# Chaos and long-lived modes in the dynamics of nematic liquid crystals

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(Submitted 23 June 1983)
Zh. Eksp. Teor. Fiz. **86**, 147–156 (January 1984)

The possible role of inertial terms in the dynamics of liquid crystals is discussed. A hypothetical case of a liquid crystal for which the moment of inertia per unit volume is not small is examined. It is shown that, at sufficiently high frequencies in such a system, the interaction between orientational degrees of freedom and the hydrodynamic flow of the liquid crystal, and also all spatial inhomogeneities, can be neglected. The change in orientation may then occur quasiperiodically for a particular long-lived mode. The conditions for the chaotic behavior of orientation are also discussed.

**1.** Studies of the dynamics of systems with a complex order parameter, for example, liquid crystals or superfluid $He^3$ phases, have attracted considerable attention in recent years.[1–4] The general equations describing the dynamics of such systems are now well established. Their distinctive feature is the role played by the degrees of freedom associated with the order parameter.

Thus, for example, in $He^3 - B$, an abrupt change in a constant external field[1–3] is accompanied after a period of the order of a few hundred microseconds by the appearance of a long-lived oscillatory mode of magnetization, with a slowly-varying frequency. On the other hand, chaotic motion of magnetization may arise under parametric excitation if the frequency and amplitude of the exciting field have suitable values (see Ref. 1 for further details).

According to currently established general ideas, the structure of the order parameter is determined by the corresponding symmetry group. In the case of the superfluid $He^3 - B$, this is the $SO(3)\times SO(3)$ group of rotations in orbital and spin spaces. A partial breaking of this group is, in fact, described by the order parameter. The breaking of this symmetry is the reason for the dynamic features of magnetization, mentioned above.

The breaking of the symmetry of the $SO(3)$ space rotations occurs in nematic liquid crystals (NLC). The resulting order parameter is described by the real unit vector $\mathbf{n}$, where $\mathbf{n}$ and $-\mathbf{n}$ are indistinguishable, i.e., the space of the order parameter is a sphere in three-dimensional space with identified antipodal points. This relatively complex manifold can, under certain definite conditions (see below), ensure the existence of long-lived modes and chaotic regimes.

**2.** The form of the dipole-dipole energy with two minima in the angular variable of the order parameter plays an important role in the dynamics of $He^3 - B$ magnetization. A liquid crystal does not have an analog of dipole energy, but the interaction between the director in the NLC and the external field gives rise to a certain effective potential in the dynamic description of the system, which has similar properties. Let us examine the form of this effective potential.

The dynamic equations for the NLC are well known.[4] For the presentation given below, we shall need only the equation of motion for the director $\mathbf{n}$, which is conveniently written in the form

$$\frac{d}{dt}\mathbf{n}=[\Omega\times\mathbf{n}],\quad I\frac{d}{dt}\left[\mathbf{n}\times\frac{d}{dt}\mathbf{n}\right]=-\gamma\Omega+\chi_a(\mathbf{Hn})[\mathbf{n}\times\mathbf{H}],\quad (1)$$

where we have taken into account the fact that, since $\mathbf{n}$ is a unit vector, $\mathbf{n}^2 = 1$ and any change in $\mathbf{n}$ reduces to rotation, described by the "angular frequency" $\Omega$. The parameter $I$ has the significance and dimensions of the moment of inertia per unit volume, $\gamma$ is the rotational viscosity, and $\chi_a$ is the anisotropic part of the susceptibility.

