1,7,9] compared with the ideal Hall conductivity $\sigma_{xy}^{(0)} = n_e ec/B$: solid curve—$T_1$, dashed—$T_2$, $T_1 < T_2$, B = const.

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states. Let, for example, an integer number of lower ($N' < \overline{N}$) Landau levels as well as a certain number of localized electronic states of the level N be completely filled at $T = 0$. The wave function of completely filled lower Landau levels is nondegenerate, and these levels, at least for weak perturbations, are separated by a gap from the states of the partly filled Landau level. One should then expect the contribution of these levels to the Hall mobility of the system to be close to ideal. On the other hand the contribution of the states of the partly filled Landau levels should decrease abruptly with decreasing temperature, for in this case the electrons land, by definition, on localized states. This gives the electronic part of the plateaus ($n_e \lesssim pn_0$) of Fig. 1. To explain the existence of plateaus at $n_e \perp pn_0$ it must be taken into account that in this region of the electron densities the N th Landau level is almost filled to the limit. It is then convenient to describe the properties of the system in the language of the Landau-level vacancies, which are constructed by filling the missing number of electronic states and introducing on the Landau level an equal number of positively charged particles. The situation then becomes analogous to that considered above: the contribution of a filled Landau level to the conductivity of the system is ideal, while the contribution of the vacancies, with account taken of their localization near the upper edge of the Landau level, decreases steeply with decreasing temperature. This gives the hole part of the plateaus ($n_e \lesssim pn_0$) of Fig. 1. Thus the quantum Hall effect is substantially connected with the localization of the electronic states near the edges of the broadened Landau levels.

To corroborate these qualitative premises we must prove, first, that the Hall conductivity of completely filled Landau levels coincides with the corresponding ideal value $\sigma_{xy}^{(0)}(p)$, so that the correction $\Delta\sigma_{xy} = 0$. Second, we must verify the additivity of the contributions of the filled levels and of the partly filled Landau level to the conductivity of the system. Finally, we must show that the contribution of the electrons or of the vacancies of a completely filled Landau level to the conductivity of the system tends to zero under the conditions described above in the limit as $T \to 0$. This program is carried out in §§3 and 4 of the present paper. The calculation is based on the modified Kubo formula for the Hall conductivity,[15] which is convenient in the limit of a strong magnetic field. A direct derivation of this formula is therefore presented in §2 from the usual Kuba formula.

The important role of electron-state localization in the formation of plateaus on the $\sigma_{xy}(n_e)$ plot was noted earlier in Refs. 3 and 5. There, however, the theorem that there are no corrections to the Hall conductivity of filled Landau levels was proved only in lower orders of perturbation theory. Unfortunately, the proof presented by Thouless[4] of the quantization of the Hall conductivity of filled Landau levels is based on an incorrect formula for the Hall current. We note that when the QHE phenomenon is explained on the basis of linear-response formulas it is apparently impossible to avoid the use of unbounded operators. We emphasize in this connection that an indispensable part of the entire presented reasoning must be a check on the existence of expressions that contain such operators.

§2. DERIVATION OF THE MODIFIED KUBO FORMULA

We shall investigate the properties of a quasi-two-dimensional electron + impurity system in the limit of a weakly perturbing potential with amplitude $V_0 \ll \hbar\omega_c$. For bounded perturbations $|V(\mathbf{r})| \leqslant V_0$ the broadening of the degenerate Landau levels should be small compared with the distance between the unperturbed Landau levels. For the following it suffices to assume that in the intervals between the broadened Landau levels there exists an energy region (which can be narrow) in which the density of states of the system is zero. In this case, the exact wave functions and energies of the system can again be characterized by the Landau quantum number N, as well as by the quantum number α that numbers the states of the specified broadened Landau level: $\Psi = |N\alpha\rangle$; $E = E_{N\alpha}$. The wave functions $|N\alpha\rangle$ will be understood to be superpositions

$$|N\alpha\rangle = \sum_{LX_0} C_{N\alpha}(LX_0)\,|LX_0\rangle \tag{2}$$

of unperturbed wave functions $|LX_0\rangle$ taken in the Landau representation, where X_0 is the quantized position of the oscillator center. The quantum number X_0 in (2) varies over the periodicity interval of the system in the x direction, $0 \leqslant x_0 \leqslant L_x$, with L_x assumed to be much larger than the characteristic lengths of the system, such as the average distance between the impurities, the characteristic dimension of the electronic states, etc. As for the coefficients $c_{N\alpha}(LX_0)$ in (2), we shall assume that by virtue of applicability of perturbation theory they decrease rapidly with increasing difference $|L - N|$, and that the wave functions (2) are normalized by the condition

$$\sum_{LX_0} |C_{N\alpha}(LX_0)|^2 = 1. \tag{3}$$

We analyze now the behavior of the coefficient $C(X_0) = C_{N\alpha}(LX_0)$ as a function of the parameter X_0. For localized states whose wave-function modulus decreases in all directions in the xy plane, the values of the function $|C(X_0)|$ decrease rapidly outside a certain interval ΔX_0 near the point $\overline{X}_0$; this interval characterizes the position of the x coordinate of the localization center of the given state. We classify as delocalized wave functions those whose modulus does not decrease in at least one direction in the xy plane. In particular, one should include among the delocalized states the basis functions of the unperturbed set $|LX_0\rangle$, as well as the wave functions (2) with $|C(X_0)| = \text{const}$, which do not decrease in the y and x directions respectively. In general one should expect that for delocalized states the values of the function $|C(X_0)|$ are distributed on the average uniformly in any sufficiently large specified interval of values of ΔX_0. We shall present below a more exact definition of localized and delocalized states.

