Pressure of a gas of fast charged particles that diffuse in a medium with a stochastic magnetic field

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The force exerted by a collisionless gas of fast charged particles on a medium with a frozen-in random magnetic field is calculated in the diffusion approximation.

1. INTRODUCTION

The motion of fast charged particles (cosmic rays) that do not experience Coulomb collisions in cosmic plasma can frequently be treated in the diffusion approximation. The spatial diffusion is due here to scattering of particles that interact with random electromagnetic fields. The first example of a consistent calculation of the diffusion of cosmic rays was that of Dolginov and Toptygin.[1] It was assumed that the scattering was by random magnetic-field inhomogeneities "frozen" in the infinitely conducting medium.[1)]

The energy density of fast particles in outer space can be relatively large. Thus, in the interstellar medium the energy density of the cosmic rays is of the order of the energy density of the magnetic field and of the energy density of the turbulent motions of the medium. This raises the question of the dynamic action of the fast particles on the cosmic plasma. In those cases when the spatial scale of the corresponding slow hydrodynamic motions of the medium is high compared with the transport range of the fast particles, it seems natural to use the concept of "fast particle pressure" $P=\frac{1}{3}\int d^3pvf_0$ ($f_0(t,\mathbf{r},\mathbf{p},)$ is the isotropic part of the fast-particle momentum distribution) and to introduce a corresponding force density $-\nabla P$ into the equation of motion for the continuous medium. This approach was used without a rigorous basis, in particular, in problems on the dynamics of the galactic wind,[4] the interaction of cosmic rays with solar wind,[5] propagation of long MHD waves in the interstellar medium,[6] and the stability of the galactic halo.[7]

This paper is devoted to a calculation of the density of the force exerted by a gas of fast particles on a continuous conducting medium with a frozen-in random magnetic field. What is specifically calculated is the quantity

$$f=\frac{1}{c}\langle[\delta\mathbf{j},\times\delta\mathbf{H}_{cr}]\rangle; \tag{1}$$

here $\delta\mathbf{j}$ is the fluctuating current connected with the presence of a random field in the medium; $\delta\mathbf{H}_{cr}$ are the magnetic field fluctuations due to the fluctuating current of the scattered fast charged particles. The averaging is over an ensemble of random magnetic fields. It turns out that under conditions of applicability of the diffusion approximation for the fast particles the force density (1) actually reduces to a pressure gradient $\mathbf{f}=-\nabla P$, so that the formally introduced pressure of a collisionless gas of fast particles can be used in the equations of motion of a continuous medium. No additional conditions, other than validity of the diffusion approximation, are needed in this case.

2. FLUCTUATION OF A CURRENT OF FAST PARTICLES AS THEY DIFFUSE IN A RANDOM MAGNETIC FIELD

We consider the motion of fast charged particles in a medium with a static random magnetic field $\mathbf{H}(\mathbf{r})(\langle\mathbf{H}\rangle=0)$ whose correlation scale is much smaller than the particle gyroradius: $L_c\ll r_H$. The fast-particle distribution function $f(t,\mathbf{r},\mathbf{p})$ satisfies the equation

$$\frac{\partial f}{\partial t}+(v\nabla)f+q\left[\frac{\mathbf{v}}{c}\times\mathbf{H}\right]\frac{\partial f}{\partial\mathbf{p}}=0, \tag{2}$$

where q and v are the particle charge and velocity.

The condition $L_c\ll r_H$ allows us to regard the last term of the left-hand side of (2) as a perturbation and to use the methods of the theory of a weakly turbulent plasma[8] (see also Ref. 9). Putting

$$f=\langle f\rangle+\delta f,\quad \langle\delta f\rangle=0,\quad \langle f\rangle\gg|\delta f|, \tag{3}$$

we find[2)]

$$\delta f(t,\mathbf{r},\mathbf{p})=\int_{-\infty}^{t}dt_0\,H_i(\mathbf{r}-\mathbf{v}(t-t_0))D_i\langle f\rangle(t,\mathbf{r},\mathbf{p}),\qquad \mathbf{D}=\frac{q}{c}\left[\mathbf{v}\times\frac{\partial}{\partial\mathbf{p}}\right] \tag{4}$$

and obtain an equation for the averaged slowly varying distribution function

$$\partial\langle f\rangle/\partial t+(\mathbf{v}\nabla)\langle f\rangle=D_jM_{ji}D_i\langle f\rangle(t,\ \mathbf{r},\ \mathbf{p}), \tag{5}$$

where

$$M_{ji}=\int_{-\infty}^{t}dt_0\langle H_j(\mathbf{r})H_i(\mathbf{r}-\mathbf{v}(t-t_0))\rangle. \tag{6}$$