For our purposes, the important frequencies are those satisfying the inequality

$$\omega\gg(\chi_a I^{-1}H^2)^{1/2}.\quad (2)$$

The quantity $I$ is, at present, unknown for liquid crystals. It is usually considered[4] that $I$ is a molecular quantity, in which case $I\sim 10^{-14}$ g/cm. If this is so, the molecules rotate individually in the liquid crystal, in the same way that they do in an isotropic liquid. However, the introduction of the director $\mathbf{n}$ that distinguishes the liquid crystal from an isotropic liquid in itself implies averaging over physically infinitesimal volumes (which, nevertheless, contain a large number of molecules). To the extent to which one can write a hydrodynamic equation for $\mathbf{n}$ with an inertial term, the parameter $I$ may be regarded as being much greater than the molecular value. The justification for the small value $I\sim 10^{-14}$ g/cm is the experimental confirmation of the absence of inertial effects in NLC dynamics. We shall show below that, even for the much greater values $I\sim 10^{-7}$–$10^{-8}$ g/cm, very special conditions are necessary for the observation of such effects and, in any case, they can be observed only in very short-lived transient states. Recent observations of propagating waves in the distribution of the director[5] may be looked upon as an indication that such gigantic values of $I$ may occur. The velocity of these waves has been estimated[5] as being $\sim 1$ cm/s. However, this velocity depends only on the ratio $(K/I)^{1/2}$ from which it follows that $I\sim 10^{-7}$ g/cm ($K$ is Frank's modulus, $\sim 10^{-7}$ dyne/cm).

It is possible that $I$ is much greater in lyotropic liquid crystals. It is now well known[6] that the structural units responsible for nematic order in such systems are not the individual molecules, but molecular aggregates that are cylindrical in shape and contain a large number of particles.

In any case, it seems to us that the basic formulation of

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the problem of high-frequency NLC dynamics, with the parameter values chosen above, makes sense in relation to the possible synthesis of new materials with previously unattainable physical constants.

Let us estimate the terms in the second equation in (1):

$$I\omega^2 \sim -\gamma\omega + \chi_a H^2.$$

Numerical calculations (see below) show that the suitable parameter values are such that, with the left-hand side $\sim 1$, the coefficients on the right-hand side of the symbolic equation (2) are $5\times 10^{-3}$ and $10^{-4}$, respectively. Hence, it follows that

$$\gamma/(I\omega) \sim 5\cdot 10^{-3}, \quad \chi_a H^2/(I\omega^2) \sim 10^{-4}. \tag{3}$$

Taking $\gamma \sim 10^{-3}$ P, $I \sim 10^{-7}$ g/cm, $\chi_a \sim 5\times 10^{-7}$ cgs, and $H \sim 5\times 10^4$ G, find from (3) that the frequencies that are important for our purposes are

$$\omega \sim 10^7\ \mathrm{s}^{-1}.$$

We are entitled to neglect spatial derivatives in any specific problem that involves frequencies exceeding the frequencies of the natural NLC modes. It is known that there are two modes in nematic crystals,[4] namely,

$$\omega_s \sim Kq^2/\eta, \quad \omega_f \sim \eta q^2/\rho, \tag{4}$$

where $\rho$ is the density (in g/cm$^3$) and $\eta$ is the effective viscosity. The slow mode (frequency $\omega_s$) is connected with the relaxation of the director, and the fast mode $\omega_f$ determines the viscous damping of shear waves, as in ordinary liquids. For the above parameter values, $\omega_f/\omega_s \sim 10^4$, so that the fast mode is the most "hazardous." To satisfy the homogeneous approximation, we must have

$$\omega_f \ll 10^7\ \mathrm{s}^{-1}! \tag{5}$$

This inequality actually signifies that the entire process of changing the state of a liquid crystal can be divided into two time intervals. Initially, up to time $\sim \rho L^2/\eta$ ($L$ is the thickness of the nematic crystal), we can neglect gradients and use (1). For longer times, on the other hand, we can usually neglect the inertia $I$ of the rotation of the director and everything is determined by the gradient terms.

Condition (5) is satisfied when wave vectors $q < 10^2$–$10^5$ cm$^{-1}$ are important. Thus, sufficiently thick cells ($L \gtrsim 10^{-2} - 10^{-3}$ cm) with ordered NLC must be used.

Suppose that (5) is satisfied and let us analyze (1) for $\gamma = 0$. The latter simplification is necessary to enable us to find the form of the effective potential. A general solution, obtained with the aid of a computer, will be considered in the next section.