We now describe briefly the properties of the operators used in our paper. We have assumed from the very outset that the potential-energy operator $V(\mathbf{r})$ is bounded. We shall assume in addition that the potential $V(\mathbf{r})$ has continuous lower-order partial derivatives; the operators $\partial\hat{V}/\partial x_\mu$

($\mu = x, y$) are then also bounded and the matrix elements of these operators over the wave functions (1) are well defined. On the contrary, the operators $\hat{x}_\mu$ of the coordinates, $\hat{v}_\mu$ of the velocities in the magnetic field, $\hat{x}_\mu$ of the coordinates of the center of the cyclotron motion, which take in the Landau gauge $\mathbf{A} = (0, Bx, 0)$ the form

$$\hat{v}_x = -\frac{i\hbar}{m}\frac{\partial}{\partial x}, \quad \hat{v}_y = -\frac{i\hbar}{m}\frac{\partial}{\partial y} + \omega_c x, \tag{4a}$$

$$\hat{X} = \hat{x} - \frac{\lambda^2}{\hbar} m\hat{v}_y = i\lambda^2 \frac{\partial}{\partial y}, \quad \hat{Y} = \hat{y} + \frac{\lambda^2}{\hbar} m\hat{v}_x = \hat{y} - i\lambda^2 \frac{\partial}{\partial x}, \tag{4b}$$

are unbounded, but not all for the same reason. The operators $\hat{v}_\mu$ are unbounded because the nonzero matrix elements of these operators in the unperturbed basis $\langle LX_0|\hat{v}_\mu|L \pm 1, X_0\rangle$ increase with increasing number L in proportion to $\sqrt{L}$. But these matrix elements do not depend on the quantum number X_0, and the coefficients $C_{N\alpha}(LX_0)$ are assumed to be rapidly decreasing functions of the differences $|L - N|$. Therefore matrix elements of the operators $\hat{v}_\mu$ always exist in the basis (2). The fact that the operators $\hat{v}_\mu$ are unbounded is of no great importance in the following, so long as all the expressions contain these operators together with the statistical-averaging operator $\exp\{-\beta\hat{H}\}$, which automatically cuts off all the high cyclotron levels. With the foregoing taken into account, we can treat the operators $\hat{v}_\mu$ in most cases as ordinary bounded operators. We assign the operators $\hat{v}_\mu$ and $\partial\hat{V}/\partial x$ whose matrix elements exist in basis (2) to the first class. On the contrary, the matrix elements of the operators $\hat{x}_\mu$ and $\hat{X}_\mu$ do not exist generally speaking in basis (2), and are assigned to the second class. Indeed, in the unperturbed basis the operator $\partial/\partial y$ has only diagonal matrix elements

$$\langle LX_0|\partial/\partial y|LX_0\rangle = -iX_0/\lambda^2,$$

with arbitrarily large modulus as $|X_0| \to \infty$. Therefore the matrix elements of the operator $\hat{X}$

$$\langle N\alpha|\hat{X}|N'\alpha'\rangle = \sum_{LX_0} X_0 C_{N\alpha}^*(LX_0) C_{N'\alpha'}(LX_0) \tag{5}$$

should be regarded as diverging, at any rate for states delocalized in the x direction. The situation is analogous for the operator $\hat{x}$, since its matrix elements that are diagonal in L

$$\langle LX_0|\hat{x}|LX_0'\rangle = X_0\delta_{X_0, X_0'}$$

have likewise large modulus at large $|X|$. (We note that for the operator $\hat{v}_y$ (4a), which is a definite linear combination of the operators $\partial/\partial y$ and $\hat{x}$, the corresponding diagonal contributions cancel one another exactly.) Convergence in (5) can be ensured only in the case of sufficiently rapid decrease of at least one of the coefficients $C(X_0)$ as a function of $|X_0 - \overline{X}_0|$, i.e., in the case when at least one of the states $|N\alpha\rangle$ or $|N'\alpha'\rangle$ is localized. This is sufficient also for the existence of matrix elements of the operators $\hat{y}$ and $\hat{Y}$. We regard this circumstance as the definition of localization. Namely, we consider a state $|N\alpha\rangle$ to be localized if for a specified $|N\alpha\rangle$ the matrix elements of the operators $\hat{x}_\mu$ and $\hat{X}_\mu$ of the second class exist for any choice of the states $|N'\alpha'\rangle$; in the opposite case the state $|N\alpha\rangle$ is assumed delocalized.

It follows from the foregoing that it is necessary to verify the existence of expressions containing the operators $\hat{x}_\mu$ and $\hat{X}_\mu$, whereas many authors use such expressions without due caution. For example the initial expression used by Thouless[4] for the current of a filled Landau level is in general divergent. We shall prove in §3 a theorem that there is no correction $\Delta\sigma_{xy}$ for the ideal Hall conductivity of filled Landau level. Our justification is a modified Kubo formula[15] that gives an explicit expression for this correction in terms of bounded operators $\partial\hat{V}/\partial x_\mu$. To this end, we present here a direct derivation of the modified Kubo formula and obtain, as a corollary, some interesting sum rule.

