Metal–insulator transition in a disordered system

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(Submitted 21 June 1983)
Zh. Eksp. Teor. Fiz. **86**, 367–380 (January 1984)

A study is made of the metal–insulator transition for interacting electrons in a disordered system in the presence of a magnetic field or magnetic impurities. The renormalizability in first order in $\varepsilon = d - 2$ is demonstrated, and renormalization-group equations are given with full allowance for the Coulomb interaction. It is shown that in the critical region of the metal–insulator transition the dependence of the conductivity, polarizability, and other quantities on the frequency of the external field or on the temperature is governed by the magnitude of the charge z arising upon renormalization of the frequency coefficient in the diffusion propagator.

1. INTRODUCTION

The metal–insulator transition in disordered systems has of late been studied by the methods of the theory of second-order phase transitions. Experiments confirm the correctness of this approach: according to recent data the static conductivity goes zero in a continuous manner[1–3] at the metal–insulator transition, and not in a jump. In a disordered system both the Anderson localization effect[4–6] and the Coulomb correlations of the diffusing electrons[7,8] are important. Definite progress has been made[9,10] in the description of noninteracting electrons in a metal containing impurities, but a systematic theory incorporating both these effects has yet to be constructed for this transition. In the very important paper of McMillan[11] a scheme is proposed for describing the transition, but this scheme has certain shortcomings. The present author has ascertained[12,13] that the renormalization-group equations for interacting electrons in a disordered metal differ from those used by McMillan[11] chiefly in the following respects:

a) The single-particle density of states $N(E)$ does not influence the renormalization of the conductivity. The Einstein relation linking the conductivity with the diffusion coefficient contains not $N(E)$ but $\partial n/\partial\mu$, which does not have diffusion corrections.

b) In contradiction to the assumption of McMillan, the relationship between the energy and length scales is not governed by the single-particle density of states. The relation between these scales is actually governed by a new charge z arising upon renormalization of the frequency coefficient in the diffusion propagator. This charge plays an important role in the description of the transition: in the critical region the dependence of the conductivity[13] and dielectric constant on the frequency ω of the external field and on the temperature T is governed by the size of the parameter z near the fixed point of the renormalization-group equations.

In Refs. 12 and 13 the renormalizability in first order in $\varepsilon = d - 2$ was demonstrated and the renormalization-group equations were given for cases in which the corrections from the Cooper channel[5,6] are suppressed but the corrections due to the Coulomb interaction of the diffusion electrons[7,8] remain. The Cooper channel can be suppressed by a magnetic field[14] or by magnetic impurities,[15] since interactions which break the time-inversion symmetry lead to cutoff of the diffusion pole in the Cooper propagator.

In the present paper we consider the conductivity, dielectric constant, and single-particle density of states in the vicinity of the metal–insulator transition in a magnetic field or in the presence of magnetic impurities. The structure of this paper is as follows. Section 2 gives the effective free-energy functional of interacting electrons in a disordered system. In Sec. 3 the Einstein relation is discussed and it is shown that the presence of a charge z in the renormalization-group equations is necessary in order for the equations to be consistent with conservation of particle number. In Sec. 4 the renormalization-group equations describing the metal–insulator transitions in a magnetic field and in the presence of magnetic impurities are given in first order in ε. Since the Coulomb interaction is not small, the electron–electron interaction is taken into account exactly in these equations. In Sec. 5 the frequency and temperature dependences of the conductivity and dielectric constant are found and arguments are presented as to the value of the static polarizability of the insulator in the vicinity of the transition. Section 6 takes up the problem of the single-particle density of states. In the Conclusion (Sec. 7) the results of this paper are compared with the theory of McMillan, and certain experiments are discussed.

2. FREE-ENERGY FUNCTIONAL

To find the renormalization-group equations it is useful to construct the effective Lagrangian of the diffusion modes.[9,10,12] In such an approach the integration over large electron momenta $\sim p_F$ is done right from the start, whereupon the remaining problem of the interaction of diffusion modes is one in which only distances greater than the mean free path are important. The effective functional permits the use of the standard methods of field theory in treating this part of the problem. In deriving the effective Lagrangian it is necessary to average the free energy over the randomly distributed impurities. This averaging is done by the method of replicas.[16] In the problem of localization of noninteracting electrons the effective Lagrangian was first derived with the aid of a functional integral over Bose fields[9] and later with an integration over Fermi fields.[10] To obtain the correct statis-

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tics the integration in Ref. 12 was done over Fermi fields, as in Ref. 10.

The parameter describing the local properties of the electrons in a disordered metal is the matrix $\hat{Q}$, which has frequency, spin, and replica indices. Since for the cases of metal–insulator transitions examined in this paper the Cooper channel is unimportant, the elements of $\hat{Q}$ are complex numbers (in the opposite case the elements of $\hat{Q}$ are quaternions). The matrices $\hat{Q}$ satisfy the following conditions:

$$\hat{Q}=\hat{Q}^{+}, \quad \mathrm{Sp}\,\hat{Q}=0, \quad \hat{Q}^{2}=\hat{I}, \tag{1}$$

where $\hat{I}$ is the unit matrix.

The effective functional describing noninteracting electrons in a disordered system is given in terms of $\hat{Q}$ by the expression[9,10,12]

$$\mathcal{F}_{D}=T\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla\hat{Q})^{2}-4\,\mathrm{Sp}(\hat{\varepsilon}\hat{Q})]\,dr, \tag{2}$$

where D is the electron diffusion coefficient, the constant ν is the density of states without allowance for diffusion corrections, and $\hat{\varepsilon}$ is a matrix whose components are the Fermi frequencies:

$$\varepsilon_{nm}^{\alpha\beta}=\varepsilon_{n}\delta_{nm}\delta_{\alpha\beta}\delta_{ij}, \quad \varepsilon_{n}=(2n+1)\pi T, \quad i=1,\ldots,N, \quad N\to 0. \tag{3}$$

The lower indices of the matrices $\hat{Q}, \hat{I}, \hat{\varepsilon}$, etc. will correspond to the Matsubara frequencies, while the upper indices combine the spin (α,β) and replica (i,j) indices. Unlike the treatment in papers on the localization of noninteracting electrons,[9,10] where it is sufficient to consider only two levels with energies $\pm\omega/2$, the functional $\mathcal{F}_{D}$ incorporates the enitre set of energy levels of the diffusing electrons, since Sp in (2) presupposes, in addition to summation over spins and replicas, a summation over the electron energies in the interval $|\varepsilon|\lesssim\tau^{-1}$ (τ is the mean free time).

