Critical density of electrons on the surface of liquid helium

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A number of topics of the theory of a multielectron dimple on the surface of liquid helium are discussed in the harmonic approximation. In this approximation it is possible to describe the structure of an elliptic-symmetry dimple. An interesting analogy is noted between the Hertz contact problem of elasticity theory and the theory of a multielectron dimple on a helium surface. Qualitative ideas concerning the problem of dimple-disintegration mechanism in a strong electric field are advanced. Solution of the last problem gives an ideal of the real scale of the critical density of the electrons at the center of the dimples and concerning methods of increasing this density.

INTRODUCTION

One of the essential features of a charged liquid-helium surface is its instability to small oscillation when the electron density exceeds a certain critical value. Various concrete consequences of this instability, such as singularities of the dispersion of small oscillations of the charged surface of the liquid, reconstruction of the charged helium surface, and the appearance of a dimple crystal or of individual multielectron dimples, have been by now investigated in detail both theoretically[1–4] and experimentally.[5–8] Obviously, the instability of a charged surface of bulk helium will continue to stimulate efforts to clarify details of this effect. Yet the upper bound imposed on the surface-electron density by the instability of the charged surface of liquid helium is a serious obstacle to the development of independent trends in the investigation of the two-dimensional electron system on the surface of liquid helium, and in particular to the problem of Wigner crystallization. In fact, the Coulomb crystallization observed by Grimes and Adams[9] in the electron system on a helium surface, and the subsequent measurements of the phase diagram for this phase transition, observed so far only in a two-dimensional charged system over helium,[10–12] were peformed only in a relatively narrow electron-density interval, $\sim 10^8$ cm^{-2}. The real obstacle to an advance to higher density, which would be most interesting, e.g., from the viewpoint of the properties of the phase diagram of a Wigner crystal, is the capillary instability of a charged surface of helium. All the foregoing considerations lead to the problem of increasing the critical electron density over helium, a problem approached at present from two directions.

In the first it is proposed to use a thin helium film as the substrate for the electron system. The presence of additional van der Waals forces in such films increases drastically the capillary constant $\varkappa$ of the helium, leading automatically to an increase of the critical density n_{cr} of the electrons on the film compared with its bulk value. This idea was reinforced by calculations.[13–15] The first experimental facts[5,6–18] also suggests that this idea is reasonable. In this case, however, if we are dealing with Coulomb crystallization, we face in essence a new problem, inasmuch as on a thin helium film we have not a Coulomb but a dipole gas (the distance between the electrons on a thin helium film is comparable with the distance to the solid substrate, as a result of which each electron interacts mainly with its image and not with neighboring electrons). Progress in the solution of this problem is satisfactory,[15] but so far only from the theoretical viewpoint.

The next possibility was indicated recently by the experiments of Savignac and Leiderer[19] and Volodin and Édel'man.[20] It was shown in the first of the cited papers that a multielectron dimple produced on the surface of solid helium in the range of critical parameters (in particular, of the critical electric field $E_\perp^{\min}$) corresponding to loss of stability of the charged surface of bulk helium, can exist in a range of electric fields that exceed noticeably (by about three or four times) the field $E_\perp^{\min}$ at which such a dimple appears. Recognizing that the radius R of the charged core of a multielectron dimple in the region $\varkappa R \ll 1$ is inversely proportional to the square of E [this circumstance will be demonstrated below, see Eq. (13)], and that the scale of the electron density $n(0)$ at the center of the dimple is $n(0) \propto R^{-2} \propto E_\perp^4$, it is easily seen that the density, which increases with field like $E_\perp^4$, can substantially exceed the critical homogeneous density of the electrons above bulky helium. In other words, a multielectron dimple is a suitable object with which to produce on a helium surface electron densities that exceed the critical homogeneous density. Of course, this must be "paid for" by the onset of an electron-density inhomogeneity in the core of the multielectron dimple. This inhomogeneity, however, is relatively small relative to the corresponding period of the possible electron crystal in the dimple, and hopefully is of no principal signficance from the viewpoint of observing Coulomb crystallization.

An independent result, but conceptually close to that of Ref. 19, was obtained by Volodin and Édel'man.[20] They have shown that by using for the helium film a special furrowed substrate with groove depth h on the order of the capillary constant of the helium and with a characteristic period noticeably smaller than this constant it is possible to increase drastically the critical instability field, but preserve at the same time on the helium film the electron mobility typical of the bulk situation. It must be noted that the idea of artificial-

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ly suppressing the capillary instability of the charged surface of bulk helium with the aid of boundary conditions that hinder the development of oscillation of the charged surface of a liquid with small wave numbers (of the order of the capillary constant) is contained in the review by Williams.[21] We are thus dealing with a useful idea and its successful realization. It is also obvious that the results of Refs. 19 and 20, notwithstanding their apparent difference, indicate that multielectron dimples, especially artificial ones, are promising objects having an increased critical electron density.