We start from the usual Kubo formula for the Hall conductivity 16:

$$\sigma_{xy} = -\frac{e^2}{S}\lim\int_0^\infty dt e^{-\varepsilon t}\Phi_{xy}(t), \tag{6}$$

where $\Phi_{xy}(t)$ is a response function that takes in the basis (2) the form

$$\Phi_{xy}(t) = \sum_{\alpha,\gamma}\langle\alpha|\hat{v}_y|\gamma\rangle\langle\gamma|\hat{v}_x|\alpha\rangle\frac{f_\alpha - f_\gamma}{E_\alpha - E_\gamma}\exp\left\{-i\frac{E_\alpha - E_\gamma}{\hbar}t\right\}. \tag{7}$$

Here and below $|\alpha\rangle = |N\alpha\rangle$, $E_\alpha = E_{N\alpha}$, and f_α are the filling ratios of the state $|\alpha\rangle$.

To derive the modified Kubo formula we use the operator identities

$$\hat{v}_y = -\frac{\lambda^2}{\hbar}\frac{\partial\hat{V}}{\partial x} + \frac{i}{\hbar\omega_c}[\hat{v}_x, \hat{H}], \tag{8a}$$

$$\hat{v}_x = \frac{\lambda^2}{\hbar}\frac{\partial V}{\partial y} - \frac{i}{\hbar\omega_c}[\hat{v}_y, \hat{H}], \tag{8b}$$

where $\hat{H} = \hat{H}_0 + \hat{V}$ is the complete single-particle Hamiltonian of the system. We note that relation (8) contains only operators of the first class, which obviously include also the Hamiltonian $\hat{H}$. Therefore the product of the velocity matrix elements in (7) can be expressed in term of a product of matrix elements from the right-hand sides of relations (8). Calculating next the time integral in (6) and taking the limit as $\varepsilon \to 0$, we get

$$\begin{aligned}\sigma_{xy} = &-i\frac{e^2}{S\hbar\omega_c^2}\sum_{\alpha,\gamma}\langle\alpha|\hat{v}_x|\gamma\rangle\langle\gamma|\hat{v}_y|\alpha\rangle(f_\alpha - f_\gamma)\\ &-i\frac{e^2\lambda^4}{S\hbar}\lim_{\varepsilon\to 0}\sum_{\alpha,\gamma}\frac{\langle\alpha|\partial V/\partial x|\gamma\rangle\langle\gamma|\partial V/\partial y|\alpha\rangle}{(E_\alpha - E_\gamma)(E_\alpha - E_\gamma - i\hbar\varepsilon)}(f_\alpha - f_\gamma)\\ &+\frac{e^2\lambda^4 m}{S\hbar^2}\sum_{\alpha,\gamma}\left\{\langle\alpha|\hat{v}_x|\gamma\rangle\langle\gamma|\frac{\partial V}{\partial y}|\alpha\rangle - \langle\alpha|\frac{\partial V}{\partial x}|\gamma\rangle\langle\gamma|\hat{v}_y|\alpha\rangle\right\}\\ &\frac{f_\alpha - f_\gamma}{E_\alpha - E_\gamma}.\end{aligned} \tag{9}$$

By virtue of the aforementioned properties of the operators $\hat{v}_\mu$, the first sum in (9) can be written in the form of the mean value of the commutator $[\hat{v}_x, \hat{v}_y] = i\hbar\omega_c/m$ (cf. the analysis of the corresponding sum in Ref. 5), which yields an ideal Hall conductivity $\sigma_{xy}^{(0)} = -ecN_e/BS$. The second term in (9) is the sought correction $\Delta\sigma_{xy}$ to this quantity, which can be easily written in the form[15]

$$\Delta\sigma_{xy} = \frac{e^2}{S}\lim_{\varepsilon\to 0}\int_0^\infty dt e^{-\varepsilon t}\int_0^\beta dz\,\mathrm{Sp}\{\hat\rho\dot{\hat Y}(-i\hbar z)\dot{\hat X}(t)\}, \quad (10)$$

where $\hat\rho$ is the equilibrium density-matrix operator, and $T=\beta^{-1}$ is the temperature. We shall show now that the last term in (9) is identically zero. We use the operator Kubo identity[16]

$$[e^{-\beta\hat H}, \hat A] = i\hbar e^{-\beta\hat H}\int_0^\beta dt e^{t\hat H}\dot A e^{-t\hat H}, \quad (11)$$

$$\dot A = [\hat A, \hat H]/i\hbar$$

to transform an operator $\hat A$ of the second class (such as $\hat x_\mu$ or $\hat X_\mu$) into an operator $\mathring{\mathrm{A}}$ of the first class (such as $\hat v_\mu$ or $\partial\hat V/\partial x_\mu$), whose matrix elements in the basis (2) are well defined. We assume here that the commutator in the left-hand side of (11) is also an operator of the first class. This assumption is in fact always made in the derviation of linear-response formulas. For any operator $\hat B$ from the first class there exists hence

$$\mathrm{Sp}\,[e^{-\beta\hat H}, \hat A]\hat B = i\hbar\int_0^\beta dt\,\mathrm{Sp}\{e^{-\beta\hat H}e^{t\hat H}\dot A e^{-t\hat H}\hat B\}. \quad (12)$$

