Spatial dispersion of conductivity in one-dimensional conductors

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The dependence of the conductivity on the wave number q in a $1D$ conductor is calculated and analyzed with account taken of electron localization and of the fluctuations of the electron collision frequency. Geometric-resonance oscillations of the dissipative conductivity are predicted at low frequencies; these are due to the hopping character of the electron motion in an inhomogeneous wave field, with a fixed hop length. At high frequencies, the imaginary part of the conductivity is described by the Drude formula, and the real part is influenced by localization and differs substantially from the Drude formula.

1. INTRODUCTION

The dependence of the complex conductivity $\sigma(\omega, q)$ of a one-dimensional ($1D$) conductor on the frequency ω under conditions of localization of the electronic states has been sufficiently well investigated in a uniform electric field ($q = 0$), in the limiting cases of high and low frequencies.[1–4] At low frequencies, when $\omega\tau_2 \ll 1$ (τ_2 is the electron backscattering time), a one-dimensional system of electrons is more readily a dielectric than a semiconductor, in view of the localization, and the real part of the conductivity is small compared with the imaginary part. The behavior of the function Re $\sigma(\omega, q)$ in this frequency region can be determined with the aid of the qualitative arguments advanced by Mott.[5] Berezinskiĭ's theory[2] confirms Mott's conclusions and permits in principle to determine the conductivity of $1D$ conductors even in inhomogeneous fields. Berezinskiĭ, however, uses complicated finite-difference equations, for which asymptotic solutions could be obtained so far[1–4] only in the limiting cases of high and low frequencies and only for a uniform field, when the parameter $\kappa = ql_2$ is equal to zero (q is the external wave vector, $l_2 = v\tau_2$, and v is the Fermi velocity). For this reason, there are no results whatever at present concerning the effect of spatial dispersion on the conductivity of a $1D$ conductor.

Yet the spatial dispersion of the conductivity is of primary interest for the investigation of the propagation of waves of various types in $1D$ conductors. This pertains first and foremost to the propagation of sound, since $ql_2 \gg \omega\tau_2$ for a sound wave.

It is known[1,5] that localization of electronic states in disordered systems is due to anomalously large fluctuations of the phases of the electron wave functions. A brilliant mathematical formalism that embodies this physical idea was developed in the book by I. Lifshitz, Gredeskul, and Pastur[1] (see also the review by Abrikosov and Ryzhkin[4]). Phase fluctuations lead in fact also to another important effect—fluctuations of the frequency of the electron relaxation, the latter a macroscopic kinetic characteristic. This effect was predicted theoretically by one of us.[6]

The purpose of the present paper is an investigation of the Berezinskiĭ equations at finite values of the parameter $\varkappa$, and a calculation of the conductivity under conditions of strong ($\varkappa \gg 1$) and weak ($\varkappa \ll 1$) dispersion with allowance for localization and fluctuations of the free path time of the electrons.

We derive in this paper for the conductivity $\sigma(\omega, q)$ a relatively simple interpolation formula that gives correctly the functional dependence both on ω and on q at arbitrary frequencies and wavelengths. A relatively small error occurs only in certain numerical coefficients, while the others are obtained accurately. The correct values of these coefficients are determined below for various limiting cases.

2. SELF-SIMILAR APPROXIMATION

The conductivity $\sigma(\omega, q)$ of a $1D$ conductor (at $T = 0$) as a function of ω and q can be calculated according to Ref. 2 from the formula

$$\sigma(\omega, q) = \frac{\sigma_0}{2} \sum_{n=0,\pm}^{\infty} Q_n(\omega, \pm q) P_n(\omega). \tag{2.1}$$

Here $\sigma_0 = ne^2\tau_2/m$ is a factor that coincides formally with the static conductivity of a $3D$ metal (n is the number of electrons per unit volume and m is the electron effective mass), while $Q_n(\omega, q)$ and $P_n(\omega)$ are functions that depend only on the dimensionless parameters x and $\varkappa$ which correspond to ω and q. The parameter corresponding to ω is chosen to be

$$x = -2i\omega\tau_2 \tag{2.2}$$

and when solving the equations we shall regard x as real and positive, $x > 0$. This corresponds to a choice of ω on the axis Im $\omega > 0$ in the upper complex ω half-plane. In the final results an analytic continuation is carried from the axis Im $\omega > 0$ to the axis Re $\omega > 0$.

The fact that (2.1) contains the even-in-q part of $Q_n(\omega, q)$ is the result of averaging over the electron-velocity directions, i.e., over the Fermi surface, since Q_n depends only on the product qv.

The function $P_n(x)$ is equal to[2]

$$P_n(x) = x \int_0^{\infty} dt e^{-xt} t^n (1+t)^{-n-1}, \tag{2.3}$$

and $Q_n(\omega, q)$ is determined from the finite-difference equation[2]

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$$[(n+1)^2(Q_n-Q_{n+1})-n^2(Q_{n-1}-Q_n)+xnQ_n]$$
$$-i(\omega-qv)\tau_2 Q_n=P_n(x). \qquad (2.4)$$

The boundary conditions are that Q_n be finite at all $n = 0, 1, 2, \ldots$ and that they decrease rapidly as $n\to+\infty$, so that the series

$$\sum_{n=0}^{\infty} Q_n z^n \qquad (2.5)$$

converges in the circle $|z|\leqslant 0$.

We consider first the region of low frequencies, $\omega\tau_2\ll 1$. To obtain the principal term in the expansion of the conductivity in this limit it is necessary to transform to the self-similar variable $p = nx$ (Ref. 2) and, letting x go to zero, assume p to be finite. In this self-similar approximation the difference operator in the square brackets of (2.4) goes over into a differential one and the parameter x drops out of it[2]:

$$\left[-\frac{d}{dp}\left(p^2\frac{d}{dp}\right)+p\right]Q(p)-i(\omega-qv)\tau_2 Q(p)=2xK_0(2\sqrt{p}). \qquad (2.6)$$

Equation (2.6) must be solved on the axis $0\leqslant p<\infty$ with boundary conditions that $Q(p)$ decrease as $p\to+\infty$ and integrability be ensured at zero.[7]

A term proportional to $\omega\tau_2$ is retained in the left-hand side of (2.6); this is justified if account is taken of the corrections to (2.6) from (2.4).