In the description of the structure of a multicharged dimple in sufficiently strong clamping fields, when the inequality $\varkappa R \ll 1$ holds (R is the radius of the charged core of the dimple, $\varkappa$ is the capillary constant of the liquid helium), it becomes possible to determine independently the radius R and the deformation of the helium surface at distances from the dipole center that are large compared with R. To perform concrete calculations under these conditions it is convenient and legitimate to use a harmonic approximation in which the deformation $\xi(r)$ of the helium surface in the vicinity of the dimple surface is approximated by a Taylor series with accuracy up to quadratic or higher powers of the distance r to the center of the dimple. The use of this device yields a self-consistent description not only of cylindrically symmetric but also of elliptic dimples. The latter circumstance is of considerable interest, since there are a number of physical reasons for the appearance of elliptic dimples, but there is still no appropriate theory.

In view of foregoing, the plan of our exposition is the following. In the first part we present a linear theory for an arbitrary multielectron dimple in the harmonic approximation. The cause of the ellipticity is assumed to be anisotropy of the surface tension of the helium substrate; this is possible when the dimple is produced on the interface between liquid and solid helium. In the limiting case of zero anisotropy, the results of this theory can be compared with those already known for a dimple with cylindrical symmetry. It turns out that, apart from numerical factors, the results of the "harmonic" theory agree with those previously obtained for a cylindrical dimple. This raises hope that a generalized harmonic approximation is qualitatively correct for the description of nonlinear effects that occur in the behavior of a multielectron dimple in the presence of a strong electric field. The second section of the paper is an exposition of the results of the generalized theory for a dimple. The actual calculations could be carried though to conclusion only in the cylindrically symmetric case. It became clear that besides the minimum electric field $E_\perp^{\min}$ starting with which the existence of a dimple becomes energywise favored, the theory admits of another characteristic electric field $E_+{}^{\max}$ that bounds from above the region of existence of dimples. The field $E_\perp^{\max}$ is defined by Eq. (35). In contrast to $E_\perp{}^{\min}$, the quantity $E_\perp{}^{\max}$ is an essential function of the total charge Q of the dimple. Another interesting end result of this section is the definition (34) of the radius R of the charged-region spot at the center of the dimple. The dependence of R on the problem parameters $E_\perp$ and Q has a regular asymptotic behavior in the region of moderate fields $E_\perp^{\min} < E_\perp < E_\perp^{\max}$ and a nontrivial behavior as $E_\perp$ comes close to $E_\perp^{\max}$.

1. ELLIPTIC DIMPLES IN THE HARMONIC APPROXIMATION

The problem of elliptic multielectron dimples arises, for example, in the investigation of the instability of a charged interface between liquid and solid helium (photographs of anisotropic multicharged dimples on such an interface are given in Ref. 19). A similar situation can occur in the investigation of the properties of an electron system in a trough between vertical plates immersed in liquid helium (a variant of the problem of Volodin and Édel'man[20]). Finally, a dimple becomes elliptic in the presence of a magnetic field parallel to the liquid-helium surface. The corresponding possibilities of the theory will be demonstrated below, mainly in the framework of the so-called harmonic approximation that permits the use of an analogy with the Hertz contact problem in elasticity theory.[22] A brief indication of this analogy is contained in Ref. 23.

The initial system of equations for the description, e.g., of an elliptic multicharged dimple on an anisotropic solid-helium substrate is, according to Refs. 3 and 4, of the form

$$eE_\perp \xi(x,y) + e^2 \int \frac{n(x_1,y_1)\,dx_1\,dy_1}{[(x-x_1)^2+(y-y_1)^2]^{1/2}} = \lambda, \tag{1}$$

$$\alpha_1 \xi_x'' + \alpha_2 \xi_y'' + \Delta\rho g \xi = eE_\perp n(x,y), \tag{2}$$

$$\xi_x'|_0 = \xi_y'|_0 = 0, \quad \xi\Big|_{\substack{x\to\pm\infty \\ y\to\pm\infty}} \to 0, \tag{3}$$

$$\int n(x,y)\,dx\,dy = N. \tag{4}$$

Here $\xi(x,y)$ is the self-consistent deformation of the interface between the liquid and solid helium in the vicinity of the charged dimple, $n(x,y)$ is the local charge density in the dimple, $E_\perp$ is the clamping field, $\Delta\rho$ is the difference between the densities of the liquid and solid helium, g is the free-fall acceleration, and λ is a Lagrange multiplier that appears in the problem because of the normalization of $n(x,y)$ to N [Eq. (4)], where N is the total number of charges in the dimple. The physical meaning of the multiplier λ will be discussed below. In Eq. (2) it is assumed that the x and y axes are chosen along two mutually perpendicular crystallographic axes of the solid-helium surface, and α_1 and α_2 are the corresponding surface-tension components.