It is readily seen from the first equation in (1) that

$$\left[\mathbf{n}\times\frac{d}{dt}\mathbf{n}\right] = \boldsymbol{\Omega}.$$

Thus, the set of equations given by (1) with $\gamma = 0$ reduces to

$$\frac{d\mathbf{n}}{dt} = [\boldsymbol{\Omega}\times\mathbf{n}], \quad I\frac{d}{dt}\boldsymbol{\Omega} = \chi_a(\mathbf{Hn})[\mathbf{n}\times\mathbf{H}]. \tag{6}$$

This has the following constants of motion:

$$\mathbf{H}\boldsymbol{\Omega} = \mathrm{const}, \quad \boldsymbol{\Omega}^2 - \delta(\mathbf{Hn})^2 = \mathrm{const},$$

where $\delta = \chi_a/I$. These constants can be used to rewrite (1) in the form of the equation of motion of a particle in a certain potential. In fact, it follows from (6) that

$$\left[\frac{d}{dt}(\mathbf{Hn})\right]^2 = (\boldsymbol{\Omega}[\mathbf{n}\times\mathbf{H}])^2. \tag{7}$$

We can now transform the right-hand side with the aid of the last equation:

$$(\boldsymbol{\Omega}[\mathbf{n}\times\mathbf{H}])^2 = \mathbf{H}^2\boldsymbol{\Omega}^2 - (\mathbf{H}\boldsymbol{\Omega})^2 - \boldsymbol{\Omega}^2(\mathbf{Hn})^2.$$

When $\mathbf{H}\boldsymbol{\Omega} \equiv C_2$ and $\boldsymbol{\Omega}^2 = C_1 + \delta(\mathbf{Hn})^2$, we obtain the following equation of motion for the variable $\mathbf{H}\cdot\mathbf{n} = y$:

$$(dy/dt)^2 + V_{\mathrm{eff}} = \mathrm{const}, \tag{8}$$

$$V_{\mathrm{eff}} = \delta y^4 + (C_1 - \delta H^2)y^2, \quad H^2C_1 - C_2^2 = \mathrm{const}. \tag{9}$$

It is clear from (9) that, when $C_1 - \delta H^2 < 0$, and $\delta > 0$, the effective potential (regarded as a function $y$) has minima at the two points $\pm[(\delta H^2 - C_1)/\delta]^{1/2}$ (see Fig. 1). The essential point is that the effective potential has one minimum and one degenerate minimum when it is looked upon as a function in the space of the order parameter, i.e., a sphere with identical diametrically opposite points. The two minima in the variable $y$ coalesce in the space of the order parameter because of the indistinguishability $\mathbf{n} \leftrightarrow -\mathbf{n}$. The maximum $y = \mathbf{H}\cdot\mathbf{n} = 0$ corresponds to a circle, i.e., a set of directors that are perpendicular to $\mathbf{H}$. This result requires some elucidation. In fact, in statics, there is only one minimum $\delta > 0$ ($y = 0$, i.e., $\mathbf{H}\cdot\mathbf{n} = 0$). When $\delta < 0$, the minimum is absent altogether, and the smallest value occurs for $\mathbf{H}\|\mathbf{n}$, i.e., $y = H$. The situation illustrated in Fig. 1 is connected with the motion of the director. When the vector $\mathbf{n}$ executes rapid "rotation," the minimum effective-potential configuration occurs for $y \neq 0$.

To find the stationary modes of (1), let us rewrite it in terms of the new scalar variables

$$y = \mathbf{Hn}; \quad \boldsymbol{\Omega}^2; \quad D = \mathbf{H}[\boldsymbol{\Omega}\times\mathbf{n}] \tag{10}$$

from which, after some simple rearrangement, we obtain (for $\gamma \neq 0$)

$$\frac{d}{dt}(\mathbf{Hn}) = D, \quad \frac{d}{dt}\boldsymbol{\Omega}^2 = 2(\mathbf{Hn})D - 2\gamma\boldsymbol{\Omega}^2,$$
$$\frac{d}{dt}D = -(\mathbf{Hn})\boldsymbol{\Omega}^2 + (\mathbf{Hn})\mathbf{H}^2[1 - \mathbf{H}^{-2}(\mathbf{Hn})^2] - \gamma D. \tag{11}$$

As before, let us substitute $\gamma = 0$. In terms of these variables, we then have the equilibrium stationary solution

$$\mathbf{Hn} = 0; \quad D = 0; \quad \boldsymbol{\Omega}^2 = \mathbf{H}^2 - (\mathbf{Hn})^2. \tag{12}$$

Moreover, the conditions $\mathbf{H}\cdot\mathbf{n} = 0$, $D = 0$, $\boldsymbol{\Omega}^2$ arbitrary, correspond to neutral unstable equilibrium. Figure 2 shows the phase diagram in terms of the variables $\mathbf{H}\cdot\mathbf{n}$, $D$, and $\boldsymbol{\Omega}^2$.