If one could neglect the frequency terms in $\mathcal{F}_{D}$, homogeneous unitary transformations (rotations) of the matrix $\hat{Q}$ which are allowed by conditions (1) would not change the free energy at all. The term containing $\hat{\varepsilon}$ in (2) breaks the symmetry of $\mathcal{F}_{D}$ with respect to rotations of $\hat{Q}$ (Ref. 9) and fixes the equilibrium position $\hat{Q}=\hat{\Lambda}$:

$$\Lambda_{nm}^{\alpha\beta}=\mathrm{sign}\,\varepsilon_{n}\delta_{nm}\delta_{\alpha\beta}\delta_{ij}. \tag{4}$$

At low frequencies ε the matrix $\hat{Q}$ is fixed weakly and the transverse deviations of $\hat{Q}$ is fixed weakly and the transverse deviations of $\hat{Q}$ and $\hat{\Lambda}$ are substantial. We shall show that these deviations correspond to diffusion modes. The matrices $\hat{Q}$ satisfy (1) if $\hat{Q}=\hat{U}^{-1}(r)\hat{\Lambda}\hat{U}(r)$, where the $\hat{U}$ are unitary matrices which are conveniently written in the form[10]

$$\hat{U}=\exp\left(\frac{\hat{W}}{2}\right), \quad W_{\varepsilon\varepsilon'}=\begin{pmatrix} \overset{\varepsilon'>0}{0} & \overset{\varepsilon'<0}{B} \\ -B^{+} & 0 \end{pmatrix}\begin{matrix}\varepsilon>0\\ \varepsilon<0\end{matrix}, \quad \hat{Q}=\hat{\Lambda}\exp\hat{W}. \tag{5}$$

Retaining in $\mathcal{F}_{D}$ only terms quadratic in B, we obtain

$$\mathcal{F}_{0}=T\frac{\pi\nu}{2}\int\sum_{\substack{n>0\\ n'<0}}[D\nabla B_{nn'}^{ij}\nabla B_{n'n}^{+ji}+(\varepsilon_{n}-\varepsilon_{n'})B_{nn'}^{ij}B_{n'n}^{+ji}]\,dr.$$

It follows that the correlator $\langle BB^{+}\rangle_{0}$ is the propagator of the diffusion mode:

$$\langle B_{\varepsilon_1\varepsilon_2}^{ij}B_{\varepsilon_3\varepsilon_4}^{+kl}\rangle_{0}=\frac{2}{\pi\nu}\frac{I_{\varepsilon_1\varepsilon_4}^{il}I_{\varepsilon_2\varepsilon_3}^{kj}}{Dk^{2}+\Omega}, \quad \Omega=\varepsilon_{1}-\varepsilon_{2}>0. \tag{6}$$

In the diffusion propagator (6) there is no factor $-i$ in front of the frequency because we are working in the temperature technique.

The Coulomb interaction of the electrons will be considered with allowance for the singularity at small momenta which is due to the long-range character of the Coulomb forces. For describing the interaction of the electrons, two amplitudes, Γ and Γ_{2}, were used in Ref. 12. Amplitude Γ describes the small-angle scattering, and Γ_{2} the large-angle scattering. It is important that these amplitudes correspond to different structures of the spin indices:

$$\Gamma\psi_{p}^{+\alpha}\psi_{p+k}^{\pm\beta}\psi_{p}^{\beta}\psi_{p+k}^{\alpha}+\Gamma_{2}\psi_{p}^{+\alpha}\psi_{p+k}^{\pm\beta}\psi_{p+k}^{\beta}\psi_{p}^{\alpha}$$
$$=\psi_{p}^{+\alpha}\psi_{p+k}^{\pm\beta}\psi_{p}^{\gamma}\psi_{p+k}^{\delta}(\Gamma\delta_{\alpha\delta}\delta_{\beta\gamma}-\Gamma_{2}\delta_{\alpha\gamma}\delta_{\beta\delta}). \tag{7}$$

The effective functional for interacting diffusing electrons was obtained in Ref. 12 as

$$\mathcal{F}\{Q\}=T\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla\hat{Q})^{2}-\nu\Gamma(\hat{Q}\gamma_{1}\hat{Q})$$
$$+\nu\Gamma_{2}(\hat{Q}\gamma_{2}\hat{Q})-4z\,\mathrm{Sp}(\hat{\varepsilon}\hat{Q})]\,dr, \tag{8}$$

where γ_{1} and γ_{2} are tensors which differ in their spin structure:

$$(\hat{Q}\gamma_{1}\hat{Q})=2\pi T\sum Q_{n_1n_2}^{\alpha\alpha}Q_{n_3n_4}^{\beta\beta}\delta(n_1+n_3,n_2+n_4)\,\delta^{i},$$
$$(\hat{Q}\gamma_{2}\hat{Q})=2\pi T\sum Q_{n_1n_2}^{\alpha\beta}Q_{n_3n_4}^{\beta\alpha}\delta(n_1+n_3,n_2+n_4)\,\delta^{i}; \tag{9}$$