The main task in the solution of the system (1)–(4) is the inversion of Eq. (1) relative to $n(x,y)$. In the general case of arbitrary $\xi(x,y)$ this problem can be solved only in the one-dimensional variant.[24] As for the two-dimensional situation, a simple enough solution exists here in the harmonic approximation, when the deformation $\xi(x,y)$ in the vicinity of the origin can be represented in the form

$$\xi(x,y) = \xi_0 + {}^1/_2\xi_x''(0)\,x^2 + {}^1/_2\xi_y''(0)\,y^2 + \ldots = \xi_0 + Ax^2 + By^2 + \ldots \tag{5}$$

Equation (1) with $\xi(x,y)$ expanded as in (5) coincides with the basic equation of the Hertz contact problem in elasticity theory.[22] Using this analogy, we have directly from (1) and (5)

$$n(x,y) = \frac{3N}{2\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}, \tag{6}$$

$$eE_\perp A = \frac{3}{4} Ne^2 \int_0^\infty \frac{d\xi}{(a^2+\xi)[(a^2+\xi)(b^2+\xi)\xi]^{1/2}}, \tag{7}$$

$$eE_\perp B=\frac{3}{4}Ne^2\int_0^\infty\frac{d\xi}{(b^2+\xi)[(a^2+\xi)(b^2+\xi)\xi]^{1/2}},\qquad(8)$$

$$\lambda-eE_\perp\xi(0)=\frac{3}{4}Ne^2\int_0^\infty\frac{d\xi}{[(a^2+\xi)(b^2+\xi)\xi]^{1/2}}.\qquad(9)$$

Expression (6) determines the charge distribution in the center of the elliptic multicharged dimple, while Eqs. (7) and (8) give two connections between the four unknown constants A, B, a, and b (a and b are the semiaxes of the charged ellipse); relation (9), finally, determines the value of the chemical potential λ in this problem. In principle, the instant at which the constant λ vanishes sets the conditions starting with which the multicharged dimple becomes energywise favored.

To obtain additional relations between the constants A, B, a, and b we must use the equation of mechanical equilibrium with appropriate boundary conditions. We consider in this connection several concrete cases.

A. Cylindrically symmetric situation. In this variant, when $\alpha_1=\alpha_2=\alpha$, the semiaxes of the ellipse are equal, $a=b=R$, and the mechanical-equilibrium equation takes the form

$$\Delta\xi-\varkappa^2\xi=\frac{eE_\perp}{\alpha}n(r),\qquad(2a)$$

$$n(r)=\frac{3N}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{1/2},\quad r^2=x^2+y^2,\quad \varkappa^2=\frac{\Delta\rho g}{\alpha}.\qquad(10)$$

Assuming that $\varkappa R\ll 1$, we have from (2a) the relation

$$\xi_0''\approx eE_\perp\frac{3N}{4\pi R^2\alpha}.\qquad(11)$$

As for the definitions (7) and (8), they agree with each other and reduce inthe cylindrical-symmetry case to the equality

$$\xi_0''=3\pi Q/4E_\perp R^3,\quad Q=eN.\qquad(12)$$

From (11) and (12) follows the definition

$$R=\pi^2\alpha/E_\perp^2.\qquad(13)$$

The result (13) for R is equal, apart from a number, to the definition of R in Ref. 3 in the limit $\varkappa R\ll 1$.

Relation (9) with allowance for the definition of the self-consistent deformation of the helium surface in the vicinity of the dimple, and in particular of the dimple depth $\xi(0)$, takes the form

$$\lambda=\frac{e^2E_\perp^2N}{2\pi\alpha}\left(\frac{3}{2\pi}-\ln\frac{1}{\varkappa R}\right),\quad \varkappa R\ll 1.\qquad(14)$$

The condition for λ to vanish in this case is

$$\varkappa R\leqslant 0{,}625\quad\text{or}\quad E_\perp^2>(E_\perp^{\min})^2,\qquad(E_\perp^{\min})^2=15{,}78\varkappa\alpha.\qquad(15)$$

The correct criterion obtained in Refs. 3 and 4 using more accurate algebra, yields $(E_\perp^{\min})^2=16.52\varkappa\alpha$. The definition (10) of $n(r)$ coincides with the first term of the corresponding expansion of $n(r)$ in Ref. 4.