For a statistically isotropic and homogeneous random field with a Gaussian correlation function[1]

$$\langle H_j(\mathbf{r}_1)H_i(\mathbf{r}_2)\rangle=B_{ji}(\mathbf{x})=\frac{\langle H^2\rangle}{12}\left(\frac{\partial^2}{\partial x_j\,\partial x_i}-\delta_{ji}\frac{\partial^2}{\partial x_s\,\partial x_s}\right)\Phi(x), \tag{7}$$

$$\Phi(x)=L_c^2\exp(-x^2/L_c^2), \tag{8}$$

where $\mathbf{x}=\mathbf{r}_1-\mathbf{r}_2$, we obtain

$$M_{ji}=\frac{\pi^{1/2}}{12}\frac{\langle H^2\rangle L_c}{v}\left(\delta_{ji}+\frac{v_jv_i}{v^2}\right). \tag{9}$$

Equation (5) with account taken of relation (9) takes the form

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$$\frac{\partial\langle f\rangle}{\partial t}+(\mathbf{v}\nabla)\langle f\rangle=\frac{\pi^{1/2}}{12}\langle H^2\rangle L_c D_i v^{-1} D_i\langle f\rangle \tag{10}$$

(a similar equation was obtained in Ref. 1 by another method). The particle scattering described by the right-hand side of (10) leads to isotropization of the distribution function $\langle f\rangle$ after a certain characteristic time τ. The evolution of the function $\langle f\rangle$ over a scale $R\gg v\tau$ and over a time $T\gg\tau$ can be described by the diffusion equation.

We write the distribution function in the form

$$\langle f\rangle(t,\mathbf{r},\mathbf{p})=f_0(t,\mathbf{r},p)+3v^{-2}\mathbf{vJ}(t,\mathbf{r},p), \tag{11}$$

where the second term in the right-hand side of the equation describes the weak deviation of the function $\langle f\rangle$ from isotropic. From (10) we obtain an equation for the isotropic part of the distribution function $f_0=\int d\Omega\,\langle f\rangle/4\pi$ and an expression for the diffusion flux J (see Ref. 1):

$$\partial f_0/\partial t-\nabla\varkappa\nabla f_0=0, \tag{12}$$

$$\mathbf{J}=-\varkappa\nabla f_0. \tag{13}$$

The diffusion coefficient is equal to

$$\varkappa=2c^2p^2v/\pi^{1/2}q^2L_c\langle H^2\rangle. \tag{14}$$

From (4), (11), (13), and (14) we find the current connected with the fast-particle functions that diffuse in the medium with the random magnetic field

$$(\delta j_{cr})_i=q\int d^3p\,v_i\,\delta f(t,\mathbf{r},\mathbf{p})$$

$$=-(6c/\pi^{1/2}L_c\langle H^2\rangle)\int_0^\infty d\tau\int d^3p\,pv_i[\mathbf{v}\nabla]_s f_0 H_s(\mathbf{r}-\mathbf{v}\tau). \tag{15}$$

3. FORCE ACTING ON THE MEDIUM

To calculate the force (1) we must find the magnetic field $\delta\mathbf{H}_{cr}(t,\mathbf{r})$ due to the fluctuating cosmic-ray current $\delta\mathbf{j}_{cr}(t,\mathbf{r})$. Since the current (15) varies relatively slowly in time, we can use Maxwell's equations in the quasi-stationary approximation:

$$4\pi c^{-1}\delta\mathbf{j}_{cr}=\operatorname{rot}\delta\mathbf{H}_{cr},\quad \nabla\delta\mathbf{H}_{cr}=0. \tag{16}$$