The structure of Eqs. (2.4) and (2.6) allows us to call the operator in the square brackets of each the "collision-frequency operator"; the results that follow will justify this designation.

To solve (2.6) we expand $Q(p)$ in the total orthonormal system of eigenfunctions $\Phi_\mu(p)$ of the collision-frequency operator:

$$Q(p)=\int_0^\infty d\mu\Phi_\mu(p)Q_\mu, \qquad \Phi_\mu(p)=\left(\frac{\mu\,\mathrm{sh}\,\pi\mu}{\pi^2 p}\right)^{1/2}K_{i\mu}(2\sqrt{p}), \qquad (2.7)$$

where $K_{i\mu}(z)$ is the Macdonald function and

$$\left[-\frac{d}{dp}\left(p^2\frac{d}{dp}\right)+p\right]\Phi_\mu(p)=\nu(\mu)\Phi_\mu(p), \qquad (2.8)$$

the eigenvalue being

$$\nu(\mu)=(1+\mu^2)/4,\ 0\leqslant\mu<\infty. \qquad (2.9)$$

The functions $\Phi_\mu(p)$ are real at all real μ and p and satisfy the completeness relation on the axis $0\leqslant p<\infty$:

$$\int_0^\infty d\mu\Phi_\mu(p)\Phi_\mu(p')=\delta(p-p'),$$
$$\int_0^\infty dp\Phi_\mu(p)\Phi_{\mu'}(p)=\delta(\mu-\mu'). \qquad (2.10)$$

The first of these relations follows from the formulas for the Kontorovich-Lebedev transformation,[8,9] a variant of which is (2.7), while the second can be proved with the aid of the Nicholson formula from the theory of Bessel functions (it was obtained in somewhat different form in Ref. 3).

Substituting the expansion (2.7) in (2.6) and using (2.8) we get

$$Q(p)=x\int_0^\infty d\mu\Phi_\mu(p)p(\mu)[\nu(\mu)-i(\omega-qv)\tau_2]^{-1}, \qquad (2.a)$$

where $p(\mu)$ is the coefficient in the expansion of the right-hand side of (2.6) in the functions $\Phi_\mu(p)$:

$$p(\mu)=2\int_0^\infty dp\Phi_\mu(p)K_0(2\sqrt{p})=(\mu\,\mathrm{sh}\,\pi\mu)^{1/2}\frac{\pi}{2\,\mathrm{ch}^2(\pi\mu/2)}. \qquad (2.11)$$

The conductivity $\sigma(\omega, q)$ in the self-similar approximation is obtained by substituting (2.10a) in (2.1) with allowance for the fact that the sum over n is replaced by an integral with respect to p, with

$$P_n(x)\to 2xK_0(2\sqrt{p}). \qquad (2.12)$$

Integrating with respect to p, we obtain

$$\sigma_0(\omega,q)=-i(\omega\tau_2)\sigma_0\sum_{\pm}\int_0^\infty d\mu w(\mu)[\nu(\mu)-i(\omega\pm qv)\tau_2]^{-1}. \qquad (2.13)$$

The function

$$w(\mu)=p^2(\mu)=\frac{\pi^2}{2}\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^3(\pi\mu/2)} \qquad (2.14)$$

is the square of the modulus of the coefficient $p(\mu)$ in the expansion of the "state function" in the right-hand side of (2.6) in the system of functions $\Phi_\mu(p)$. It is non-negative and normalized to unity, the latter circumstance being mathematically the equivalent of the Parseval equation. Therefore the positive-definite eigenvalues (2.9) of the operator (2.8) can be interpreted in accordance with (2.13) as possible values of the dimensionless frequency of the collisions of the electron with the scatterers, where μ plays the role of a "quantum number." Integration with respect to μ in (2.13) corresponds to averaging of the energy denominator $[\nu - i(\omega - qv)]^{-1}$ over the fluctuations of the collision frequency ν with the probability distribution function $w(\nu)$ obtained from (2.14):

$$w(\nu)=\pi^2\tau_2\theta\left(\nu\tau_2-\frac{1}{4}\right)\frac{\mathrm{sh}\,\pi(\nu\tau_2-1/4)^{1/2}}{\mathrm{ch}^3\,\pi(\nu\tau_2-1/4)^{1/2}}. \qquad (2.15)$$

We emphasize that the integration with respect to μ in (2.13) is carried out at a fixed quantum state of the electron and summation over different electronic states can have no meaning.

In other words, in a $1D$ conductor at $\omega\tau_2\ll 1$ the transport frequency of the collisions of an electron having a fixed momentum (and spin) has no definite value and fluctuates in analogy with a dynamic quantity in quantum mechanics. In particular, the collision frequency corresponds to a self-adjoint operator whose eigenvalues determine the possible collision frequencies, and the treatment of $w(\mu)$ as a probability distribution function is based on the fact that $w(\mu)$ is the squared modulus of a coefficient in the expansion of the state

function $P_n(x)$ in the complete system of eigenfunctions of the collision-frequency operator.

The origin of the fluctuations of the collision frequency can be understood from simple physical considerations. In fact, localization of the electronic states takes place in a $1D$ system because there exists a finite probability of electron backscattering per unit time, with an order of magnitude $1/\tau_2$. On the average, the electron moves in one direction (for a time on the order of τ_2) and is then reflected backwards. But τ_2 is only the average time of motion in one direction. These times will be different for different electrons even if for no other reason that only the backward scattering probability is specified. It is the spread of the scattering times that leads to the relaxation-frequency fluctuations. These fluctuations are phase-connected, since the finite relaxation time is due to fluctuations of the wave-function phases.[5] This is precisely why the collision-frequency fluctuations are of quantum nature.[6]

From (2.13) we obtain an expression for the static dielectric constant of a $1D$ conductor

$$\varepsilon(q)=1+4\pi\sigma_0\tau_2\int_0^\infty d\mu w(\mu)2\nu(\mu)[\nu^2(\mu)+(ql_2)^2]^{-1}, \quad (2.16)$$

which shows that $\varepsilon(q)$ is a monotonically decreasing function of q. At $q=0$ we obtain

$$\varepsilon(0)=1+8\pi\sigma_0\overline{\nu^{-1}}=1+4\zeta(3)(\omega_p\tau_2)^2 \quad (2.17)$$

in exact agreement with Ref. 7 [$\zeta(z)$ is the Riemann zeta function, ω_p is the plasma frequency of the electron, a superior bar means averaging with the distribution function (2.15)].