It is worthy of note that the formalism developed in Ref. 4 to describe the structure of a cylindrically symmetric dimple makes it possible, in principle, to take into account also higher powers of the expansion (5) of $\xi(r)$. Claiming only that $\varkappa R\ll 1$, the corresponding calculations can be carried out analytically, dealing ultimately with a finite system of algebraic equations. The details of this calculation are given in Appendix 1. As a result, the definition of $n(r)$ accurate to terms $\propto r^4$ in the expansion (5) of $\xi(r)$ takes the form

$$n(r)\approx\frac{3N_1}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{1/2}+\frac{5N_2}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{3/2},\qquad R=0{,}82\frac{\pi^2\alpha}{E_\perp^2},\quad N_1+N_2=N,\quad N_2=-0{,}09N_1.\qquad(10a)$$

Thus, the next term of the expansion of $n(r)$ in polynomials of the form $(1-r^2/R^2)^{s+1/2}$ are numerically small.

In their original paper[4] the authors were interested in the general situation of arbitrary $\varkappa R$. The corresponding system of algebraic equations was therefore of high order and was investigated by numerical methods.

B. Anisotropic substrate. The equation (2) for mechanical equilibrium at the center of the charged core of the dimple takes the form

$$2\alpha_1A+2\alpha_2B=eE_\perp n(0).\qquad(16)$$

This can be used as one of the additional relations between the sought coefficients A, B, a, and b.

The last of the sought relations

$$2\alpha_1A=eE_\perp n(0)\delta/(\gamma+\delta),\quad \gamma^2=\alpha_2/\alpha_1,\quad \delta=b/a,\qquad(17)$$

was obtained by solving Eq. (2) and then calculating $\xi_x''(0)$ (see Appendix 2). The expression used for the distribution $n(x, y)$ was then

$$n(x,y)=\frac{N}{\pi ab}\exp\left(-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

The justification for this simplification is the simplicity of (17). The corresponding calculations for the distribution $n(x, y)$ [Eq. (6)] leads to an integral connection between the coefficients of interest to us rather than to (17).

The system (7), (8), (16), and (17) can be reduced to a single relation between γ and δ:

$$\gamma\delta=F(\delta),\quad F(\delta)=\int_0^\infty ds/(1+s)[(1+s)(\delta^2+s)s]^{1/2}\Big/\int_0^\infty ds/(\delta^2+s)[(1+s)(\delta^2+s)s]^{1/2}.\qquad(18)$$

Thus, to each value of the parameter γ corresponds a definite value of the parameter δ. In the limit as $\gamma\to 1$, naturally, $\delta\to 1$.

The integrals in (18) can be expressed in terms of elliptic ones. The resultant equations, however, are quite unwieldy. It is therefore sensible to make do with an analytic relation between γ and δ in the limits of either small or large ellipticity.

Assuming with this in mind

$$\delta^2=1+\Delta,\quad \Delta\ll 1,\qquad(19)$$

we have from (18)

$$\gamma\approx 1+{}^1/_4\Delta,\quad \Delta\ll 1.\qquad(20)$$

In the opposite limiting case $\delta\gg 1$ we have from (18)

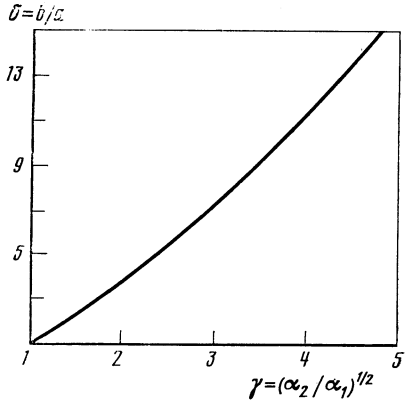

FIG. 1.

$$\gamma \approx \delta/\ln \delta^2. \tag{20a}$$

The relation between γ and δ is shown for the general case in Fig. 1.