The existence of traces of the type (12) requites in the present case a separate consideration, by virtue of the unique character of the spectrum of the complete Hamiltonian $\hat H$ of the system. The eigenvalues $E_{N\alpha}$ of this operator increase only with increasing number N of the Landau level, and at a specified N they remain bounded functions of the quantum number α. Therefore the operator $\exp\{-\beta\hat H\}$ ensures only convergence of the sum over different Landau levels N. Within the limits of a specified Landau level the corresonding sums can increase with increasing size of the system, either in proportion to the number of places $N_0 = S/2\pi/\lambda^2$ on the Landau level, or in proportion to a certain power $n>1$ of this number. In the former case, obviously, there exists a thermodynamic limiting approach and it is natural to assume that the corresponding trace exists. In the latter case there is no thermodynamic limit. As noted above, the matrix elements of the operators of the first class, $\langle N\alpha|\hat v_\mu|N'\alpha'\rangle$, are bounded at given N and N' as functions of the quantum numbers α and α'. Therefore traces of the type (12) exist for operators of the first class. For operators of the second class such traces, generally speaking, do not exist. For example $Sp\,\hat\rho_0\hat X_\mu^2$, where $\hat\rho_0$ is the equilibrium density matrix of the unperturbed Hamiltonian $\hat H$, must be regarded as diverging in the thermodynamic limit, although in this case one can choose a representation in which matrix elements of the operator $\hat X_\mu$ exist.

Returning to Eq. (12), we note that under the conditions imposed on the operators $\hat A$ and $\hat B$, the commutator $[\hat A,\hat B]$ is a bounded operator and exists

$$\mathrm{Sp}\,e^{-\beta\hat H}[\hat A, \hat B] = \mathrm{Sp}\,[e^{-\beta\hat H}, \hat A]\hat B. \quad (13)$$

The equality in (13) follows, in fact, from the existence of both traces in the sense indicated above, and from the fact that it is valid for arbitrary bounded operators for which cyclic permutation under the trace sign is legitimate. Kubo justified earlier equalities of the type (13) by integrating by parts.

The operators $\hat A$ and $\hat B$ in (11)–(13) must be understood as second-quantization operators:

$$\hat A = \int d\mathbf{r}\hat\Psi^+(\mathbf{r})\hat a\hat\Psi(\mathbf{r}); \quad \hat B = \int d\mathbf{r}\hat\Psi^+(\mathbf{r})\hat b\hat\Psi(\mathbf{r}), \quad (14)$$

where $\hat\Psi(\mathbf{r})$ is the electron field operator in the basis (2). Using (12)–(14) we get

$$\sum_\alpha\langle\alpha|[\hat a,\hat b]|\alpha\rangle f_\alpha = i\hbar\beta\sum_\alpha\langle\alpha|\dot{\hat a}|\alpha\rangle f_\alpha\sum_\gamma\langle\gamma|b|\gamma\rangle f_\gamma - i\hbar\sum_{\alpha,\gamma}\langle\alpha|\dot{\hat a}|\gamma\rangle\langle\gamma|\hat b|\alpha\rangle\frac{f_\alpha - f_\gamma}{E_\alpha - E_\gamma}. \quad (15)$$

Putting $\hat a = \hat x_\mu$ and $\hat b = \hat v_\mu$ in (15) and recognizing that $\mathrm{Sp}\,\hat\rho\hat v_\mu = 0$ at thermodynamic equilibrium, we obtain the following sum rule:

$$\sum_{\alpha,\gamma}\langle\alpha|\hat v_\mu|\gamma\rangle\langle\gamma|v_{\mu'}|\alpha\rangle\frac{f_\alpha - f_\gamma}{E_\alpha - E_\gamma} = -\frac{Ne}{m}\delta_{\mu,\mu'}. \quad (16)$$

Next, putting $\hat a = \hat X_\mu$ and $\hat b = \hat v_\mu$ in (15) we obtain a second sum rule:

$$\sum_{\alpha,\gamma}\langle\alpha|\frac{\partial\hat V}{\partial x_\mu}|\gamma\rangle\langle\gamma|\hat v_{\mu'}|\alpha\rangle\frac{f_\alpha - f_\gamma}{E_\alpha - E_\gamma} = 0. \quad (17)$$

It follows from (17) that the last term in (9) is identically equal to zero. This concludes the derivation of the modified Kubo formula (9) and (10). We note that with the aid of the sum rules (16) and (17) one can write the modified Kubo formula in other forms equivalent to (9) and (10). The correction $\Delta\sigma_{xy}$ can then be expressed in terms of the operators $\partial\hat V/\partial x$ and $\hat v_x$ or in terms of the operators $\partial\hat V/\partial y$ and $\hat v_y$. To this end it suffices to replace in the response function (7), using the corresponding identity in (8), only one of the velocity matrix elements. In analogy with the foregoing, it is easy to derive a modified Kubo formula for the component σ_{xx}. All the foregoing derivations remain obviously valid also in the three-dimensional case.

We consider now a situation when an integer number $\overline N$ of broadened Landau levels in the system is completely filled with electrons at $T=0$, so that the Fermi level lies in the gap between the Landau levels $\overline N - 1$ and $\overline N$, where the density of states vanishes by definition. The correction to the conductivity of the filled Landau levels, as follows from (10), is then proportional to

$$F\left(\frac{\partial\hat V}{\partial x}, \frac{\partial\hat V}{\partial y}\right) = \sum_{\alpha,\gamma,N<\overline N,N'\geqslant\overline N}\frac{\langle N\alpha|\partial\hat V/\partial x|N'\gamma\rangle\langle N'\gamma|\partial\hat V/\partial y|N\alpha\rangle - \text{c.c.}}{(E_{N\alpha} - E_{N'\gamma})^2}. \quad (18)$$