3. GENERATING FUNCTION FOR $Q_n(\omega, q)$ IN THE SELF-SIMILAR REGION

Equation (2.13) yields the principal term of the expansion of the conductivity at $\omega\tau_2\ll 1$, with full account taken of the spatial dispersion, while the coefficient of $(-i\omega)$ in (2.13) is the polarizability of the $1D$ conductor. To obtain the dissipative conductivity, we transform to a generating function[2] for $Q_n(\omega, q)$:

$$y(t)=\frac{1}{1+t}\sum_{n=0}^{\infty}Q_n(\omega,q)\left(\frac{t}{1+t}\right)^n. \quad (3.1)$$

The conductivity is given in terms of $y(t)$ by means of the formula[2]

$$\sigma(\omega,q)=\frac{1}{2}\sigma_0 x\int_0^\infty dte^{-xt}y(t)+(q\to -q), \quad (3.2)$$

which is obtained by substituting (3.1) in (2.1). Here and below the symbol $(q\to -q)$ denotes the preceding term with the sign of the parameter in the brackets reversed).

An equation for $y(t)$ follows from (2.4) (Ref. 2):

$$-\frac{d}{dt}\left[t(1+t)\frac{dy}{dt}\right]-i(\omega-qv)\tau_2 y+xt\frac{d}{dt}[(1+t)y]=xp_0(t), \quad (3.3)$$

where

$$p_0(t)=-e^{x(1+t)}\,\mathrm{Ei}\,(-x(1+t)), \quad (3.4)$$

and $\mathrm{Ei}(z)$ is the integral exponential function. Equation (3.3) must be solved on the axis $0\leqslant t<\infty$ with boundary conditions that $y(t)$ be finite at $t=0$ and that $y(t)$ decrease to zero as $t\to+\infty$ (Ref. 2). It is known that it is impossible to solve (3.3) for arbitrary values of the parameter $\omega\tau_2$. We consider first the case of low frequencies: $\omega\tau_2\ll 1$.

We introduce the variable $\xi=x(1+t)$ and obtain $y(t)$ in the self-similar region, where ξ is finite as $z\to 0$. We rewrite (3.3) in terms of ξ:

$$-\frac{d}{d\xi}\left(\xi^2\frac{dy}{d\xi}\right)+\xi\frac{d}{d\xi}(\xi y)-i(\omega-qv)\tau_2 y+x\frac{d}{d\xi}\left[\xi\left(\frac{dy}{d\xi}-y\right)\right]$$

$$=xp_0(\xi). \quad (3.5)$$

Equation (3.5) can be solved by iteration with respect to x. In the first approximation we leave out the last term in the left-hand side of (3.5):

$$-\frac{d}{d\xi}\left(\xi^2\frac{dy_0}{d\xi}\right)+\xi\frac{d}{d\xi}(\xi y_0)-i(\omega-qv)\tau_2 y_0=xp_0(\xi). \quad (3.6)$$

This equation is solved exactly. We seek $y_0(\xi)$ in the form

$$y_0(\xi)=\frac{1}{\xi^{1/2}}e^{\xi/2}\int_0^\infty d\mu K_{i\mu/2}\left(\frac{\xi}{2}\right)g(\mu). \quad (3.7)$$

We substitute (3.7) in (3.6) and use the relation

$$\left[-\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\right)+\xi\frac{d}{d\xi}\xi\right]\frac{e^{\xi/2}}{\xi^{1/2}}K_{i\mu/2}\left(\frac{\xi}{2}\right)$$

$$=\nu(\mu)\frac{e^{\xi/2}}{\xi^{1/2}}K_{i\mu/2}\left(\frac{\xi}{2}\right). \quad (3.8)$$

It shows that $y_0(\xi)$ is expanded in (3.7) in the eigenfunctions of the differential operator in the left-hand side of (3.7), whose eigenvalues $\nu(\mu)$ are determined by the previous relation (2.9). As a result we obtain an integral equation for the function $g(\mu)$:

$$\int_0^\infty d\mu g(\mu)[\nu(\mu)-i(\omega-qv)\tau_2]K_{i\mu/2}\left(\frac{\xi}{2}\right)=xe^{-\xi/2}\xi^{1/2}p_0(\xi). \quad (3.9)$$

Its solution is obtained by the inversion formula[9]

$$g(\mu)=\frac{x}{2\pi^2}\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\nu(\mu)-i(\omega-qv)\tau_2}\int_0^\infty\frac{d\xi}{\xi^{1/2}}e^{-\xi/2}p_0(\xi)K_{i\mu/2}\left(\frac{\xi}{2}\right). \quad (3.10)$$

The integral here can be calculated exactly and is equal to

$$\int_0^\infty\frac{d\xi}{\xi^{1/2}}e^{-\xi/2}p_0(\xi)K_{i\mu/2}\left(\frac{\xi}{2}\right)=\frac{\pi^{1/2}}{\mathrm{ch}^2(\pi\mu/2)}. \quad (3.11)$$

Substituting (3.11) in (3.10) and next $g(\mu)$ in (3.7) we obtain a generating function in the self-similar region in the form

$$y_0(\xi)=\frac{x}{2}\left(\frac{\pi}{\xi}\right)^{1/2}e^{\xi/2}\int_0^\infty d\mu\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^2(\pi\mu/2)}\frac{K_{i\mu/2}(\xi/2)}{\nu(\mu)-i(\omega-qv)\tau_2}. \quad (3.12)$$

The correction to $y_0(\xi)$ for the discarded term in (3.5) can be obtained by iteration, having expression (3.12) and knowing the Green's function $G(\xi, \xi')$ of Eq. (3.6)

$$G(\xi,\xi')=\frac{1}{2\pi^2}\int_0^\infty d\mu\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\nu(\mu)-i(\omega-qv)\tau_2}$$

$$\times e^{(\xi-\xi')/2}(\xi\xi')^{-1/4}K_{i\mu/2}\left(\frac{\xi}{2}\right)K_{i\mu/2}\left(\frac{\xi'}{2}\right). \qquad (3.13)$$

We note that (3.12) can be obtained also by another method, using the results of the preceding section. Indeed, in the self-similar region we have $t\gg 1$ and from (3.1) we get

$$y_0(\xi)=\frac{1}{\xi}\int_0^\infty dpQ(p)e^{-p/\xi}=\frac{1}{\xi}\tilde{Q}'\left(\frac{1}{\xi}\right), \qquad (3.14)$$

where $Q(p)$ is given by (2.11) and $\tilde{Q}(z)$ is the Laplace transform of $Q(p)$. Calculating the Laplace transform of (2.11) and substituting the result in (3.14) we arrive at (3.12).