A variant close to the above treatment of an elliptic dimple is obtained if the source of the ellipticity is a one-dimensional external potential that clamps the dimple in one of the directions. If, for the sake of argument, this additional action is directed along the x axis and is given by

$$V(x)=V_0+Cx^2+\ldots, \quad C={}^1/_2 V_x''(0), \tag{21}$$

the definition (6) of $n(x, y)$ and relations (8) and (9) remain unchanged, but the definitions (7), (16), and (17) become

$$eE_\perp A+C=\frac{3}{4}Ne^2\int_0^\infty \frac{d\xi}{(a^2+\xi)[(a^2+\xi)(b^2+\xi)\xi]^{1/2}}, \qquad 2A=\frac{eE_\perp}{\alpha}n(0)\frac{\delta}{1+\delta}, \quad 2\alpha(A+B)=eE_\perp n(0). \tag{22}$$

As a result, the analog of (18) is now

$$C^*=\frac{F(\delta)-\delta}{1+\delta}, \quad C^*=C\left(\frac{e^2E_\perp{}^2 n(0)}{2\alpha}\right)^{-1}, \tag{23}$$

where $F(\delta)$ is the right-hand side of (18).

2. DISINTEGRATION OF DIMPLE IN A STRONG ELECTRIC FIELD

The existing theory of a multielectron dimple on the surface of liquid helium[3,4] is valid if the following inequalities hold:

$$\nabla\xi \ll 1, \quad E_\perp > 2\pi e n(0). \tag{24}$$

The first of these inequalities allows us to linearize the surface-tension term in the mechanical-equilibrium equation. The second is needed for the dimple to be electrostatically stable (it is necessary that the external clamping field exceed the intrinsic field of the charged core of the dimple). Neither inequality is automatically satisfied. Thus, if we use for $n(0)$ the definitions that follow from (10) and (13), the second inequality of (24) is replaced by

$$3E_\perp{}^3 Q/\pi^4\alpha^2 < 1, \quad Q=eN, \tag{24a}$$

which, given Q, is a restriction on the value of $E_\perp$. A similar inequality can be obtained on the basis of the requirement $\nabla\xi < 1$ by using for $\nabla\xi$ the explicit expression for the gradient in the region $r \sim R$, where $\nabla\xi$ is a maximum. This raises the question: what happens to the dimple when inequalities (24) become uncertain or even wrong?

A. To shed light on the features of this problem it is useful to discuss first an auxiliary one-dimensional problem that can, in principle, be also of interest as the limiting case of a strongly elongated elliptic dimple. The general mechanical-equilibrium equation for a one-dimensional multielectron dimple is

$$\frac{\xi''}{[1+(\xi')^2]^{3/2}}-\varkappa^2\xi=p_{el}(x)\alpha^{-1}, \tag{25}$$

$$\int_{-\infty}^{+\infty} p_{el}(x)\,dx=eE_\perp N, \quad \xi'(0)=0, \quad \xi(\pm\infty)\to 0;$$

$$n(x)>0 \text{ at } -R\leqslant x\leqslant +R \text{ and } n(x)=0 \text{ at } |x|>R;$$

N is the total number of electrons per unit length of the dimple.

Taking into account the inequality $\varkappa R \ll 1$, which is assumed to be satisfied, it is convenient to rewrite (25) in homogeneous form with appropriate boundary conditions

$$\xi''[1+(\xi')^2]^{-3/2}-\varkappa^2\xi=0, \quad |x|\gtrsim R, \tag{26}$$

$$\xi(\pm\infty)\to 0, \tag{27}$$

$$\xi'[1+(\xi')^2]^{-1/2}|_{x=\pm R}=\pm(eE_\perp N/2\alpha+\varkappa^2\xi_0 R). \tag{28}$$

To obtain the boundary condition (28) we must integrate (25) in the vicinity of small x (at $-R\leqslant x\leqslant +R$) and take into account the requirement

$$\int_{-R}^{\pm R} p_{el}(x)\,dx=eE_\perp N.$$

It is also obvious that

$$\varkappa^2\int_{-R}^{+R}\xi(x)\,dx\approx 2\varkappa^2\xi_0 R,$$

where ξ_0 is the depth of the dimple.

The first integral of (26) is

$$(1+(\xi')^2)^{-1/2}=A_0-{}^1/_2\varkappa^2\xi^2. \tag{29}$$

From the condition (27) at infinity it is clear tht the constant $A_0 = 1$. Using now the integral (29) at distances $x \sim R$ from the dimple center and recognizing that $\nabla\xi$ is a maximum in this region, we easily see that under conditions when ξ_{max} becomes large enough ($\xi' > 1$) the maximum value of ξ_0 tends to $\xi^0_{max}\to 2^{1/2}\varkappa^{-1}$. This estimate of ξ^0_{max} leads to the conclusion that the right-hand side of (28) can be simplified by virtue of the inequality