δ^{i} indicates that all the replica indices of the matrices $\hat{Q}$ coincide, while $\delta(n,m)=\delta_{mn}$; $\mathcal{F}\{Q\}$ has the meaning of the effective free-energy functional of the interacting diffusing electrons. We note that in (8) a factor z not present in (2) precedes the frequency term [the last term in (8)]. This factor stems from the fact that, unlike the case of free electrons, for which $z\equiv 1$, the coefficient multiplying the frequency term in the free-energy functional for interacting electrons changes during the renormalization process.[12] As will be demonstrated in the following section, the introduction of a new charge z in the renormalization-group equations is necessary for these equations to be consistent with the condition of conservation of the number of particles.[12]

3. EINSTEIN RELATION. CONSERVATION OF PARTICLE NUMBER AND THE INVARIANT CHARGE z

The Einstein relation connects the conductivity σ with the diffusion coefficient D_{e} for the density of interacting electrons:

$$\sigma/e^{2}=(\partial n/\partial\mu)D_{e}, \tag{10}$$

where $\partial n/\partial\mu$ is a quantity which determines the compressibility of the electron gas. By definition,

$$\frac{\partial n}{\partial\mu}=2T\sum_{\varepsilon}\int\frac{\partial}{\partial\mu}\mathcal{G}(\varepsilon,p)\frac{d^{d}p}{(2\pi)^{d}}, \tag{11}$$

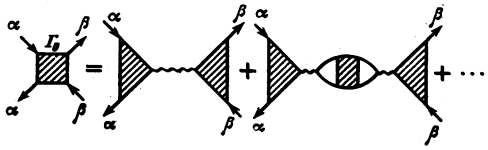

FIG. 1.

where π is the single-particle Green function of the electrons. Near the Fermi surface, differentiation with respect to μ is equivalent to differentiation with respect to the momentum modulus p. Keeping this in mind, it can be verified after integration by parts in (11) that the magnitude of $\partial n/\partial\mu$ is governed by the large-momentum region, where diffusion corrections, in which only small momentum transfers are important, do not arise.[12,17] With allowance for the Fermi-liquid renormalization one has $\partial n/\partial\mu = 2\nu/(1+F_0)$, where F_0 is the standard constant in the theory of the Fermi liquid. By virtue of the well-known Ward identity

$$\partial n/\partial\mu=\pi(\Omega=0)=2\nu/(1+F_0), \tag{12}$$

where $\pi(\Omega = 0)$ is the static part of the polarization operator.

To find the diffusion coefficient D_e in (10), let us evaluate the polarization operator $\pi(k,\Omega)$. For this purpose let us separate from the amplitude Γ the statically screened Coulomb interaction Γ_0:

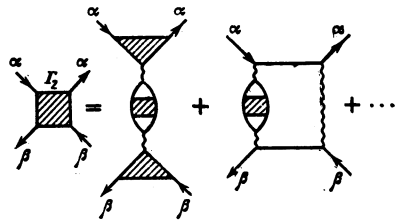

FIG. 3.

$$\Gamma=\Gamma_0+\Gamma_1, \quad \Gamma_0=\frac{1}{(1+F_0)^2}\frac{v_C(k)}{1+v_C(k)\pi(\Omega=0)}. \tag{13}$$

Here $v_C(k)$ is the Coulomb potential, and the factor $1/(1+F_0)^2$ takes into account the Fermi-liquid corrections to the triangular vertices (Fig. 1). The amplitude Γ_1 represents the set of diagrams with spin structure $\delta_{\alpha\delta}\delta_{\beta\gamma}$ [see (7)] which are not separable by the breaking of only one Coulomb line (Fig. 2). Examples of diagrams contributing to the amplitude Γ_2, which has the spin structure $\delta_{\alpha\gamma}\delta_{\beta\gamma}$, are shown in Fig. 3. It has been established[12] that in the polarization operator the diffusion corrections to the triangular vertices and to the Green functions cancel each other (Fig. 4). Therefore, to find $\pi(k,\Omega)$ it is sufficient to evaluate the ladder diagrams of Fig. 5, in which the amplitudes Γ_1 and Γ_2 and the diffusion propagators should be normalized. As a result we obtain

$$\pi(k,\Omega>0)=\pi(\Omega=0)-\frac{2\nu}{(1+F_0)^2}\frac{\mathscr{D}_0(k,\Omega)\pi T\sum_n[\operatorname{sign}(\varepsilon_n+\Omega)-\operatorname{sign}\varepsilon_n]}{1+(-2\nu\Gamma_1+\nu\Gamma_2)\mathscr{D}_0(k,\Omega)\pi T\sum_n[\operatorname{sign}(\varepsilon_n+\Omega)-\operatorname{sign}\varepsilon_n]} \tag{14}$$

$$\mathscr{D}_0(k,\Omega)=1/(Dk^2+z\Omega).$$

After summation over frequencies we have

$$\pi(k,\Omega>0)=\frac{\partial n}{\partial\mu}\left(1-\frac{1}{(1+F_0)}\frac{\Omega}{Dk^2+(z-2\nu\Gamma_1+\nu\Gamma_2)\Omega}\right). \tag{15}$$

The condition that the number of particles be conserved requires that $\pi(k=0,\Omega)$ vanish. It follows from (15) that this requires that the renormalized quantities z, $\nu\Gamma_1$, and $\nu\Gamma_2$ obey the relation

$$z-2\nu\Gamma_1+\nu\Gamma_2=1/(1+F_0). \tag{16}$$

Here

$$\pi(k,\Omega)=\frac{\partial n}{\partial\mu}\frac{D_e k^2}{D_e k^2+|\Omega|}, \quad D_e=(1+F_0)D. \tag{17}$$

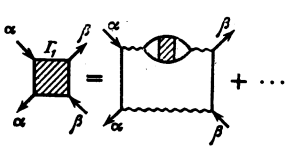

FIG. 2.