The contribution from the self-similar region to $\sigma(\omega, q)$ is

$$\sigma_{SS}(\omega,q)=\frac{1}{2}\sigma_0 e^x\int_x^\infty d\xi e^{-\xi}y_0(\xi)+(q\to -q). \qquad (3.15)$$

Substituting here $y_0(\xi)$ from (3.12), we find

$$\sigma_{SS}(\omega,q)=\frac{\pi^{1/2}}{4}\sigma_0 x e^x\sum_{\pm}\int_0^\infty d\mu$$

$$\times\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^2(\pi\mu/2)}\frac{1}{\nu(\mu)-i(\omega\pm qv)\tau_2}\int_x^\infty\frac{d\xi}{\xi^{1/2}}e^{-\xi/2}K_{i\mu/2}\left(\frac{\xi}{2}\right). \qquad (3.16)$$

The principal term, linear in x, of the expansion of $\sigma_{SS}(\omega, q)$ coincides with the function (2.13). Since the right-hand side of (2.4) is taken into account exactly and not approximately as in (2.13), the function $\sigma_{SS}(\omega, q)$ contains also terms of higher order of smallness in x. The fact that the lower limit of the integral with respect to ξ differs from zero allows us to obtain for $\mathrm{Re}\,\sigma(\omega, q)$ an expression that coincides in form at $q = 0$ with that obtained in Ref. 2, but having different numerical coefficients.

In the principal approximation, we assume a zero lower limit of the integral with respect to ξ in (3.16) and obtain

$$\sigma_{SS}^{(0)}(\omega,q)=\frac{\sigma_0}{2}xe^x\sum_{\pm}\int_0^\infty d\mu w(\mu)[\nu(\mu)-i(\omega\pm qv)\tau_2]^{-1}. \qquad (3.17)$$

It was noted above that terms of order higher than the first in x can be retained in (3.17), strictly speaking, only if account is taken also of the corrections for the terms discarded in the calculation of (2.17), as will be done below.

At large values of ql_2 we obtain from (3.17) the asymptotic form of the term linear in x

$$\sigma_{SS}^{(0)}(\omega,q)=\sigma_0\frac{x}{2(ql_2)^2}\left[1+O\left(\frac{1}{q^2l_2^2}\right)\right], \qquad (3.18)$$

which is used to monitor the accuracy of the expansions obtained below.

4. SPATIAL DISPERSION OF DISSIPATIVE CONDUCTIVITY

The main contribution to $\mathrm{Re}\,\sigma(\omega, q)$ is made by the region $0\leqslant t\ll 1/x$ (Ref. 2). To calculate $y(t)$ in this region, which we shall name logarithmic, we use Eq. (3.3) without the last term of its left-hand side:

$$-\frac{d}{dt}\left[t(1+t)\frac{dy_1}{dt}\right]-i(\omega-qv)\tau_2 y_1=xp_0(t). \qquad (4.1)$$

Just as in going from (3.5) to (3.6), an error is incurred in the collision-frequency operator, but exact account is taken of the state function in the right-hand side of (3.3) (details follow).

To solve (4.1) we use the Mehler-Fock integral transformation[10] and represent $y_1(t)$ in the form of an expansion in cone functions:

$$y_1(t)=\int_0^\infty d\mu P_{-(1+i\mu)/2}(1+2t)f(\mu). \qquad (4.2)$$

We substitute (4.2) in (4.1) and use the fact that the cone functions are eigenfunctions for the differential operator in the left-hand side of (4.1):

$$-\frac{d}{dt}\left[t(1+t)\frac{d}{dt}\right]P_{-(1+i\mu)/2}(1+2t)=\nu(\mu)P_{-(1+i\mu)/2}(1+2t) \qquad (4.3)$$

with eigenvalues (2.9). For $f(\mu)$ we obtain the integral equation

$$\int_0^\infty d\mu P_{-(1+i\mu)/2}(1+2t)[\nu(\mu)-i(\omega-qv)\tau_2]f(\mu)=xp_0(t). \qquad (4.4)$$

Its solution is given by the inversion formula[10]

$$f(\mu)=\frac{x}{2}\frac{\mu\,\mathrm{th}(\pi\mu/2)}{\nu(\mu)-i(\omega-qv)\tau_2}p_0(\mu), \qquad (4.5)$$

$$p_0(\mu)=-\int_0^\infty dt e^{x(1+t)}\,\mathrm{Ei}(-x(1+t))P_{-(1+i\mu)/2}(1+2t)=\left(\frac{\pi}{x}\right)^{1/2}$$

$$\times\frac{e^{x/2}}{\mathrm{ch}(\pi\mu/2)}K_{i\mu/2}\left(\frac{x}{2}\right). \qquad (4.6)$$

Substitution of (4.5) in (4.2) yields the expression

$$y_1(t)=\frac{(\pi x)^{1/2}}{2}e^{x/2}\int_0^\infty d\mu\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^2(\pi\mu/2)}\frac{K_{i\mu/2}(x/2)}{\nu(\mu)-i(\omega-qv)\tau_2}$$

$$\times P_{-(1+i\mu)/2}(1+2t). \qquad (4.7)$$

In the logarithmic region it approximates well enough the true generating function $y(t)$. We note that (4.2) can be regarded as an expansion of $y_1(t)$ in the complete orthonormalized system of the eigenfunctions $\varphi_\mu(t)$ of the operator (4.3):

$$\varphi_\mu(t)=\left(\frac{\mu}{2}\,\mathrm{th}\frac{\pi\mu}{2}\right)^{1/2}P_{-(1+i\mu)/2}(1+2t),\quad 0\leqslant t<\infty, \qquad (4.8)$$

$$\int_0^\infty d\mu\varphi_\mu(t)\varphi_\mu(t')=\delta(t-t'),\quad \int_0^\infty dt\varphi_\mu(t)\varphi_{\mu'}(t)=\delta(\mu-\mu'). \qquad (4.9)$$