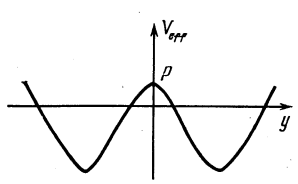


FIG. 1. The form of the effective potential $V_{\mathrm{eff}}$.

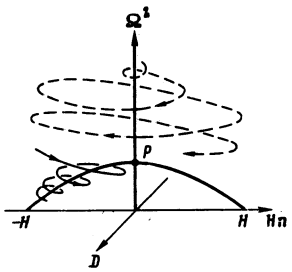


FIG. 2. Thick lines show stationary solutions. The $\Omega^2$ axis is unstable when dissipation is taken into account, and the parabola given by (12) is stable. The point $P$ corresponds to the region of unstable director dynamics. The thin line shows a typical trajectory, attracted by the stable stationary solution given by (12). The broken line shows a typical trajectory corresponding to high frequencies $\Omega$ (above $P$) that breaks away from the unstable stationary solution.

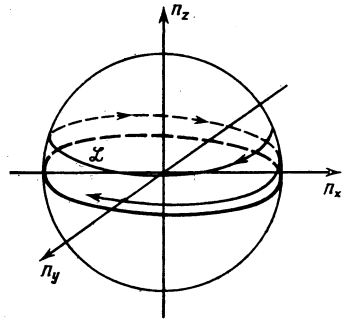


FIG. 3. Values of the director $\mathbf{n}$ correspond to points on a unit sphere. The $\mathbf{n} \leftrightarrow -\mathbf{n}$ symmetry is taken into account by the indistinguishability of antipodal points. The maximum of $V_{\text{eff}}$ corresponds to points on the equator shown by the thick line. The thin line $\mathscr{L}$ shows a trajectory corresponding to an approach to $P$ in the space of the variable $\Omega^2$, $\mathbf{H}\cdot\mathbf{n}$, $D$. The equator is a limit cycle for $\mathscr{L}$.

We note that the stationary solution $\mathbf{\Omega}^2 = \mathbf{H}^2 - (\mathbf{Hn})^2$ is similar to the so-called *WP*-mode in the dynamics of $He^3 - B$. Moreover, the effective potential (subject to the above restrictions) resembles the dipole energy.

Hence, by analogy with the situation in $He^3 - B$,[1-3] we may expect nontrivial dynamics, described by (11), even for $\gamma = 0$.

In the next section, we shall give a numerical analysis of the dynamics of nematic liquid crystals under the conditions described above (i.e., in an external field of about $10^4$ that is sufficiently strong to produce an effective potential with two minima, and such that we can neglect all spatial distribution gradients). Actual physical effects will be examined in Sec. 4.

**3.** Equations (11) provide us with a graphic description of the dynamics of nematic crystals in the space of the variables $\mathbf{\Omega}^2$, $\mathbf{H}\cdot\mathbf{n}$, $D = \mathbf{H\Omega}\times\mathbf{n}$. The first step is to consider stationary solutions that are obtained by omitting dissipative terms proportional to $\gamma$ and by equating the right-hand sides of (11) to zero (see Fig. 2).

Numerical analysis shows that there are two distinct states of motion, namely, those above the point $P$ at which stationary solutions cross, and those below this point. In the region above $P$, the system tends to reach the neighborhood of the stable stationary solution (12) in a short time $\tau$ and then to oscillate around it in the form of a gradually damped, long-lived mode. In our calculations, the material constants $I$, $\gamma$, $\chi_a$ were taken to be $I = 10^{-7}$ g/cm, $\gamma = 10^{-3}$ P and $\chi_a = 5\times 10^{-7}$ cgs. The time $\tau$ was of the order of 100 $\mu$s. The lifetime of the long-lived mode was 400 $\mu$s. The number of complete periods observed during this time was 3–4. For the more favorable parameter values $\gamma = 0.005$ P $\chi_a = 10^{-6}$ cgs, the time $\tau$ was 65 $\mu$s, the lifetime was 150 $\mu$s, and the total number of periods was about 15. The dynamic characteristics of the state of motion are thus more clearly defined in this case.