The continuity equation enables one to find σ if $\pi(k,\Omega)$ is known:

$$\frac{\sigma(k,\Omega)}{e^2}=\frac{\Omega}{k^2}\pi(k,\Omega). \tag{18}$$

This equation implies the Einstein relation (10), and with allowance for (17) and (12) we obtain a relation between σ and D:

$$\frac{\sigma}{e^2}=\frac{\partial n}{\partial\mu}(1+F_0)D=2\nu D. \tag{19}$$

By renormalizing the functional $\mathscr{F}\{Q\}$, one can find the renormalized value of the coefficient D and, hence, of σ as well.

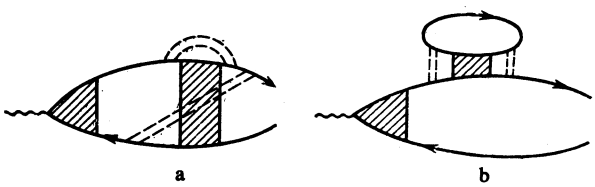

FIG. 4.

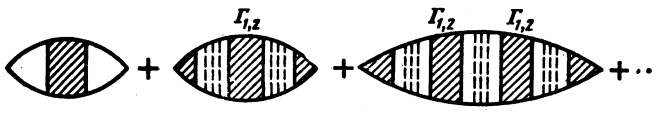

FIG. 5.

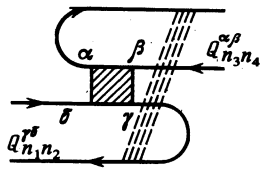

FIG. 7.

In the theory of the Fermi liquid, Eq. (16) corresponds to the familiar equation

$$1-2\nu\Gamma_1^0+\nu\Gamma_2^0=1/(1+F_0),$$

where Γ_1^0 and Γ_2^0 are the Coulomb amplitudes without diffusion corrections. Satisfaction of (16) with allowance for the diffusion corrections ensures that the renormalization-group equations will be consistent with the condition of conservation of particle number. The following procedure was adopted in Ref. 12: The renormalization-group equations for z and for the Coulomb amplitudes were derived independently, and satisfaction of (16) was used as a check. We shall have no further need of separating Γ into Γ_0 and Γ_1. Since

$$\nu\Gamma_0=\frac{\nu}{(1+F_0)^2}\frac{v_c(k)}{1+v_c(k)\pi(\Omega=0)}=\frac{1}{2(1+F_0)}$$

we can write (16) as

$$z=2\nu\Gamma-\nu\Gamma_2. \tag{20}$$

4. RENORMALIZATION-GROUP EQUATIONS

To derive the renormalization-group equations, let us follow Ref. 18 and integrate $\exp(-\mathcal{F}\{\tilde{Q}\}/T)$ over the rapidly changing variables. To do this, let us separate the matrix U [see (5)] into a product of rapidly and slowly varying parts[19,10]:

$$U=U_0\tilde{U}, \tag{21}$$

where U_0 and $\tilde{U}$ are unitary matrices; U_0 is the rapid part and $\tilde{U}$ the slow. If we now integrate $Q_0=U_0^+\Lambda U_0$ over the rapid variables, we obtain a renormalized functional $\tilde{\mathcal{F}}\{\tilde{Q}\}$ describing the slowly varying field $\tilde{Q}=\tilde{U}^+\Lambda\tilde{U}$:

$$\tilde{\mathcal{F}}\{\tilde{Q}\}=-T\ln\int\exp(-\mathcal{F}\{Q\}/T)\,dQ_0. \tag{22}$$

In the cases under consideration here, one finds that in first order in $\varepsilon=d-2$ integration (22) recasts the functional $\tilde{\mathcal{F}}\{\tilde{Q}\}$ in the form (8) with renormalized coefficients D, $\nu\Gamma$, $\nu\Gamma_2$, and z.

The technical details of performing the integration in (22) and a comparison of this procedure with the diagrams in the impurity technique are described in sufficient detail in an earlier paper[12] and will not be discussed here. Let us merely note two important points.

1. Since the quantities z, $\nu\Gamma$, and $\nu\Gamma_2$ to be renormalized are not small, these charges should be taken into account exactly. This turns out to be possible because in the problem under study the order of the renormalization-group equations is determined solely by the number of integrations over the momenta of fast diffusion modes. In first order in $\varepsilon=d-2$ it is sufficient to consider only those diagrams in which there is just one momentum integration. To obtain exact (in the sense of incorporating the electron–electron interaction) renormalization-group equations it is therefore sufficient to augment the renormalized Coulomb amplitudes in the skeleton diagrams by the sum of the ladder diagrams (Fig. 6). Upon such a modification the number of integrations over the momenta of diffusion modes does not change.

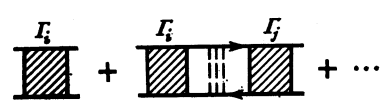

FIG. 6.

2. Two types of diagrams are encountered in the renormalization of amplitudes $\nu\Gamma$ and $\nu\Gamma_2$. In the diagram shown in Fig. 7, the integration is done only over the momentum. In addition, there are other diagrams in which the integration is done over both the momentum and frequency of the fast diffusion mode. Such diagrams have the shape of a ring within which the fast momentum passes. There are several different ring diagrams, but all of them are paired; the diagrams of a pair differ by the manner in which the ring is broken. An example of such a pair is given in Fig. 8 (in accordance with what we have said above, the interior amplitudes in diagrams 8a,b should be augmented by the ladder diagrams of Fig. 6). It turns out that in contrast to the situation studied in Ref. 12, in the cases considered in the present paper the diagrams of a pair cancel each other, so that the renormalization of the Coulomb amplitudes is brought about only by the diagrams of Fig. 7.