The first of the completeness relations (4.9) is a consequence of the Mehler-Fock transformation formulas,[10] and the second (which is given in a different form in Ref. 3) can be proved with the aid of (2.13) and of the known integral representation of the cone function:

$$P_{-(1+i\mu)/2}(u)=\frac{2^{1/2}}{\pi^{3/2}}\operatorname{ch}\frac{\pi\mu}{2}\int_0^\infty dz z^{-1/2}e^{-zu}K_{i\mu/2}(z). \tag{4.10}$$

The Green's function $\mathcal{G}(t, t')$ of Eq. (4.1)

$$\mathcal{G}(t,t')=\int_0^\infty d\mu\varphi_\mu(t)\varphi_\mu(t')[\nu(\mu)-i(\omega-qv)\tau_2]^{-1} \tag{4.11}$$

makes it possible in principle to find the corrections to $y_1(t)$ by iterating with respect to the term discarded in the left-hand side of (3.3).

The error incurred in the calculation of the conductivity with the aid of $y_1(t)$ [Eq. (4.7)] can be estimated by comparing the two approximate expressions for $y(t)$: from (3.12) and from (4.7). We see first that the two coincide at $t=0$. This means that $y_0(0)=y_1(0)$ is the exact value, for all ω and q, of the true generating function at this point. On the other hand, it follows from the definition (3.1) that $y(0)$ is equal to $Q_0(\omega, q)$.

Thus, the exact value of $Q_0(\omega,q)$ is

$$Q_0(\omega,q)=\frac{(\pi x)^{1/2}}{2}e^{x/2}\int_0^\infty d\mu\frac{\mu\operatorname{sh}(\pi\mu/2)}{\operatorname{ch}^2(\pi\mu/2)}\frac{K_{i\mu/2}(x/2)}{\nu(\mu)-i(\omega-qv)\tau_2}. \tag{4.12}$$

In particular, in the high-frequency case $x\gg 1$

$$Q_0(\omega,q)=\frac{\pi}{2}\int_0^\infty d\mu\frac{\mu\operatorname{sh}(\pi\mu/2)}{\operatorname{ch}^2(\pi\mu/2)}\frac{1}{\nu(\mu)-i(\omega-qv)\tau_2}\left[1+O\left(\frac{1}{x}\right)\right]. \tag{4.13}$$

We compare next $y_0(\xi)$ and $y_1(t)$ in the logarithmic region. To find the asymptotic form of $y_1(t)$ [Eq. (4.7)] in this region we use the expression for the cone functions in terms of hypergeometric functions:

$$P_{-(1+i\mu)/2}(1+2t)=(1+t)^{-(1+i\mu)/2}\frac{\Gamma(-i\mu)}{\Gamma^2((1-i\mu)/2)}\times F\left(\frac{1+i\mu}{2},\frac{1+i\mu}{2};1+i\mu;\frac{1}{1+t}\right)+(\mu\to-\mu) \tag{4.14}$$

and the ensuing asymptotic expression for large t:

$$P_{-(1+i\mu)/2}(1+2t)=\frac{\Gamma(i\mu)}{\Gamma^2((1+i\mu)/2)}(1+t)^{(i\mu-1)/2}\times\left[1+\frac{1-i\mu}{4}\frac{1}{1+t}+\ldots\right]+(\mu\to-\mu). \tag{4.15}$$

Substituting (4.15) in (4.7) we obtain $y_1(t)$ at $x=0$ in terms of the variables ξ:

$$y_1(t)\cong e^{x/2}\left\{\frac{x}{2}[\ln^2\xi+2(C-1)\ln\xi+A]+\frac{x^2\ln x}{\xi}-x^2\frac{\ln\xi+1}{\xi}\right\}, \tag{4.16}$$

$$A=1+(1-C)^2+\pi^2/2,$$

where C is Euler's constant. Transformation in (4.16) to a new variable $u=xt$ followed by differentiation with respect to u makes the expression in the curly brackets equal to the corresponding expression of Ref. 2.

On the other hand, the asymptotic form (3.12) of $y_0(\xi)$ takes in the same region the form

$$y_0(\xi)=e^{\xi/2}\frac{x}{2}[\ln^2\xi+2(C-1)\ln\xi+B]+O(\xi), \tag{4.17}$$

$$B=1+(1-C)^2+\pi^2/6. \tag{4.18}$$

Expressions (4.16) and (4.17) have in common only the principal terms $\ln^2\xi$ and $\ln\xi$ of their expansions, i.e., they agree only with logarithmic accuracy. Calculations with the aid of the function $y_1(t)$ can yield the coefficient of x^2 in Re $\sigma(\omega, q)$ only near its sigularity, i.e., only with logarithmic accuracy. In other words, the use of (4.7) as the generating function for the calculation of the conductivity leads to an error on the order of x^2 in Re $\sigma(\omega, q)$. To determine the correct numerical coefficient of x^2 we must take into account the correction for the discarded term in (3.3).

The condition for the applicability of the expression obtained for Re $\sigma(\omega, q)$ by using the function (4.7) is thus that the coefficient of x^2 in Re $\sigma(\omega, q)$ (which depends on ω and q) be large compared with unity.

Finally, in the self-similar region ($t\sim 1/x$) the relative difference between the functions $y_0(\xi)$ and $y_1(t)$ becomes of the order of unity. Therefore calculation of the principal term of the conductivity (polarizability) at $\omega\tau_2\ll 1$ with the aid of (4.7) leads to a relative error of the order of unity in the numerical factor (see below).