In this expression the denominator $(E_{N\alpha} - E_{N'\gamma})^2 \geqslant \Delta^2$, where Δ is the width of the gap between $(\overline N - 1)$th and $\overline N$th Landau levels. Using this circumstance, we can obtain certain sum rules for quantities of the type (18), by going in (16) and (17) to the limit as $T\to 0$ and using the commutation

relations (8). In particular, we can prove the identity $F(\partial\hat{V}/\partial x, \hat{v}_y) = 0$, as well as $F(\partial\hat{V}/\partial y, v_x) = 0$. The equality $F(\partial\hat{V}/\partial x, \partial\hat{V}/\partial y) = 0$, however, does not follow from (16) and (17). It can only be shown that

$$\frac{\lambda^2}{\hbar} F\left(\frac{\partial\hat{V}}{\partial x}, \frac{\partial\hat{V}}{\partial y}\right) = F\left(\frac{\partial\hat{V}}{\partial x}, \hat{v}_x\right) = F\left(\frac{\partial\hat{V}}{\partial y}, \hat{v}_y\right), \qquad (19)$$

which agrees with the statement that different forms of the modified Kubo formula are equivalent. Nonetheless, we shall show in §3 that when (18) is averaged over random coordinates of the impurity centers the contribution of each finite order of perturbation theory vanishes.

§3. IDEAL CONDUCTIVITY OF FILLED LANDAU LEVELS

We draw in the complex E plane an integration contour C that encloses the filled Landau levels (see Fig. 2) and express the quantity (18), which is proportional to the correction to the ideal conductivity of completely filled Landau levels, in the form

$$F = \frac{1}{2\pi i} \oint_C dz \,\mathrm{Sp}\, \hat{R} \frac{\partial\hat{V}}{\partial x} \hat{R}^2 \frac{\partial\hat{V}}{\partial y}, \qquad (20)$$

where $\hat{R} = (z - \hat{H})^{-1}$ is the operator of the resolvent of the total Hamiltonian of the system. In the situation considered, the Fermi level coincides at $T = 0$ with the upper edge of the last of the filled Landau levels. In the presence of a gap between neighboring Landau levels, however, the contour of the integration in (20) can be shifted into the interior of the forbidden energy region, as shown in Fig. 2. All the operators in (20) are then bounded on the contour C so that cyclic permutations under the trace sign are legitimate. It is convenient to symmetrize (20) by recognizing that $\hat{R}^2 = -d\hat{R}/dz$ and integrating by parts:

$$F = -\frac{1}{4\pi i} \oint_C dz \,\mathrm{Sp}\, \hat{R} \left(\frac{\partial\hat{V}}{\partial x} \hat{R} \frac{\partial\hat{V}}{\partial y} - \frac{\partial\hat{V}}{\partial y} \hat{R} \frac{\partial\hat{V}}{\partial x} \right) \hat{R}. \qquad (21)$$

The term outside the integral sign vanishes because it is single-valued on the integration contour.

Expression (21) must be averaged over the random coordinates of the impurity centers. This operation will be denoted by a superior bar. We shall prove that the contribution of any finite order in the perturbation-theory expansion of $\overline{F}$ vanishes. This expansion is obtained with the aid of the Dyson equation

$$\hat{R} = \hat{R}_0 + \hat{R}_0 \hat{V} \hat{R}, \qquad (22)$$

where $\hat{R}_0 = (z - \hat{H}_0)^{-1}$ is the operator of the resolvent of the unperturbed Hamiltonian. From the structure of (21) it is clear that the nth order contribution F_N will be equal the sum of C_N^2 terms of the type

$$\frac{1}{4\pi i} \oint_C dz \,\mathrm{Sp} \left\{ \hat{R}_0 \hat{V} \hat{R}_0 \ldots \hat{R}_0 \frac{\partial\hat{V}}{\partial x} \hat{R}_0 \hat{V} \ldots \hat{R}_0 \frac{\partial\hat{V}}{\partial y} \hat{R}_0 \hat{V} \ldots \hat{V} \hat{R}_0 - \left(\frac{\partial\hat{V}}{\partial x} \leftrightarrow \frac{\partial\hat{V}}{\partial y} \right) \right\}, \qquad (23)$$

in which the operators $\partial\hat{V}/\partial x$ and $\partial\hat{V}/\partial y$ are arranged in all possible ways into a series of $N-2$ operators $\hat{V}$ subject to the condition that in the first product of the operators in (23) the operator $\partial\hat{V}/\partial x$ is always to the left of $\partial\hat{V}/\partial y$, while in the second product they change places. We shift in each product of (23) the last operator $\hat{R}_0$ to the first place and denote the resultant expression by $P^{(2)}_{nm}$. The first subscript $n < m$ indicates the position of the operator $\partial\hat{V}/x$, and the second the position of $\partial\hat{V}/\partial y$ in the series of operators $\hat{V}$ of the first product of the operators (23). The superscript means that the first in the series of operators $\hat{R}_0$ is $\hat{R}_0^2$. By virtue of the foregoing the contribution of the Nth order of perturbation theory for F can be written in the form

$$F_N = -\sum_{1 \leqslant n < m \leqslant N} P^{(1)}_{n,m}. \qquad (24)$$

The purpose of the reasoning that follows is to find for the quantities $\overline{P}^{(1)}_{n,m}$, a sufficient number of linear equations that result in $\overline{F}_N = 0$. We introduce first of all the quantities $P^{(\alpha)}_{n,m}$ with arbitrary superscript α, which indicates the position of the operator $\hat{R}_0^2$ in the series of operators $\hat{R}_0$, and also lift the restriction $n < m$ on the subscripts of these quantities, assuming that at $n \geqslant m$

$$P^{(\alpha)}_{n,m} = -P^{(\alpha)}_{m,n}; \quad P^{(\alpha)}_{n,n} = 0. \qquad (25)$$