$$eE_\perp N\alpha^{-1} \gg 2\varkappa^2\xi_0 R = 2^{3/2}\varkappa R.$$

In fact, in sufficiently strong fields the combination $eE_\perp N\alpha^{-1}$ can take on values larger than 1. As for the parameter $\varkappa R$, it is by definition less than unity and decreases with increasing clamping field. Thus, the boundary condition (28) for ξ'_R takes the form

$$\xi_R'[1+(\xi_R')^2]^{-1/2}\approx eE_\perp N/2\alpha \tag{30}$$

or

$$\xi_R'=\Gamma^2/(1-\Gamma^2), \quad \Gamma=eE_\perp N/2\alpha. \tag{30a}$$

Obviously, this boundary condition is meaningful only in the region

$$\Gamma<1. \tag{31}$$

The inequality (31) is the sought upper bound of $E_\perp$ at fixed N.

From the foregoing analysis we can draw two qualitative conclusions:

1. Just as in the region of relatively weak electric fields, mechanical equilibrium at the center of a charged dimple is ensured mainly by the competition between the electron pressure and the surface-tension forces (the discussion that follows Eq. (29) shows the gravitational term to be negligible).

2. In the linear region, where the denominator of the surface term becomes influential, it becomes impossible for the electron pressure to be offset by the surface tension forces. These qualitative conclusions should hold also in the cylindrical-symmetry problem.

B. In the more general, two-dimensional, case the average curvature of the dimple surface is defined by

$$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{2}\left[1+\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\right]^{-3/2}$$
$$\times\left\{\frac{\partial^2\xi}{\partial x^2}\left[1+\left(\frac{\partial\xi}{\partial y}\right)^2\right]-\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2\xi}{\partial x\partial y}+\frac{\partial^2\xi}{\partial y^2}\left[1+\left(\frac{\partial\xi}{\partial x}\right)^2\right]\right\}. \tag{32}$$

The structure of this definition is quite complicated and does not admit of calculations similar to the one-dimensional ones. In the presence of cylindrical symmetry, however, and if the harmonic approximation (5) is used for $\xi(x, y)$, the situation is simpler, for in the vicinity of the origin the average curvature takes the form

$$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\approx\frac{2\xi_0''+(\xi_0'')^3r^2}{2[1+(\xi_0'')^2r^2]^{3/2}},\quad r^2=x^2+y^2. \tag{32a}$$

Proceeding now in analogy with the one-dimensional case, i.e., integrating the general mechanical-equilibrium equation with the Laplace term from (32a) over the region $r\leqslant R$ and neglecting the gravitational term (this is more justified in the cylindrical-symmetry case than in the one-dimensional one), a relation reminiscent of (28) can be obtained:

$$\xi_0''R^2[1+(\xi_0'')^2R^2]^{-1/2}=E_\perp Q/2\pi\alpha,\quad Q=eN. \tag{33}$$

We note also that the definition (33) is a generalization of (11) to include the case of arbitrary ξ_0''. Assuming now that the connection (12) between ξ_0'' and R which follows from the solution of Eq. (1), retains its meaning, and using it jointly with (33), we arrive at the following definition of R:

$$(R_c/R)^2=\tfrac{1}{2}\Lambda^{-2}(1-(1-4\Lambda^2)^{1/2}),$$
$$\xi_0''=R_c^2/R^3,\quad \Lambda=E_\perp{}^3Q/3\pi^3\alpha^2,\quad R_c^2=3\pi Q/4E_\perp. \tag{34}$$

In the limiting case $\Lambda\ll 1$ the structure of R is similar to that of (13)

$$R\approx 3\pi^2\alpha/2E_\perp{}^2. \tag{34a}$$

The difference between the numerical coefficients in the definitions of R from (13) and (34a) is a measure of the accuracy of the harmonic approximation in its various modifications.[1]

If, however, Λ becomes comparable with unity, the situation changes. According to (34) the real expression for R holds only under the conditions

$$\Lambda<\tfrac{1}{2}\quad\text{or}\quad E_\perp<E_\perp^{max},\quad E_\perp^{max}=(3\pi^2\alpha^2/2Q)^{1/3}. \tag{35}$$

The inequality (35) is analogous in character to inequality (31) for the one-dimensional problem, and defines the scale of the limiting electric field in which equilibrium between the surface-tension forces and the electron pressure is possible at the dimple center for fixed Q.