1. Magnetic field

In describing the neighborhood of the transition, we are interested in the region[1)]

$$\omega,\ T<g_L\mu_BH, \tag{23}$$

where g_L is the Landé factor, H is the magnetic field, and μ_B is the Bohr magneton. Here one should be mindful of the Zeeman splitting. Magnetization of the spins corresponds to the appearance of the following term in the functional of the diffusing electron:

$$\mathcal{F}_{g_L}=T\pi\nu\int g_L\frac{i\mu_BH}{2}\mathrm{Sp}(\sigma_z\tilde{Q})\,dr. \tag{24}$$

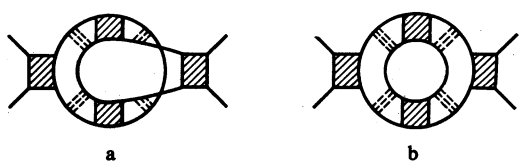

FIG. 8.

Expanding $\mathscr{F}_{gL}$ to second order in B, we obtain (indices 1 and 2 correspond to the spin projections)

$$\mathscr{F}_{g_L} \approx T\frac{\pi\nu}{2}\int \sum_{\varepsilon,\varepsilon'}(-ig_L\mu_B H)\,(B^{12}_{\varepsilon\varepsilon'}B^{+21}_{\varepsilon'\varepsilon}-B^{21}_{\varepsilon\varepsilon'}B^{+12}_{\varepsilon'\varepsilon})\,dr. \qquad (25)$$

Thus the Zeeman splitting leads to a cutoff of the pole in the diffusion propagator with opposite projections of the electron spins.[20,21] As a result, at large distances these modes drop out, and $W^{\alpha\beta}$ and $Q^{\alpha\beta}$ turn out to be diagonal in the spin indices:

$$W^{\alpha\beta}=\begin{pmatrix}\hat{W}_\uparrow & 0\\ 0 & \hat{W}_\downarrow\end{pmatrix},\quad \hat{Q}=\hat{\Lambda}\exp\hat{W}=\begin{pmatrix}\hat{Q}_\uparrow & 0\\ 0 & \hat{Q}_\downarrow\end{pmatrix}. \qquad (26)$$

Then the effective functional for interacting electrons (8) in region (23) can be written in the form

$$\mathscr{F}\{Q\}=T\frac{\pi\nu}{4}\int\{D\,\mathrm{Sp}[(\nabla\hat{Q}_\uparrow)^2+(\nabla\hat{Q}_\downarrow)^2]-4z\,\mathrm{Sp}[\hat{\varepsilon}(\hat{Q}_\uparrow+\hat{Q}_\downarrow)]-\nu\Gamma[(\hat{Q}_\uparrow+\hat{Q}_\downarrow)\gamma(\hat{Q}_\uparrow+\hat{Q}_\downarrow)]+\nu\Gamma_2[(\hat{Q}_\uparrow\gamma\hat{Q}_\uparrow)+(\hat{Q}_\downarrow\gamma\hat{Q}_\downarrow)]\}dr,$$

$$(\hat{Q}\gamma\hat{Q})=2\pi T\sum Q^{ii}_{n_1n_2}Q^{ii}_{n_3n_4}\delta(n_1+n_3,n_2+n_4), \qquad (27)$$

where i is the replica index. The renormalization of the coefficients of functional (27) is given by the following expressions:

$$\delta D=-\frac{1}{\pi\nu}\int\mathscr{D}_2\mathscr{D}_0\left(\frac{\nu\Gamma}{Dq^2}-\frac{\nu\Gamma_2}{Dq^2+z\Omega}\right)\frac{4D^2q^2}{d}\frac{d^dq}{(2\pi)^d}d\Omega, \qquad (28a)$$

$$\delta z=-\delta\nu\Gamma_2=-\frac{1}{\pi\nu}\int\mathscr{D}_2\mathscr{D}_0^{-1}\left(\frac{\nu\Gamma}{Dq^2}-\frac{\nu\Gamma_2}{Dq^2+z\Omega}\right)\frac{d^dq}{(2\pi)^d}, \qquad (28b)$$

$$\delta\nu\Gamma=0, \qquad (28c)$$

where

$$\mathscr{D}_2=[Dq^2+(z+\nu\Gamma_2)\,|\Omega|]^{-1},\quad \mathscr{D}_0=[Dq^2+z|\Omega|]^{-1}.$$

It follows from (28b,c) that the relation $z=2\nu\Gamma-\nu\Gamma_2$ remains valid in the renormalization process.

Let us introduce a dimensionless quantity G which, together with z, $\nu\Gamma$, and $\nu\Gamma_2$, will be an invariant charge of the renormalization group:

$$G=\frac{\sigma}{e^2}\lambda^{2-d}\frac{\pi}{4k_d}. \qquad (29)$$

Here λ is the momentum cutoff, which decreases during renormalization, and $k_d=2^{-d+1}\pi^{-d/2}/\Gamma(1/2d)$. Formulas (28) correspond to the following renormalization-group equations of first order in ε:

$$\frac{dG}{dx}=\varepsilon G-\frac{1}{d}\left(2+\frac{z+\nu\Gamma_2}{\nu\Gamma_2}\ln\frac{z}{z+\nu\Gamma_2}\right), \qquad (30a)$$

$$d\nu\Gamma/dx=0, \qquad (30b)$$

$$\frac{dz}{dx}=-\frac{1}{4G}(z-\nu\Gamma_2),\quad \frac{d\nu\Gamma_2}{dx}=-\frac{dz}{dx}, \qquad (30c)$$

where $x=\ln(\lambda_0/\lambda)$ and $\lambda_0=2\pi/l_0$ (l_0 is the mean free path). Equations (30) have an unstable fixed point corresponding to the metal–insulator transition in a magnetic field:

$$G^*=(1-\ln 2)/\varepsilon,\quad z^*=\nu\Gamma_2^*=(1+\nu\Gamma_2^0)/2. \qquad (31)$$

We note that the parameter z is nonzero at this point.