Using the Fourier transform

$$\delta\mathbf{j}_{cr}(t,\mathbf{k})=\int d^3r\,e^{-i\mathbf{kr}}\,\delta\mathbf{j}_{cr}(t,\mathbf{r}),$$

we obtain from (16) and (15)

$$(\delta\mathbf{H}_{cr})_s(t,\mathbf{k})$$

$$=-\frac{24\pi^{1/2}i}{L_c\langle H^2\rangle k^2}\int_0^\infty d\tau\int d^3p\,p[\mathbf{kv}]_s[\mathbf{v}\nabla]_l\,e^{-i\mathbf{kv}\tau}f_0H_l(\mathbf{k}). \tag{17}$$

The current density $\delta\mathbf{j}$ in (1) is obtained from the equation

$$\delta\mathbf{j}=(c/4\pi)\operatorname{rot}\mathbf{H}, \tag{18}$$

where $\mathbf{H}$ is a random magnetic field with a correlation function (7), (8).

With the aid of (17) and (18) we determine the density of the force acting on the medium:

$$\frac{1}{c}\langle[\delta\mathbf{j},\times\delta\mathbf{H}_{cr}]\rangle_s$$

$$=\frac{1}{c}\iint\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}\langle[\delta\mathbf{j}(\mathbf{k}_1),\times\delta\mathbf{H}_{cr}(\mathbf{k}_2)]_s\rangle$$

$$=\frac{6}{\pi^{1/2}L_c\langle H^2\rangle}\int_0^\infty d\tau\iint\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\int d^3p\,e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}p$$

$$\times\left\langle[\mathbf{k}_1\times\mathbf{H}_1(\mathbf{k}_1)]_j\,e_{sjq}\frac{[\mathbf{k}_2\times\mathbf{v}]_q}{k_2^2}[\mathbf{v}\times\nabla]_l f_0 e^{-i\mathbf{k}_2\mathbf{v}\tau}H_l(\mathbf{k}_2)\right\rangle=-\nabla_s P, \tag{19}$$

where $P=\frac{1}{3}\int d^3ppvf_0$.

In the derivation of (19) we used a relation that follows from expressions (7) and (8)

$$\langle H_s(\mathbf{k}_1)H_l(\mathbf{k}_2)\rangle$$

$$=\frac{\pi^{3/2}}{3}\langle H^2\rangle L_c^3\exp\left\{-\frac{k^2L_c^2}{4}\right\}\left(\delta_{sl}-\frac{k_{1s}k_{2l}}{k_1^2}\right)(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2). \tag{20}$$

4. CONCLUSION

The main result of the foregoing calculation is that under the condition when the diffusion approximation is valid the pressure of the fast charged particles is transferred to the medium via interaction of currents that are due to the presence of a stochastic magnetic field in the conducting medium and of a random magnetic field due to the fluctuations of the scattered fast particles.

If the medium moves, the pessure force does work. The corresponding power in the case of nonrelativistic motion with velocity u is equal to $-\mathbf{u}\cdot\nabla P$. We note that in Ref. 10 an analysis of the transport of the energy density of cosmic rays that diffuse in a medium with a frozen-in random magnetic field has established that the energy is transferred from the medium to cosmic rays with a power $\mathbf{u}\cdot\nabla P$ per unit volume, in full agreement with the calculation above. A microscopic calculation of the force acting on a gas of fast particles should consist of a calculation of the force density $c^{-1}\langle[\delta\mathbf{j}_{cr}\times\mathbf{H}]\rangle$ ($\mathbf{H}$ is the random field), which is equal, apart from the sign, to the force (1) acting on the medium.

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[1)]An interesting example of later investigations in this field is that of Vaĭnshteĭn and Kichatinov.[2] Hydrodynamics of a plasma with infrequent Coulomb collisions in stochastic fields was considered by Gurevich *et al.*[3]

[2)]Integration of (4) from a lower limit $-\infty$ presupposes adiabatic turning-on of the field $\mathbf{H}$.

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