Since cyclic permutation of the bounded operators under the trace sign is legitimate, we have

$$P^{(\alpha)}_{n,m} = P^{(\alpha+1)}_{n+1,m+1} = P^{(\alpha-1)}_{n-1,m-1}. \qquad (26)$$

Uniqueness of the quantities $P^{(\alpha)}_{n,m}$ is fixed here by the condition that each product of operators under the trace sign begins with an operator $\hat{R}_0$ or $\hat{R}_0^2$ and ends with an operator $\hat{V}$ or $\partial\hat{V}/\partial x$. In addition, we assume that if any of the superscripts or subscripts l is larger than N it must be replaced in accord with the rule $l \to l - N$. If it turns out that $l < 1$ we must substitute $l \to N - |l|$. This rule of cyclic replacement of the indices is assumed hereafter satisfied in all the encountered equations.

The first group of relations for the quantities $P^{(1)}_{n,m}$ is obtained by replacing the operators $\hat{R}_0^2$ that enter in the definitions of these quantities by $-d\hat{R}_0/dz$ and integrating by parts (the term outside the integral drops out as before). This yields

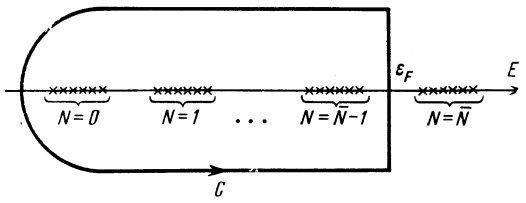

FIG. 2. Integration contour C.

$$P^{(1)}_{n,m}=-\sum_{\alpha=2}^{N} P^{(\alpha)}_{n,m}, \tag{27a}$$

which by virtue of (26) can be written in the form

$$\sum_{l=0}^{N-1} P^{(1)}_{n-l,m-l}=\sum_{l=1}^{N} P^{(1)}_{l,m-n+l}=0. \tag{27b}$$

Putting here $m-n=s$; $s=1,2,...[(N-1)/2]$ independent relations for the quantities $P^{(1)}_{n,m}$. Since the indices are cyclic, relations (27b) with $m-n>[(N-1)/2]$ are consequences of the preceding ones. Equations (27) are valid, of course, also for the quantities $\overline{P}^{(1)}_{n,m}$ averaged over the random coordinates of the impurity centers. For the averaged quantities, however, we have in addition the following $N-1$ relations:

$$\sum_{l=1}^{N} \overline{P}^{(1)}_{n,l}=0; \quad n=1,2,\ldots,N-1. \tag{28}$$

These equations can be verified by writing out the traces in the definitions of $P^{(1)}_{n,l}$ in the unperturbed basis $|LX_0\rangle$. It turns out then that averaging the sums in the left-hand side of (28) reduces to averaging of the expression

$$\left(\frac{\partial}{\partial x_n}\sum_{l=1}^{N}\frac{\partial}{\partial y_l}-\frac{\partial}{\partial y_n}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\right) V(\mathbf{r}_1)V(\mathbf{r}_2)\ldots V(\mathbf{r}_N). \tag{29}$$

The mean value of a product of the operators V depends only on the difference between the electron coordinates $\mathbf{r}_i$. Therefore the mean value of Eq. (29), which contains operators that act on the sum of electron coordinates, is zero. It is this which leads to Eqs. (28). Relation (28) with $n=N$ is a consequence of the preceding ones, as can be verified by summing (28) over n and using (25).

We cannot determine the C_N^2 quantities $\overline{P}^{(1)}_{n,m}$ in the expression for the mean value of the contribution $\overline{P}_N$, since we have ony $[(N-1)/2]+N-1$ relations for these quantities. It is possible nonetheless to express $\overline{F}_N$ in the form of a linear combination of relations (27) and (28):

$$-N\overline{F}_N=\sum_{s=1}^{[(N-1)/2]}(N-2s)\sum_{l=1}^{N}\overline{P}^{(1)}_{l,l+s}+\sum_{k=1}^{N-1}2k\sum_{l=1}^{N}\overline{P}^{(1)}_{N-k,l}, \tag{30}$$

from which follows $\overline{F}_N=0$. Equation (30) can be easily verified by recognizing that the first sum in (30) can be written in the form

$$\sum_{s=1}^{N-1}(N-2s)\sum_{l=1}^{N-s}\overline{P}^{(1)}_{l,l+s}=\sum_{1\leqslant n<m\leqslant N}[N-2(m-n)]\overline{P}^{(1)}_{n,m},$$

and the second in the form

$$\sum_{s=1}^{N}\sum_{l=1}^{N}2(N-s)\overline{P}^{(1)}_{s,l}=\sum_{1\leqslant n<m\leqslant N}2(m-n)\overline{P}^{(1)}_{n,m}.$$

We have shown that the mean value of the contribution of any finite order of perturbation theory for $\overline{F}$ is zero. If we discard the rather improbable possibility of existence of a non-analytic dependence of this quantity on the small parameter $V_0/\hbar\omega_c$, it will follow hence that $\overline{F}$ itself is zero. Consequently when an integer number p of broadened Landau levels is filled with electrons the conductivity of the quasi-two-dimensional electron system coincides with the corresponding ideal value (1).