Estimating the error incurred by the use of (4.7), we substitute $y_1(t)$ in (3.2) and calculate the conductivity

$$\sigma_1(\omega,q)=\frac{\sigma_0}{4}xe^x\sum_{\pm}\int_0^\infty d\mu\frac{\mu\operatorname{sh}(\pi\mu/2)}{\operatorname{ch}^2(\pi\mu/2)}\frac{K^2_{i\mu/2}(x/2)}{\nu(\mu)-i(\omega\pm qv)\tau_2}. \tag{4.19}$$

The approximate expressions (3.16) and (4.19) will be discussed below. At present we calculate the asymptotic value of $\sigma_1(\omega, q)$ as $x\to 0$. At small x,

$$K^{2\prime}_{i\mu/2}\left(\frac{x}{2}\right)=\frac{\pi}{\mu\operatorname{sh}(\pi\mu/2)}-\frac{\pi^2}{4\operatorname{sh}^2(\pi\mu/2)}\times\left[\Gamma^{-2}\left(1+\frac{i\mu}{2}\right)\left(\frac{x}{4}\right)^{i\mu}+(\mu\to-\mu)\right]+O(x) \tag{4.20}$$

We substitute this expansion in (4.19)

$$\sigma_1(\omega,q)=\frac{\pi^2}{4}\sigma_0xe^x\sum_{\pm}\left\{\frac{1}{\pi}\int_0^\infty d\mu\operatorname{ch}^{-2}\frac{\pi\mu}{2}\frac{1}{\nu(\mu)-i(\omega\pm qv)\tau_2}-\frac{1}{4}\int_{-\infty}^\infty d\mu\frac{\mu(x/4)^{i\mu}\Gamma^{-2}(1+i\mu/2)}{\operatorname{sh}(\pi\mu/2)\operatorname{ch}^2(\pi\mu/2)[\nu(\mu)-i(\omega\pm qv)\tau_2]}\right\}. \tag{4.21}$$

The first term of (4.21), as shown by comparison with (3.17), is the contribution made to $\sigma(\omega, q)$ by the self-similar region and taken into account in $\sigma_1(\omega, q)$ with the aid of

expression (4.7) for the generating function. It can also be seen that the error was reduced only to an inaccurate determination of the collision-frequency distribution function: (4.21) contains in place of the correct function (2.14) the function $\pi(1+\cosh\pi\mu)^{-1}$. The use of the latter leads to a relatively small numerical error. Thus, at $\omega = qv = 0$ we obtain from (4.21) the result (2.17) in which $\zeta(3)$ is replaced by $\zeta(2)$ ($\approx 1.5\zeta 3$)).

Thus, even if (4.7) is used outside the range of its applicability (in a "foreign" region), only a relatively small numerical error is incurred. This indicates that the results have relatively low sensitivity to the exact form of the collision-frequency operator and are determined mainly by the state function in the right-hand side of (3.3), which is taken into account exactly both in (3.16) and in (4.19). Consequently both (3.16) and (4.19) can be used as interpolation formulas for the complex conductivity $\sigma(\omega, q)$ at all ω and q and for arbitrary relations between them and τ_2.

The dissipative part of the conductivity is obtained from the second term in (4.21)

$$\Delta\sigma(\omega, q) = -x\sigma_0 \frac{\pi^2}{4}\int_{-\infty}^{\infty} d\mu \frac{\mu\Gamma^{-2}(1+i\mu/2)}{\operatorname{sh}(\pi\mu/2)\operatorname{ch}^2(\pi\mu/2)} \frac{(x/4)^{i\mu}}{\mu^2+\Delta^2},$$

$$\Delta \equiv \Delta(\varkappa) = (1+4i\varkappa)^{1/2}, \quad \operatorname{Re}\Delta > 0. \tag{4.22}$$

The asymptotic form of the integral in (4.22) as $x \to 0$ is determined by the integrand pole closest to the real axis in the lower half of the complex μ plane.

If $\varkappa = 0$, this is the third order pole $\mu = -i$. In this case (4.22) duplicates exactly Berezinskiĭ's result[2]

$$\operatorname{Re}\sigma(\omega, q) = -\sigma_0 x^2\{\ln^2|x| + (2C-3)\ln|x| + \text{const}\} + O(x^3). \tag{4.23}$$

In the case $\varkappa \neq 0$ the pole $\mu = -i$ becomes of second order, since the first-order pole $\mu = -i\Delta$ is separated from it and moves down and to the right from the point $\mu = -i$ with increasing $\varkappa$; allowance for the contributions of these two poles yields

$$\Delta\sigma(\omega, q) = \sigma_0 x^2\left\{-\frac{1}{4\varkappa^2} + \frac{1}{i\varkappa}(\ln x + C - 1) - \frac{\pi^3}{16}\left(\frac{x}{4}\right)^{\Delta-1} \frac{\Gamma^{-2}(1+\Delta/2)}{\sin(\pi\Delta/2)\cos^2(\pi\Delta/2)}\right\} + (\varkappa \to -\varkappa). \tag{4.24}$$

The analytic continuation $x \to -2i\omega\tau_2$ results in

$$(x)^{\Delta-1} \to (2\omega\tau_2)^{\Delta-1}\left(\sin\frac{\pi\Delta}{2} + i\cos\frac{\pi\Delta}{2}\right). \tag{4.25}$$

The dissipative conductivity is then found to equal

$$\operatorname{Re}\sigma(\omega, q) = 2\sigma_0(\omega\tau_2)^2\left\{\frac{1}{\varkappa^2} + \operatorname{Re}\left[\left(\frac{\omega\tau_2}{2}\right)^{\Delta(\varkappa)-1} \frac{\pi^3\Gamma^{-2}(1+\Delta(\varkappa)/2)}{4\cos^2(\pi\Delta(\varkappa)/2)}\right]\right\}, \tag{4.26}$$

$$\Delta(\varkappa) = (1+16\varkappa^2)^{1/4}\exp(1/2 i \operatorname{arctg} 4\varkappa). \tag{4.27}$$

The validity of (4.26) is restricted by the already indicated condition under which the expression in the curly brackets is much larger than unity, i.e., $\varkappa \ll 1$. At the same time, the range of validity of Berezinskiĭ's result (4.23) is determined by the possibility of expanding the expression in the square brackets of (4.26) in powers of $(\Delta - 1)$ and turns out to be much narrower:

$$\varkappa \ll 1/2|\ln\omega\tau_2| \ll 1. \tag{4.28}$$

At $\varkappa \ll 1$ we obtain from (4.26), at the required accuracy

$$\frac{\operatorname{Re}\sigma(\omega, q)}{\sigma_0} = \frac{2(\omega\tau_2)^2}{\varkappa^2}\{1 - e^{-\varkappa z_\omega}[\cos\varkappa z_\omega + 2\varkappa(C + \ln 4 - 1)\sin\varkappa z_\omega]\}, \tag{4.29}$$

$$z_\omega = 2|\ln(\omega\tau_2/2)|.$$

Under the condition (4.28), this expression goes over directly into (4.23). In the region $\varkappa^2 z_\omega < 1 < \varkappa z_\omega$ the conductivity executes oscillations

$$\frac{\operatorname{Re}\sigma(\omega, q)}{\sigma_0} = 4(\omega\tau_2)^2\left(\frac{\sin^2(\varkappa z_\omega/2)}{\varkappa^2} + \frac{z_\omega}{2}\right). \tag{4.30}$$