2. Magnetic impurities

Magnetic impurities create a magnetic field of random magnitude and direction. After an averaging over the random fields the functional $\mathscr{F}\{Q\}$ of the diffusion modes contains an additional term[10] $\sim\tau_s^{-1}\mathrm{Sp}(\sigma,\hat{Q})^2$, where τ_s is the spin-flip scattering time and σ is the Pauli matrix. Representing the matrix $W^{\alpha\beta}$ [see formula (5)] in the form

$$W^{\alpha\beta}=2^{-1/2}(\delta_{\alpha\beta}W^0+\sigma_{\alpha\beta}\mathbf{W}), \qquad (32)$$

we can infer that the magnetic impurities lead to a cutoff of the diffusion pole in the correlator of the fields $\mathbf{W}$ but does not affect the mode W^0, which is associated with diffusion of the particle density. At large distances all the modes except W^0 are "turned off."[10] As a result, in the presence of magnetic impurities one has

$$\hat{Q}=\hat{\Lambda}\exp(\hat{W}^0/\sqrt{2}), \qquad (33)$$

while the effective functional of the diffusing electrons (8) assumes the form

$$\mathscr{F}=T\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla\hat{Q})^2-(2\nu\Gamma-\nu\Gamma_2)(\hat{Q}\gamma_2\hat{Q})-4z\,\mathrm{Sp}(\varepsilon\hat{Q})]dr. \qquad (34)$$

Upon renormalization of (34) we see that $\delta z=\delta(2\nu\Gamma-\nu\Gamma_2)$, so that relation (20) is satisfied, and we finally get

$$\mathscr{F}=T\frac{\pi\nu}{4}\int[D\,\mathrm{Sp}(\nabla\hat{Q})^2-z(\hat{Q}\gamma_2\hat{Q})-4z\,\mathrm{Sp}(\hat{\varepsilon}\hat{Q})]dr. \qquad (35)$$

Renormalizing the functional $\mathscr{F}$, we obtain the relations

$$\delta D=-\frac{1}{\pi\nu}\int\mathscr{D}_0^2z\frac{2D}{d}\frac{d^dq}{(2\pi)^d}d\Omega, \qquad (36a)$$

$$\delta z=-\frac{1}{\pi\nu}\int\frac{z}{2Dq^2}\frac{d^dq}{(2\pi)^d}, \qquad (36b)$$

which correspond to the following renormalization-group equations [$x=\ln(\lambda_0/\lambda)$]:

$$dG/dx=\varepsilon G-1/d, \qquad (37a)$$

$$dz/dx=-z/4G \qquad (37b)$$

[the charge G was defined in (29)]. These equations have an unstable fixed point: $G^*=1/2\varepsilon$, $z^*=0$. The vanishing of z at the unstable fixed point of the renormalization-group equations is very important for the description of the critical region of the metal–insulator transition in the presence of magnetic impurities. We shall show below that the dependence of the conductivity[13] and other characteristics on the frequency ω of the external field and on the temperature in the critical region of the transition is determined by the value of the parameter z near the fixed point. It appears to the author that to establish the existence of a fixed point with $z^*=0$ the accuracy of the equations obtained in first order in ε is sufficient. However, since the initial value of z is not small (it is in fact equal to one), this fixed point may not bear any relationship to the transition; in this case the phase-plane portrait of the system of exact equations should display another unstable fixed point in addition to the one found. Such a situation is, generally speaking, unlikely.

5. CONDUCTIVITY AND DIELECTRIC CONSTANT

At a finite external-field frequency ω the renormalization of the charge G in the critical region is cut off at the length

$$L_\omega=(D/z\omega)^{1/2}, \tag{38}$$

and therefore in the transition region, where the correlation length obeys $\xi \gg L_\omega$, the conductivity is

$$\sigma\sim e^2 L_\omega^{2-d}. \tag{39}$$

With allowance for (19) it follows that

$$\sigma(\omega)\sim e^2\left(\frac{\partial n}{\partial\mu}z\omega\right)^{(d-2)/d}. \tag{40}$$

It was found above that $z^* = \text{const} \neq 0$ at the transition in a magnetic field [see (31)]. In this case

$$\sigma(\omega)\sim\omega^{(d-2)/d} \tag{41}$$

just as for the case of noninteracting electrons,[22] in which z is not renormalized at all ($z\equiv 1$).

When magnetic impurities are present in the system, $z^* = 0$. In this case the ω dependence of σ is governed by the exponent ζ, which describes the vanishing of z near the fixed point of equation (37):

$$z\sim(\lambda/\lambda_0)^\zeta, \tag{42}$$

where λ is the momentum cutoff, which goes to zero upon renormalization. At the distance L_ω which determines the magnitude of the conductivity in the critical region we have

$$z\sim(l_0/L_\omega)^\zeta \tag{43}$$

(l_0 is the mean free path). As a result, for the magnetic-impurity case we obtain from (38)–(40)

$$\sigma\sim\omega^{(d-2)/(d-\zeta)}. \tag{44}$$

In first order in ε we find the exponent ζ from (37): $\zeta = 1/4G^* = \varepsilon/2$. In what follows, we shall employ the exponent ζ both in the magnetic-impurity case and for description of the transition in a magnetic field:

magnetic field:

$$\zeta=0, \tag{45a}$$

magnetic impurities:

$$\zeta=\varepsilon/2+O(\varepsilon^2). \tag{45b}$$

We have discussed the dependence of σ on the frequency ω of the external field for $\omega \gg T$. If $\omega < T$, then in the cases under consideration here the renormalization of the quantity G in equations (30) and (37) is cut off at the temperature length $L_T = (D/zT)^{1/2}$, and therefore in the critical region of the transition we have

$$\sigma\sim T^{(d-2)/(d-\zeta)}. \tag{46}$$

Let us now consider the dieelctric constant ($d = 3$)

$$\varepsilon(q,\omega)=1+\frac{4\pi e^2}{q^2}\frac{\partial n}{\partial\mu}\frac{D_e q^2}{D_e q^2-i\omega}. \tag{47}$$

In the vicinity of the transition we find that the polarizability

$$\chi(\omega)=\mathrm{Re}\,[\varepsilon(q=0,\ \omega)-1]/4\pi$$

takes the form

$$\chi(\omega)\sim\omega^{-(2-\zeta)/(3-\zeta)}. \tag{48}$$

At the present time we do not have a systematic description of the dielectric phase near the metal–insulator transition, and therefore for finding the static polarizability of the insulator in the vicinity of the transition we shall use the arguments of scaling theory. Let us express the dielectric susceptibility $\chi(\omega)$ in terms of the length L_ω: $\chi\sim L_\omega^{2-\zeta}$.