It is important to note that the value used for $\gamma$ was somewhat lower than the viscosity of "normal" NLC's. Solutions of liquid crystals can be used to produce such low viscosities. The reduction in $\gamma$ will, of course, be accompanied by a reduction in $\chi_a$. However, the numerical results are much less sensitive to the value of $\chi_a$. Moreover, even for the usual viscosities of liquid crystals, the assumption of chaos and the presence of long-lived modes remain valid. All that happens is that there is a reduction in the lifetime, the number of periods, and so on.

States in which the trajectory in the $\mathbf{\Omega}^2$, $\mathbf{H}\cdot\mathbf{n}$, $D$ space enters the neighborhood of the point $P$ occupy a special position. To visualize their dynamics more clearly, we must consider motion in the space of the order parameter. As noted above, in the case of a nematic crystal, this space is the unit sphere in three-dimensional space with identical antipodal points. If we take the external field $\mathbf{H}$ along the $z$ axis, the effective potential (9) will have a maximum on the equator of the sphere, i.e., for $\mathbf{H}\cdot\mathbf{n} = 0$, and a minimum at the antipodal points at the north and south poles (see Fig. 3).

Thus, the peculiarity of the liquid-crystal situation generated by the initial $SO(3)$ symmetry is reflected in the degeneracy of the maximum of the effective potential. A similar situation occurs in $^3He - B$, where the dipole energy plays the role of the effective potential.[1] As a result of the degeneracy of the maximum of the potential, the scattering of trajectories by it does not lead to the appearance of domain walls and has the attributes of continuous scattering, as a result of which two initially closely spaced trajectories can diverge to a finite distance in a finite time. Hence, if the liquid crystal specimen is prepared so that trajectories describing the dynamics of the system in terms of the variables $\mathbf{\Omega}^2$, $\mathbf{H}\cdot\mathbf{n}$, $D$ reach the neighborhood of $P$, the spatial homogeneity of the distribution of the director will be disturbed by the small spatial inhomogeneities that can be neglected at the initial time, in accordance with the estimates made in Sec. 2, and the turbulization of the system may be expected in the space of the order parameter.

Turbulization can be encouraged by an external periodic magnetic field of frequency and amplitude corresponding to motion in the neighborhood of the point $P$. We have considered an external field $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, where $\mathbf{h} = 0{,}1\mathbf{H}_0 \cos 0.25\Omega t$, $H_0 = 10^4$ gauss, $\Omega = 10^7$ Hz. The results are shown in the form of the histogram of Fig. 4. It is important to note that the shape of the histogram reflects the chaotic nature of the state that, nevertheless, still fits into the framework of the spatially homogeneous approximation introduced in Sec. 2. In a real experimental situation, it may be

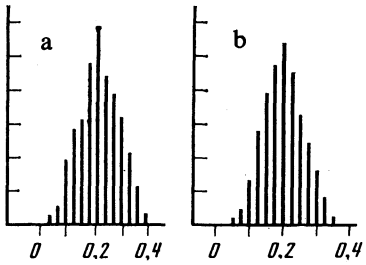


FIG. 4. Histogram showing the distribution of the initial separations between points forming a square lattice in steps of 0.0000025, with 121 points on the $(D, \Omega)$ plane. The separation between the points after 300 $\mu$s intervals is shown along the abscissa axis, and the number of pairs of points that have separated to a given distance is shown along the ordinate axis. Histogram *a* shows the distribution of separations for arbitrary pairs of points, and histogram *b* for initially neighboring pairs of points.

expected that the resulting chaos leads to the development of spatial inhomogeneities and turbulization of the system in the space of the order parameter. This form of orientational turbulence is now well known in connection with electrohydrodynamic instability.[4]

It is interesting to note the following nonlinear state that corresponds to director angular velocity greater than $\chi_a \mathbf{H}^2/I$. We shall suppose that $\Omega \gg \chi_a \mathbf{H}^2/I$, so that, in the first approximation, it may be considered that $\boldsymbol{\Omega} = \text{const}$ and the vector $\mathbf{n}$ rotates around the $\mathbf{v} = \boldsymbol{\Omega}/|\Omega|$ axis with angular velocity $\Omega \gg 1$. We now take the average over this configuration of the equation that follows from the second equation in (1):

$$I\frac{d}{dt}\boldsymbol{\Omega} = -\gamma\boldsymbol{\Omega} + \chi_a(\mathbf{Hn})[\mathbf{n}\times\mathbf{H}],$$