The period of the oscillations with respect to $\varkappa$ is $2\pi/z_\omega$, the relative amplitude is of the order of $2/\varkappa^2 z_\omega$, and the value at the minimum is $2(\omega\tau_2)^2 z_\omega$. In the region $1/z_\omega \lesssim \varkappa^2 < 1$ the oscillating terms attenuate exponentially and

$$\operatorname{Re}\sigma(\omega, q) = 2(\omega\tau_2/\varkappa)^2\sigma_0.$$

Let us clarify the physical nature of the oscillations. Equation (4.30) shows that in the inhomogenous field of the wave there appears a characteristic electronic length $l_2 z_\omega$ that determines the period of the oscillations. It is easily seen that this length can be treated as fixed (for a given ω) length of the electron hop between two localized states that differ in energy by ω. Indeed, in accord with Ref. 5 the low-frequency conductivity can be estimated in order of magnitude with the aid of the equation

$$\operatorname{Re}\sigma(\omega, q)/\sigma_0 \approx |D(\omega, q)|^2, \tag{4.31}$$

where $D(\omega, q)$ is the dimensionless Fourier transform of the matrix element of the momentum operator

$$D(\omega, q) = \frac{1}{2i}\int_{-\infty}^{\infty} d\xi \psi_{\varepsilon_F+\omega}(\xi)\left[e^{iq\xi}\frac{d}{d\xi} + \frac{d}{d\xi}e^{iq\xi}\right]\psi_{\varepsilon_F}(\xi) \tag{4.32}$$

between the wave functions of the localized states

$$\psi_{\varepsilon_F}(\xi) = \exp(-|\xi|/2) \quad \text{и} \quad \psi_{\varepsilon_F+\omega}(\xi) = \exp(-|\xi - z_\omega|/2). \tag{4.33}$$

The maxima of these functions are separated by z_ω because their energies differ by an amount $\omega \ll 1/\tau_2$ (Ref. 5). At small $\varkappa$ the main contribution to (4.32) is made by the region $0 < \xi < z_\omega$. Elementary calculation yields

$$|D(\omega, q)|^2 = e^{-z_\omega}\frac{\sin^2(\varkappa z_\omega/2)}{\varkappa^2} = \left(\frac{\omega\tau_2}{2\varkappa}\right)^2 \sin^2\frac{\varkappa z_\omega}{2} \tag{4.34}$$

which agrees, apart for a numerical factor, with the first term of (4.30).

The oscillations obtained are thus due to the oscillatory dependence of the transition matrix element on the wave phase shift $\varkappa z_\omega$ over the electron hop length $l_2 z_\omega$,

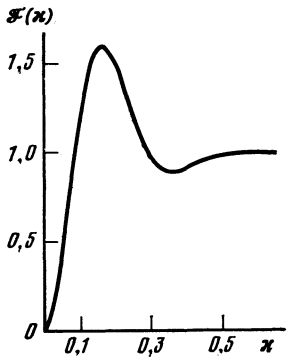

FIG. 1. Dependence of $\mathcal{F}(\varkappa)$ on $\varkappa$ for the linear relation $\omega(q) = sq$ at $s/v = 2\times 10^{-3}$.

$$\varkappa z_\omega = 2ql_2 |\ln(\omega\tau_2/2)|.$$

At $\varkappa \sim 1$ we have in order of magnitude

$$\mathrm{Re}\,\sigma(\omega, q) \sim \sigma_0(\omega/qv)^2. \tag{4.35}$$

It is simplest to obtain the asymptotic value of $\mathrm{Re}\,\sigma(\omega, q)$ at $\varkappa \gg 1$ by direct iteration of (3.3) [or (3.5)] about the principal term $i(\omega - qv)\tau_2 y$. Straightforward but somewhat lengthy calculations yield

$$\sigma(\omega, q) = \sigma_0\left[\frac{-i\omega\tau_2}{\varkappa^2} + 2\frac{(\omega\tau_2)^2}{\varkappa^4}\right]. \tag{4.36}$$

The imaginary term in (4.36) coincides with (3.18) and contains only corrections from powers of $\varkappa^{-2}$; the same holds for the corrections to $\mathrm{Re}\,\sigma(\omega, q)$.

Figure 1 shows the $\varkappa$ dependence of the function

$$\mathcal{F}(\varkappa) = \frac{\mathrm{Re}\,\sigma(\omega, q)}{2\sigma_0}\left(\frac{\varkappa}{\omega\tau_2}\right)^2, \tag{4.37}$$

where $\mathrm{Re}\,\sigma(\omega, q)$ is given by (4.29). We have assumed in the calculation that the frequency ω is linear in q, and that the phase velocity $\omega/q = s$ is small compared with the Fermi velocity v. Under the same assumption $\mathcal{F}(\varkappa)$ describes the frequency dependence of the damping of a slow wave (e.g., sound). Figure 1 shows distinctly the first oscillation of the conductivity. The succeeding periods do not appear because at the typical values $s/v \approx 2\times 10^{-3}$, even in the second period, we have $z_\omega = 2\ln(v/s\varkappa) \approx 20$, the parameter $\varkappa$ approaches unity, and the amplitude of the oscillating term in $\mathcal{F}(\varkappa)$ becomes quite small ($\sim 10^{-4}$).

5. INTERPOLATION EQUATION FOR THE COMPLEX CONDUCTIVITY

Equations (3.16) and (4.19) lead to the conclusion that the exact value of the conductivity $\sigma(\omega, q)$ of a $1D$ conductor should be, at all values of ω and q, of the form

$$\sigma(\omega, q) = \sigma_0 \sum_{\pm} \frac{1}{2}\int_0^\infty d\mu \frac{\pi}{2}\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^2(\pi\mu/2)}\frac{D_\mu(x)}{\nu(\mu) - i(\omega \pm qv)\tau_2}. \tag{5.1}$$

The summation in this equation is over the electronic states on the Fermi surface of the $1D$ conductor ($\pm p_0$) with an electron momentum distribution function equal to 1/2.