By joining the dynamic susceptibility $\chi(\omega)$ with the static susceptibility at $L_\omega\sim\xi$, we obtain for the static dielectric constant of the insulator near the transition

$$\chi\sim\xi^{2-\zeta}, \tag{49}$$

where ξ is the correlation length [recall that on the metallic side of the transition one has $\xi\sim e^2/\sigma(\omega=0)$].

6. SINGLE-PARTICLE DENSITY OF STATES

The single-particle density of states is given by

$$N(E)=-\frac{1}{\pi}\,\mathrm{Im}\int\mathcal{G}^R(E,p)\frac{d^dp}{(2\pi)^d}, \tag{50}$$

where $\mathcal{G}^R$ is the retarded single-particle Green function of the electrons. The single-particle density of states is measured in tunneling experiments,[23,24] but contrary to the assumption of McMillan,[11] this quantity does not influence the renormalization of the conductivity.

It was found previously[12] that in the variables Q expression (50) is of the form

$$N(E)=\nu(\hat{\Lambda}\hat{Q})_E \tag{51}$$

and its evaluation is analogous to finding the Debye-Waller factor:

$$(\hat{\Lambda}\hat{Q})=(\hat{\Lambda}\hat{U}^+\hat{\Lambda}e^{\hat{W}}\hat{U})\approx(\hat{\Lambda}\hat{U}^+\hat{\Lambda}\exp(\tfrac{1}{2}\langle WW\rangle)\hat{U}).$$

Augmenting the correlator $\langle WW\rangle$ by the ladder diagrams of Fig. 6 and retaining in it only the most singular term, we obtain

$$N(E)\approx\nu\exp\left\{-\frac{1}{2\pi\nu}\int\mathcal{D}_0\frac{z}{Dq^2}k_d q^{d-1}dq\,d\Omega\right\}, \tag{52}$$

where, as a result of renormalization, D and z are functions of q and Ω. In the two-dimensional case the corrections to the single-particle density of states are stronger than the corrections to the other quantities[8]; after the momentum integration the integral of $\ln(\Omega\tau)/\Omega$ remains in the argument of the exponential, so that ultimately the square of the logarithm appears. For $d = 2+\varepsilon$ we obtain after integrating over the momentum

$$N(E)\approx\nu\exp\left\{-\frac{1}{\varepsilon}\int_E\frac{k_d}{2\pi\nu D}\left(\frac{z\Omega}{D}\right)^{\varepsilon/2}\frac{d\Omega}{\Omega}\right\}. \tag{53}$$

To proceed further, let us write the renormalization-group equation for G [see (29)] in another form. We introduce G in the following way:

$$G=\frac{\sigma}{e^2}\left(\frac{D}{z\Omega}\right)^{\varepsilon/2}\frac{\pi}{4k_d}. \tag{54}$$

After integrating (28a) and (36a) over the momentum, one can obtain the following equation for the charge G in first order in ε:

a) magnetic field:

$$\frac{dG}{dy}=\frac{1}{2}\left[\varepsilon G-\frac{1}{2}\left(2+\frac{z+\nu\Gamma_2}{\nu\Gamma_2}\ln\frac{z}{z+\nu\Gamma_2}\right)\right]; \quad (55a)$$

b) magnetic impurities:

$$\frac{dG}{dy}=\frac{1}{2}\left(\varepsilon G-\frac{1}{2}\right), \quad (55b)$$

where $y=\ln(\Omega\tau)^{-1}$. Substituting (54) into (53) and making use of the fact that $\sigma/e^2=2\nu D$, we obtain

$$N(E)\approx\nu\exp\left(-\frac{1}{4\varepsilon}\int_E\frac{d\Omega}{G\Omega}\right).$$

In the critical region

$$N(E)\sim\nu(\max E\tau,\ T\tau)^{\beta}, \quad (56a)$$

while for $E,T\to 0$

$$N(0)\sim\xi^{-\theta}\sim\sigma^{\theta/(d-2)}, \quad (56b)$$

and here one can obtain the following relation for the exponents β, θ, and ζ:

$$\theta=\beta(d-\zeta). \quad (57)$$

In lowest order in ε we have $\beta=1/4\varepsilon G^*$. In a magnetic field

$$\beta=\frac{1}{4(1-\ln 2)}+O(\varepsilon), \quad (58a)$$

and in the case of magnetic impurities

$$\beta=1/2+O(\varepsilon). \quad (58b)$$

On account of the circumstance that $N(E)$ in the two-dimensional case has doubly logarithmic corrections, the series expansion of the exponent β for the single-particle density of states in powers of ε begins with a constant. What happens is that for $d=2+\varepsilon$ a factor of $1/\varepsilon$ rather than $\ln(\Omega\tau)$ arises in the argument of the exponential in (53) after the integration over momenta, and this factor cancels the charge $G^{-1}\sim\varepsilon$. The accuracy of the renormalization-group equations in first order in ε does not permit one to find the $\sim\varepsilon$ correction to the exponent (58). The entire situation described above is the result of allowing for the long-range coulomb interaction. when $v_C(q)$ is replaced by a constant in the model description the doubly logarithmic corrections do not arise, and the series expansion of the exponent β in powers of ε begins with the first power.