where the average of $f$ is given by

$$\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\,d\psi,$$

and $\psi$ is the phase of the vector $\mathbf{n}$ in the course of its rotation around $\boldsymbol{\nu}$. The result is (omitting the angle brackets)

$$I\frac{d}{dt}\boldsymbol{\Omega} = -\gamma\boldsymbol{\Omega} - \frac{1}{2}\chi_a(\mathbf{H}\boldsymbol{\Omega})\left[\frac{\boldsymbol{\Omega}\times\mathbf{H}}{\Omega^2}\right].$$

It is convenient to rewrite this equation in the form of two equations, namely, one for $\Omega = |\boldsymbol{\Omega}|$ and the other for $\boldsymbol{\nu} = \boldsymbol{\Omega}/\Omega$:

$$I\frac{d}{dt}\Omega = -\gamma\Omega, \quad I\frac{d}{dt}\boldsymbol{\nu} = -\frac{1}{2}\chi_a\Omega^{-1}(\mathbf{H}\boldsymbol{\nu})[\boldsymbol{\nu}\times\mathbf{H}].$$

From these equations, it follows immediately that: (1) the lifetime in this "average" state is of the order of $I/\gamma$, (2) the angle between $\boldsymbol{\nu}$ and $\mathbf{H}$ is constant, $\mathbf{H}\boldsymbol{\nu} = \text{const}$, and (3) the vector $\boldsymbol{\nu}$ rotates around $\mathbf{H}$ with the instantaneous angular velocity

$$\omega_\nu = -{}^1/_2\chi_a(\mathbf{H}\boldsymbol{\nu})(\Omega I)^{-1}[\mathbf{H}^2 - (\mathbf{H}\boldsymbol{\nu})^2]^{-1/2}. \tag{13}$$

The important point to note is that, for the values of $I$, $\Omega$, $\mathbf{H}$, $\chi_a$ that we have considered, the angular velocity $\omega_\nu$ is $\sim 1$, i.e., much less than $\Omega$. The configurations $\mathbf{H}\cdot\boldsymbol{\nu} = 0$ and $\mathbf{H}\|\boldsymbol{\nu}$, for which $\boldsymbol{\nu} = \text{const}$, should be particularly noted. Computer calculations performed for the original system given by (1), which is exact within the framework of our model, show that the change in $\nu$ for these values of $\gamma$ and $\chi_a$, $\mathbf{H}$ does not, in fact exceed 90–180°.

This fact is not wholly without interest in view of the fact that the average $\langle n_i n_j\rangle$ that determines the permittivity of the crystal is equal to $1/2(\delta_{ij} - \nu_i\nu_j)$.

**4.** We shall now examine possible experimental observation of the above effects. As already noted above, the necessary condition for this is the presence of a magnetic field producing an effective potential in the space of the order parameter, with one minimum and a degenerate maximum (a circle). The same field orders the liquid crystal. Moreover, since we are ignoring all spatial gradients, the field must be strong enough to exceed the frequencies of the characteristic inhomogeneous motions of the director and velocity. The estimates introduced in Sec. 2 show that these conditions are, in fact, satisfied for experimentally practicable values of $\mathbf{H}$, i.e., $10^4$–$10^5$ G and for liquid-crystal thickness of $10^{-2}$ cm. However, this effective potential is meaningful only during the motion of the director. In addition to the static field $\mathbf{H}$, we must therefore apply a perpendicular pulsed field or some other disturbance of comparable (or, better still, greater) strength. This will give rise to the rotation of the director that will produce an initial state with $\Omega^2 \neq 0$. The field is then turned off and the motion of $\mathbf{n}$, discussed in the last section, will begin. It is also important to remember that transverse (relative to the initial orientation) rotations of the director necessarily lead to the motion of the liquid-crystal mass itself, and to the appearance of nonzero gradients. However, the attenuation of the velocity $\mathbf{v}$ occurs more rapidly than the attenuation of the motion of the director.[7] This means that a state with $\boldsymbol{\Omega} \neq 0$, is possible for times that are sufficiently long in comparison with those found in the last section, but the gradients are then absent (and $\mathbf{v} = 0$). During this interval of time, the attenuation of the director has the asymptotic form $t^{-1/2}$, whereas the attenuation of the velocity is $\sim t^{-3/2}$.