The corresponding minimum radius of the charged spot at the dimple center is, obviously, given by

$$R_{min}^2=R_c^2=3\pi Q/4E_\perp. \tag{36}$$

The fact that the minimum radius R_{min} is finite means that under conditions when a multielectron dimple disintegrate the quantity ξ_0'', which is connected with R_{min} by relation (33) or (12), remains finite. This, in turn, allows us to conclude that the critical value of ξ_R', estimated at $\xi_R'=\xi_0''R$, remains finite (in contrast to the one-dimensional case, where ξ_R' became infinite at the critical point corresponding to disintegration of the dimple). According to (34) and (35), the numerical value of ξ_R' is ≈ 2. It is useful to note here that the second of the inequalities (24) discussed at the beginning of the present section, $E_\perp>2\pi en(0)$, is not violated all the way to the critical field value. In fact, this inequality, with account taken of the definition $(0)=N/\pi R^2$ and of R_{min}^2 (36), is transformed into the numerical inequality $1>8/3\pi$ which is satisfied, albeit weakly, in the required direction.

Concluding the discussion of the results of the present section, let us track the behavior of the electron density in the dimple in the critical region. Using for this purpose the definition $n(0)=N/\pi R^2$ and expressing with the aid of (35) the critical field $E_\perp^{max}$ in terms of the total charge Q of the dimple, we obtain on the basis of (36) the following definition of the critical electron density at the dimple center:

$$n^m(0)=\frac{4}{3\pi e}\left(\frac{3\alpha^2}{2Q}\right)^{1/3}. \tag{37}$$

The critical density increases thus with decreasing total charge of the dimple. This circumstance must be borne in mind when searching for possibilities of increasing the critical density of the electrons on a helium surface. The numerical value of $n^m(0)$ for $N\sim 10^5$ is of the order of $\sim 10^{10}$ cm^{-2}.

It is interesting to note that the relation $n_0^m\propto Q^{-1/3}$ can be obtained under the assumption that in the critical region the central part of the dimple recalls a multielectron bubble having a charge Q. In this case the bubble radius $R^3\propto Q^2/\alpha$ (Ref. 25), and for the density n_m^0 we get the estimate

$$n_0{}^m\sim Q/R^2\infty\frac{1}{e}(\alpha^2/Q)^{1/3},$$

which correlates with n_0^m from (37). The reasoning indicated is given in Ref. 4.

CONCLUSION

We have discussed a number of questions in the theory of a multielectron dimple on a helium surface in the harmon-

ic approximation. This approximation makes possible a description of the structure of an elliptically symmetric dimple, points out an interesting analogy between the Hertz contact problem of elasticity theory and the theory of multielectron dimples on a helium surface, and gives a qualitative interpretation of dimple disintegration in a strong electric field. Solution of the last problems gives an idea of the real scale of the density of the electrons at the center of a multielectron dimple and of methods for increasing it. In particular, the experiments of Volodin and Édel'man[20] on retaining electrons on a helium surface in small-radius artificial dimples are from this viewpoint quite promising.

APPENDIX 1

As noted in Ref. 4, if the distribution $n(r)$ is represented by the series

$$n(r)=\sum_{s=1}^{m} N_s n_s(r),$$
$$n_s(r)=\frac{2s+1}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{s-1/2} \tag{1.1}$$

and the relation

$$\int \frac{n_s(r_1)\,d^2r_1}{|r-r_1|}=\frac{1}{R}\sum_{k=0} c_{ks}\left(\frac{r}{R}\right)^{2k},$$
$$c_{ks}=(-1)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)\Big/\,\Gamma^2(k+1)\Gamma(s-k+1) \tag{1.2}$$

is used ($\Gamma(x)$ is the gamma function), the integral term of Eq. (1) of the main text is transformed in the cylindrically symmetric case into a polynomial of order m in powers of r^2. Expanding next the displacement $\xi(r)$ in powers of r^{2k} and gathering coefficients of like powers of r^{2k}, we can reduce Eq. (1) to a system of equations in terms of the coefficients N_s and $\partial^s\xi/\partial r^s|_0$. Thus, retaining two terms in expansion (1.1) and following the procedure indicated, we get

$$eE_\perp\xi_0+e^2\frac{N_1}{R}c_{01}+e^2\frac{N_2}{R}c_{02}=\lambda, \tag{1.3a}$$
$$\frac{1}{2}E_\perp\xi_0''+e\frac{N_1}{R^3}c_{11}+e\frac{N_2}{R^3}c_{12}=0, \tag{1.3b}$$
$$\frac{1}{6}E_\perp\xi_0''''+e\frac{N_2}{R^5}c_{22}=0, \tag{1.3c}$$
$$N_1+N_2=N, \tag{1.3d}$$
$$c_{01}=3/4\pi,\quad c_{02}=15/16\pi,\quad c_{11}=-3/8\pi,$$
$$c_{12}=-15/16\pi,\quad c_{22}=45/128\pi.$$