The quantity $D_\mu(x)$ describes the influence of the localization of the electronic states on the conductivity. We call $D_\mu(x)$ the "delocalization factor." Two expressions were obtained above for $D_\mu(x)$:

according to (3.16)

$$D_{\mu 0}(x) = \frac{1}{\pi^{1/2}} x e^x \int_x^\infty \frac{d\xi}{\xi^{1/2}} e^{-\xi/2} K_{i\mu/2}\left(\frac{\xi}{2}\right), \tag{5.2}$$

and from (4.19) we get

$$D_{\mu 1}(x) = \frac{1}{\pi} x e^x K^2_{i\mu/2}\left(\frac{x}{2}\right). \tag{5.3}$$

$D_\mu(x)$ tends in the high-frequency limit ($x \to \infty$) to unity, meaning a weakening of the influence of the localization on the conductivity. Indeed, from (5.3) we get

$$D_\mu(x) = 1 - \frac{2\nu(\mu)}{x} + O\left(\frac{1}{x^2}\right) \tag{5.4}$$

(whereas (5.2) yields an additional correction ($-x^{-1}$) to (5.4) and is therefore less accurate).

Even in this limiting case, however, it remains in (5.1) to integrate with respect to μ, which is tantamount to averaging of the energy denominator in (5.1) over the fluctuations of the dimensionless collision frequency $\nu(\mu)$ with a probability distribution function equal in the high-frequency limit to

$$w_\infty(\mu) = \frac{\pi}{2}\frac{\mu\,\mathrm{sh}(\pi\mu/2)}{\mathrm{ch}^2(\pi\mu/2)}. \tag{5.5}$$

This expression differs only insignificantly from the collision-frequency distribution function (2.14) in the low-frequency limit.

The function (5.5) has the following properties:

$$\int_0^\infty d\mu\, w_\infty(\mu) = 1, \quad \int_0^\infty d\mu\, w_\infty(\mu)\nu(\mu) = 1. \tag{5.6}$$

Thus, Eq. (5.1) proves the statement made in the Introduction that the conductivity of a $1D$ conductor is determined by the joint effect of two different factors: localization of the electronic states and fluctuations of the frequency of the electron collisions. The mutual relation between these factors manifests itself in the fact that the function $D_\mu(x)$ depends both on x and on μ. In the high-frequency limit ($\omega\tau_2 \gg 1$) we obtain from (5.1) with allowance for (5.4)

$$\sigma(\omega, q) = \sigma_0 \int_0^\infty d\mu\, w_\infty(\mu) \times \frac{-i\omega\tau_2}{[\nu(\mu) - i\omega\tau_2]^2 + (ql_2)^2}\left[1 + O\left(\frac{1}{\omega^2\tau_2^2}\right)\right]. \tag{5.7}$$

This expression shows that in the principal approximation in $\omega\tau_2$, when $\nu(\mu)$ can be neglected in the denominator, the imaginary part of $\sigma(\omega, q)$ is given by the Drude formula (with allowance for spatial dispersion). As for $\mathrm{Re}\,\sigma(\omega, q)$, the Drude formula yields an incorrect (half as large) coefficient in the region $ql_2 \ll 1$ and an incorrect dependence on ω and q at $qv \gg \omega$. In this latter region the high-frequency conductivity is determined by Eq. (4.36). This means that the condition for the applicability of (4.36) is $(ql_2)^2 \gg 1 + (\omega\tau_2)^2$ independently of the value of $\omega\tau_2$. An expression for $\mathrm{Re}\,\sigma(\omega, q)$ with

allowance for spatial dispersion cannot be obtained in principle with the aid of the usual kinetic equation for electrons in a $1D$ metal. The results for the conductivity at high frequencies can be qualitatively obtained if the localized electrons are regarded as charged oscillators with characteristic natural frequency τ_2^{-1}.

We note that the fluctuations of the collision frequency remain also in the high-frequency limit, but their relative contribution to the conductivity is small in the parameter $1/\omega\tau_2$. An exception is the resonance region, in which

$$|\omega - qv| \leqslant 1/\tau_2$$

and it is necessary to retain the relatively small quantity $\nu(\mu)$ in the denominator of the integrand of (5.7).

At low frequencies Eqs. (5.2) and (5.3) yield the following asymptotic expression for $D_\mu(x)$:

$$D_\mu(x) = \frac{\pi x}{\operatorname{ch}(\pi\mu/2)} - \frac{\pi x}{4\operatorname{sh}^2(\pi\mu/2)} \times \left[\Gamma^{-2}\left(1+\frac{i\mu}{2}\right)\left(\frac{x}{4}\right)^{i\mu} + \Gamma^{-2}\left(1-\frac{i\mu}{2}\right)\left(\frac{x}{4}\right)^{-i\mu}\right] + O(x^2), \tag{5.8}$$

where $\Gamma(z)$ is the gamma function. The vanishing of the delocalization factor as $x \to 0$ expresses the known fact that in this limit the localization exerts the decisive influence on the conductivity.[1,2]

Either of the equations (5.2) and (5.3) can serve in the entire frequency region as an interpolation expression for the factor $D_\mu(x)$ with which the interpolation formula (5.1) for the conductivity is constructed. The use of (5.2) or (5.3) leads to correct functional relations in all the limiting cases and for all ω and q, and is subject only to a certain error in individual numerical coefficients.

We call attention to a unique property of $1D$ conductors, namely that no collisionless Landau damping occurs in them. The reason is that the electrons on the Fermi surface actually constitute two monochromatic beams equal and opposite velocities $\pm v$. As a result, the averaging over the electronic states on the Fermi surface contains no integration over the velocity direction. As a result, at $ql_2 \gg 1$ the expression for $\sigma(\omega, q)$ does not contain the delta function $\delta(\omega - qv)$ that describes the Landau damping. Nonetheless, the effect of the spatial dispersion in the conductivity of $1D$ systems is much more complicated than might be expected by starting from an intuitive analogy with the three-dimensional case.

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