In concluding this section let us touch upon a secondary consequence of the results of this paper. Equations (55) permit one to find in the two-dimensional case exact expressions for the logarithmic corrections to the conductivity of the metal in the presence of spin-flip scattering or in a magnetic field $H>T/g_L\mu_B$, when the Zeeman splitting is important. In a magnetic field[2)] the correction is of the form

$$\delta\sigma=\frac{e^2}{2\pi^2}\left[2-\frac{1+\nu\Gamma_2^0}{\nu\Gamma_2^0}\ln(1+\nu\Gamma_2^0)\right]\ln T\tau. \quad (59)$$

To obtain this formula one must set $\varepsilon=0$, $z=1$, and $\Gamma_2=\Gamma_2^0$ in (55a) and bear in mind that $\sigma=2e^2G/\pi^2$ if $d=2$. In comparing this with the previously obtained[20,21] formula for $\delta\sigma$:

$$\delta\sigma=\frac{e^2}{4\pi^2}(2-F)\ln T\tau,\quad F=2\nu\Gamma_2^0$$

one must consider the fact that these expressions do not agree even when $F\ll 1$:

$$\delta\sigma\approx\frac{e^2}{4\pi^2}(2-F/2)\ln T\tau,$$

if $F=2\nu\Gamma_2^0\ll 1$.

It follows from (55b) that in the presence of magnetic impurities the correction to $\delta\sigma$ is particularly simple:

$$\delta\sigma=(e^2/2\pi^2)\ln T\tau. \quad (60)$$

In the three-dimensional case, to obtain the correct coefficients in the square-root corrections to σ in the metal in the presence of a magnetic field or magnetic impurities, one should set $z=1$ and $\Gamma_2=\Gamma_2^0$ in formulas (28a) and (36a) and perform the corresponding integrations.

7. CONCLUSION

The expressions found above for the conductivity, dielectric constant, and single-particle density of states for $d=3$ can be compared with the formulas of Ref. 11 if one sets

$$\zeta=\theta,\quad \theta=3-\eta$$

(the second equation is simply a redefinition of the exponent θ in terms of the exponent η used by McMillan). Then from the relation $\theta=\beta(3-\zeta)$ obtained in Sec. 6, we find for the exponent β describing $N(E)$ that $\beta=-1+3/\eta$, as in Ref. 11. An analogous agreement is found for the corresponding expressions for σ and χ [see (44), (46), and (49)]. Equality of the exponent θ for the single-particle density of states to the exponent ζ describing the relationship of the energy and length scales would constitute justification for the viewpoint of McMillan that the single-particle density of states governs the relationship between these scales. In actuality, as was shown above, the exponents ζ and θ are independent:

$$\zeta\neq\theta.$$

Without going into details, we can conclude that the main distinction between the theory of McMillan and the present study is that the relationship between the energy and length scales is governed not by the single-particle density of states, as was assumed in Ref. 11, but by the value of the charge z near the unstable fixed point of the renormalization-group equations describing the metal–insulator transition. The following types of z behavior in the critical region are possible:

a) $z\to$const, $\zeta=0$. This case is realized when the transition occurs in a magnetic field and the Zeeman splitting is important.

b) $z\to 0$. $\zeta>0$. According to the equations given in first order in $\varepsilon=d-2$, such a transition is realized in the presence of spin-flip scattering.

c) $z\to\infty$. We do not as yet have an example of such a transition.

In Sec. 5 we argue that in the metal–insulator transitions under consideration, the polarizability of the insulator in the vicinity of the transition goes as $\chi\sim\xi^{2-\zeta}$, where ξ is the correlation length. The dielectric constant of the insulator was measured[25] (in the absence of magnetic interactions)

in Si:P, and it was found that $\chi \sim \xi^{2.3 \pm 0.5}$. Although the value of the polarizability exponent is undoubtedly close to 2, the sign of the deviation from 2 is not completely clear. The author considers it possible (from the experience of Ref. 12) that in the absence of magnetic fields or impurities the metal–insulator transition corresponds to $z^* = \infty$; it is possible that here the polarizability exponent will turn out to be greater than 2.

The single-particle density of states was recently measured[24] in tunneling experiments on $Nb_x Si_{1-x}$. It was found that $N(E) \sim E^{1/3}$, while $N(0) \sim \sigma^1$, i.e., $\beta = 1/3$ and $\theta = 1$. The relation $\theta = \beta(3 - \zeta)$ is satisfied if the value of the exponent ζ is small; then $\frac{1}{3}(3 - \zeta) \approx 1$. There is justification for assuming that spin-flip scattering is important in this material. In this case it was found above that in lowest order in $\varepsilon = d - 2$ the exponent $\beta = 1/2$, while $\zeta = \varepsilon/2$. The role of spin-flip scattering might be elucidated by measuring the magnetoresistance in the metallic phase in the region of small corrections. Furthermore, it would be desirable to have data on the dielectric constant of $Nb_x Si_{1-x}$ in the vicinity of the transition.

The author wishes to thank D. E. Khmel'nitskiĭ for helpful discussions of this study.

[1)] In Ref. 12 the behavior of a two-dimensional system was considered in another region: $eDH/c > T > g_L \mu_B H$. The first inequality permits neglect of the Cooper channel, the second, the Zeeman splitting. The properties of the system when $g_L \ll 1$ turned out to be particularly interesting. In such a system the logarithmic growth of the resistance upon a further decrease in T gives way to a decline. For this reason the study of this case could be carried to conclusion. A system with $g_L \approx 0$ can be realized in compounds of the type $Al_x Ga_{1-x} As$, where the composition is chosen such that $g_L \ll 1$. In addition, there are semiconductors which for reasons of symmetry have $g_L \equiv 0$ in certain directions.

[2)] For weaker fields (see footnote 1), when the Zeeman splitting is unimportant, the correct coefficient for the logarithmic correction to σ is given in Ref. 12.

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Translated by Steve Torstveit