Two additional connections between the coefficients N_1, N_2, R, ξ_0'', and ξ_0'''' follow from the mechanical equilibrium equation (2) of the main text:

$$\xi_0''=\frac{eE_\perp}{2\alpha}\left(\frac{3N_1}{2\pi R^2}+\frac{5N_2}{2\pi R^2}\right),$$
$$\xi_0''''=\frac{eE_\perp}{2\alpha}\left(\frac{3N_1}{2\pi R^4}+\frac{15N_2}{2\pi R^4}\right). \tag{1.4}$$

The first relation (1.4) follows directly from Eq. (2) written at $r=0$, neglecting the gravitational term. The second is obtained by differentiating Eq. (2) twice. Those terms of this expansion which diverge as $r\to 0$ are mutually cancelled out. We note also that determination of the constant λ in (1.3) calls for a complete solution of the mechanical-equilibrium condition with account taken of the gravitational term, after which $\xi 0$ can be obtained in terms of the electron pressure. We shall not consider this last problem.

Combining Eqs. (1.3b)–(1.3d) and (1.4) we can reduce this system to the equations

$$R=-\frac{8\pi\alpha(c_{11}N_1+c_{12}N_2)}{E_\perp^2(3N_1+5N_2)},$$
$$N_2^2(5c_{12}-10c_{22})+N_1N_2(c_{12}+5c_{11}-6c_{22})+c_{11}N_1^2=0. \tag{1.5}$$

Solving the last equation for N_2 and using the information on c_{sk}, we get

$$N_2=(-0{,}30\pm 0{,}21)N_1. \tag{1.6}$$

The uncertainty of the sign in this relation should be resolved in favor of $+$, for otherwise expression (1.5) for R becomes negative. Taking the foregoing into account, we obtain the following final expressions for $n(r)$ and R:

$$n(r)=\frac{3N_1}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{1/2}+\frac{5N_2}{2\pi R^2}\left(1-\frac{r^2}{R^2}\right)^{3/2},$$
$$N_1+N_2=N,\quad N_2=-0{,}09N_1,\quad R=0{,}82\pi^2\alpha/E_\perp^2. \tag{1.7}$$

The results (1.7) were used in the main text.

APPENDIX 2

The equation

$$\alpha_1\xi_{xx}''+\alpha_2\xi_{yy}''+\Delta\rho g\xi=eE_\perp n(x,\,y), \tag{2.1}$$

where the distribution

$$n(x,y)=\frac{N}{\pi ab}\exp\left(-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

is normalized to N, can be solved by using Fourier transformation with respect to both variables x and y. The quantity $\xi_x''(0)$ of interest to us is then

$$\xi_{xx}''(0)=-\frac{2}{\pi}\int_0\int\frac{eE_\perp k^2 n(k,q)}{\alpha_1k^2+\alpha_2q^2}\,dk\,dq,$$
$$n(k,q)=\frac{N}{2\pi}\exp\left(-\frac{1}{4}k^2a^2-\frac{1}{4}q^2b^2\right). \tag{2.2}$$

In expression (2.2) for $\xi_x''(0)$ we left out of the denominator of the integrand the gravitational term; this does not affect adversely the convergence of the integrals at small k and q (in contrast to the expression for $\xi(0)$, where such a simplification is impossible).

After integrating in (2.2) with respect to k we have

$$\xi_{xx}''(0)=-\frac{E_\perp Q}{\pi\alpha_1}\int_0^\infty dq\exp\left(-\frac{q^2b^2}{4}\right)\left\{\frac{\sqrt{\pi}}{a}-\frac{\pi\gamma}{2}q\cdot\right.$$
$$\left.\times\exp\left(\frac{1}{4}a^2\gamma^2q^2\right)\left[1-\Phi\left(\frac{1}{2}\gamma aq\right)\right]\right\}, \tag{2.3}$$

where $\Phi(x)$ is the error function and $\gamma^2=\alpha_2/\alpha_1$.

The integration with respect to q is also carried out in explicit form. As a result,

$$\xi_x''(0)=\frac{E_\perp Q}{\alpha_1\pi ab}\frac{\delta}{\gamma+\delta},\quad \delta=\frac{b}{a}, \tag{2.4}$$

or, recognizing that $N/\pi ab = n(0)$, we arrive at the definition of ξ_x'' used in the main text.

[1]In the calculation of R (13) we used the local characteristics of the dimple, and in particular the local value of the density $n(0)$. As the definition (34a) of R, we averaged here the mechanical-equilibrium equation over a region $r \leqslant R$.

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