To avoid misunderstanding, we emphasize that the theorem proved in this section does not contradict the statements concerning the quantization of the Hall conductivity in a periodic potential,[13,17] since we used in the proof an averaging of the conductivity over a random perturbing potential.

§4. ROLE OF LOCALIZED STATES

Let now, as $T\to 0$, the Landau levels $N'=0,1,\ldots,\overline{N}-1$, be completely filled with electrons, and let a certain number of electrons land in localized state on the lower edge of the $\overline{N}$ th Landau level. Using (1), (9), and (10) we express the Hall conductivity of the system in the form

$$\sigma_{xy}=-\overline{N}\frac{e^2}{2\pi\hbar}-(N_e-\overline{N}N_0)\frac{ec}{BS}+\Delta\sigma_{xy}. \tag{31}$$

The first term in this expression is the contribution of the filled Landau levels to the ideal Hall conductivity σ_{xy}, and the second gives the contribution of $N_e-\overline{N}N_0$ electrons of the partly filled Landau level to the same quantity. To verify the qualitative arguments of §1, it must be shown that in the limit as $T\to 0$ the last two terms of (31) cancel each other exactly. First, in the low-temperature limit of interest to us we can put $f_{N'\alpha}=1$ for the filled Landau levels ($N'<\overline{N}$) and $f_{N'\alpha}=0$ for the practically empty levels with $N'>\overline{N}$. (These equations are valid, obviously, accurate to about $\exp\{-\Delta/T\}$, where $\Delta\sim\hbar\omega_c$ is the characteristic width of the gap between neighboring Landau levels.) It is then easy to verify that $\Delta\sigma_{xy}$ in (31) can be written in the form of the sum $\Delta\sigma^{(1)}_{xy}+\Delta\sigma^{(2)}_{xy}$, where the term $\Delta\sigma^{(1)}_{xy}$, which is proportional to (18), describes the contribution of the filled Landau levels to $\Delta\sigma_{xy}$ and, as we have seen in §3, vanishes on averaging. The remaining term $\Delta\sigma^{(2)}_{xy}$ contains only the occupation numbers $f_{\overline{N}\alpha}$ of the partially filled level $\overline{N}$:

$$\Delta\sigma^{(2)}_{xy}=-i\frac{e^2\lambda^4}{S\hbar}\lim_{\varepsilon\to 0}\Big\{\sum_{\alpha,\gamma}g(\overline{N}\alpha;\overline{N}\gamma)(f_{\overline{N}\alpha}-f_{\overline{N}\gamma})+\sum_{N'\neq\overline{N};\,\alpha,\gamma}g(\overline{N}\alpha;N'\gamma)f_{\overline{N}\alpha}-\sum_{N'\neq\overline{N};\,\alpha,\gamma}g(N'\alpha;\overline{N}\gamma)f_{\overline{N}\gamma}\Big\}, \tag{32}$$

where

$$g(N\alpha;N'\gamma)=\frac{\langle N\alpha|\partial\hat{V}/\partial x|N'\gamma\rangle\langle N'\gamma|\partial\hat{V}/\partial y|N\alpha\rangle}{(E_{N\alpha}-E_{N'\gamma})(E_{N\alpha}-E_{N'\gamma}-i\hbar\varepsilon)}.$$

It can be assumed in this sense that the filled Landau levels $N'<\overline{N}$ and the partly filled level $\overline{N}$ make additive contributions to the Hall mobility of the system in the low-temperature limit.

We divide now the states of the level N into localized and delocalized in accord with the definition given in §2. In the limit as $T\to 0$ the occupation numbers of the localized states $f^{(1)}_{N\alpha}$, and of the delocalized ones $f^{(2)}_{N\alpha}$, so that as $T\to 0$, by agreement, the Fermi level lands in the region of localized states on the lower edge of the Landau level $\overline{N}$. Therefore those term of (32) which contain $f^{(2)}_{N\alpha}$ tend to zero as $T\to 0$ and can be left out. In the remaining expression, at least one

of the states $|\overline{N}\alpha\rangle$, $|\overline{N}\gamma\rangle$ is localized. These terms can be simplified using the relations

$$\left\langle \alpha \left| \frac{\partial \hat{V}}{\partial x_\mu} \right| \gamma \right\rangle = \pm \frac{i}{\lambda^2}(E_\gamma - E_\alpha)\langle \alpha | \hat{X}_\mu | \gamma \rangle, \tag{33}$$

which follow from the equations of motion for the operators $\hat{X}_\mu$ and are valid under the condition that the matrix elements $\hat{X}_\mu$ exist. The resultant expression can be reduced to the form

$$-i\frac{e^2}{S\hbar}\sum_\alpha \langle \overline{N}\alpha | [\hat{X}, \hat{Y}] | \overline{N}\alpha \rangle f_{\overline{N}\alpha}^{(1)}, \tag{34}$$

where the summation extends only over the localized states of the level $\overline{N}$. The commutator $[\hat{X}, \hat{Y}] = i\lambda^2$. In addition, by agreement, as $T \to 0$ all the electrons of the level $\overline{N}$ land in localized states on the lower edge of this level, so that

$$\sum_\alpha f_{\overline{N}\alpha}^{(1)} = N_e - \overline{N}N_0$$

and expression (34), which coincides with $\Delta\sigma_{xy}$, cancels in the limit as $T \to 0$ the second term of (31). With increasing number of electrons in the system, ths cancellation will continue so long as at $T = 0$ the Fermi level remains in the region of the localized states of the lower edge of the Landau level $\overline{N}$. This produces on the $\sigma_{xy}(n_e)$ plot plateaus at values $n_e \gtrsim \overline{N}n_0$. With rising temperature the number of electrons in localized states decreases and the cancellation is upset. This explains the strong plateau-width temperature dependence observed in the experiments.[7-9] Clearly, to determine the concrete form of this dependence we must estimate the temperature dependence of $\Delta\sigma_{xy}^{(2)}$.

Let now the Fermi level land as $T \to 0$ in the region of localized states near the upper edge of the level $\overline{N} - 1$. Then the levels with $N' < \overline{N} - 1$ are completely filled with electrons, and the filling of the level $N - 1$ is close to the limit. Transforming to the representation with vacancies on the level $\overline{N} - 1$, we write the Hall conductivity of the system in a form similar to (31):

$$\sigma_{xy} = -\overline{N}\frac{e^2}{2\pi\hbar} + [\overline{N}N_0 - N_e]\frac{ec}{BS} + \Delta\sigma_{xy}, \tag{35}$$

where $\overline{N}N_0 - N_e$ is the number of vacancies of the level $\overline{N} - 1$. In the limit $T \to 0$ we can show that the quantity $\Delta\sigma_{xy}$ in (35) constitutes the sum $\Delta\sigma_{xy}^{(1)} + \Delta\sigma_{xy}^{(2)}$, where the term $\Delta\sigma_{xy}^{(1)}$ is proportional as before to (18) and describes the contribution of the filled Landau levels with $N' \leqslant \overline{N} - 1$ to $\Delta\sigma_{xy}$, a contribution that vanishes on averaging. The remaining term $\Delta\sigma_{xy}^{(2)}$ depends in this case only on the filling ratios $\bar{f}_{\overline{N}-1,\alpha} = 1 - f_{\overline{N}-1,\alpha}$ of the vacancies of the $(\overline{N} - 1)$st Landau level. In analogy with the preceding, we can establish that in the limit as $\overline{T} \to 0$ the quantity $\Delta\sigma_{xy}^{(2)}$, with allowance for the localization of the vacancies near the upper edge of the $(\overline{N} - 1)$st Landau level, cancels exactly the second term of (35). This explains the existence of the plateaus and their temperature dependence in the density region $n_e \lesssim \overline{N}n_0$.

§5. CONCLUSON

From the theorem proved in §3 it follows that in the single-electron approximation a correlation exists between the presence of gaps in the density of states of the system and the presence of a certain fraction of delocalized states on the Landau levels. It was found in experiment[9] that the fraction of the delocalized states is $\approx 3\%$ of their total number on the Landau level. The presence of so small a fraction of delocalized states is apparently, even in the single-electron approximation, not an exception but the rule. It can be easily seen, for example, that in the model of sparse impurity centers[5] the greater part of the single-electron states is localized. The situation is similar also for a slow potential,[6] where the delocalized states correspond to only a narrow energy region near the center of the Landau level. In both models, an appreciable fraction of the localized states should have a low binding energy. This can explain in principle the very low characteristic temperatures ($T \lesssim 0.1$ K) at which wide steps appear in the QHE.[7-9] Another possible explanation of these facts, namely of the large fraction of localized states and the low temperatures at which the effects are observed, is the existence of multielectron correlations.[12,13]

We note that in a strong magnetic field $B \sim 100$ kOe the electron binding energy at an individual Coulomb center, $e^2/\varkappa\lambda$, is ~ 100 K ($\varkappa \sim 10$ is the average dielectric constant of the medium). We can therefore hope to expand the range of temperatures in which the QHE can be observed, by special preparation of samples with high density of strongly split-off localized states. It would be possible to separate in this manner the contribution of multielectron correlations, which should be substantial only a low temperatures. In addition, such studies would be of interest in connection with the concept of effective conductivity of a quasi-two-dimensional system, developed in Ref. 18. Consider a sample in the form of a long narrow rectangle $L_x \gg L_y$, and let the current across the sample be $j_y = 0$. The current and the field along the sample are then connected by the condition

$$j_x = (\sigma_{xx} + \sigma_{xy}^2/\sigma_{xx})E_x = \sigma_{\text{eff}}E_x, \tag{36}$$

where the second equality defines the effective conductivity of a quasi-two-dimensional electron system. The energy-dissipation density in the system is expressed in terms of this quantity. It is easily seen that in experiment[1] σ_{eff} executes giant oscillations when the carrier density or the magnetic field is varied. If the Fermi level lies in the region of delocalized states near the maximum of the state density of one of the broadened Landau levels, the components of the system conductivity tensor are of the same order: $\sigma_{xx} \sim \sigma_{xy}$. If, however, the Fermi level lands in the region of localized states on the tail of the Landau level, the component σ_{xy} is finite but the component σ_{xx} has an activation character and decreases exponentially with decreasing temperature. In this case σ_{eff} can reach anomalously large values. In an experiment,[19] the effective relaxation time $\tau_{\text{eff}} = \sigma_{\text{eff}}m^*/ne^2$ estimated in terms of σ_{eff} was $\sim 10^{-3}$ sec at $T = 1.2$ K. By producing samples with a sufficient number of strongly split-out localized states, there is hope of increasing the activation energy of the dissipative component σ_{xx}, so that the temperature region in which the effective conductivity of the system is anomalously large can be expanded.

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