In principle, an initial state with $\boldsymbol{\Omega} \neq 0$ can also be set up in other ways. Director motion with sufficiently high frequencies will also occur[4] under the conditions of dielectric EHD instability. This will, of course, be accompanied by the motion of the liquid: $\mathbf{v} \neq 0$. However, for the reasons given above, the turning-off of the electric field is followed by a time interval (sufficient for our effects to occur) during which $\boldsymbol{\Omega} \neq 0$, but gradients are no longer present. The third way of producing the required initial state is to apply a combination of a static field and a coplanar rotating field $\mathbf{h}_0 \cos\omega t$. It is known[8,9] that, in this case, there is a critical frequency $\omega_c \sim \chi_a \mathbf{h}^2/\gamma$ such that, for $\omega \leqslant \omega_c$, the rotation of the director occurs in step with the frequency $\omega$ but, for $\omega > \omega_c$, there is a delay given by

$$\omega' = \omega - \pi\omega_c\left[\int_v^\pi \left(\frac{\omega}{\omega_c} + \frac{1}{2}\sin 2x\right)^{-1} dx\right]^{-1}.$$

It is clear from this formula that, when $\omega \gg \omega_c$, we have $\omega' = 0$, i.e., rotation of frequency much higher than $\omega_c$ is not possible. In the most favorable situation (from our point of

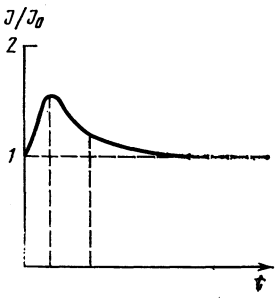


FIG. 5. Schematic curves showing the transmission coefficient as a function of time.

view), $\omega_c \sim 10^6 - 10^7$ s$^{-1}$ ($h_0 \sim 10^4 - 10^5$ G), which lies in the required frequency interval. Thus, we shall consider that the state with $\Omega \neq 0$ has been produced in some way. Next, as already noted, there are two possible types of behavior. Either the system takes up one of the branches of the stationary mode (Fig. 2) and follows it for a relatively "long" period of time, or it reaches an unstable point and "dephasing" begins (close states become separated), i.e., chaotic motion sets in. Let us begin with the first situation. This behavior can be observed in a number of ways. For example, one can measure the intensity of light scattered or transmitted by a cell. The intensity of light transmitted by the cell and the analyzer depends on the angle $\beta$ between the polarization vector in the incident beam and the original direction of the director, and the phase difference $\Delta\Phi$ between the ordinary and extraordinary rays that is acquired within the cell (see, for example, Ref. 10, where estimates are given for the times for which the different deformations of the director must be turned on and off):

$$J = J_0 \sin^2 \beta \sin^2 (^1/_2 \Delta\Phi),$$

where $J_0$ is the intensity of the linearly polarized light incident on the cell. The phase difference $\Delta\Phi$, on the other hand, is determined by the difference between the principal eigenvalues of the matrix $\varepsilon_{ij} = \varepsilon_0 \delta_{ij} + \varepsilon_a n_i n_j$. The average $\langle n_i n_j \rangle$ over the period is directly related to $\Omega$. The intensity $J$ will therefore vary in the course of motion along the stationary mode as shown schematically in Fig. 5. The same average appears in the so-called dielectric $g$ factor, on which the dielectric relaxation is found to depend. In all cases, the time dependence has the characteristic nonmonotonic segment. The simple average $\langle n_i n_j \rangle$ over the basic solution corresponding to the absence of dissipation is evaluated in the Appendix.

We note that, as $\Omega$ decreases along the stationary mode, the departure from the condition for the validity of (1) becomes greater. Finally, the same average $\langle n_i n_j \rangle$ appears in the spectral intensity of the scattered light. During the lifetime of the stationary state, the scattered line is a superposition of three Lorentz curves and not the usual two. The additional contribution is much narrower than the curves connected with the slow nematic and fast hydrodynamic modes (determined by $\Omega^2$). However, if the assumptions that enable us to neglect gradients are valid, only this additional contribution should, in fact, be noticeable.

In the case of the second type of (chaotic) behavior, close states (orientations) begin to diverge. This corresponds to spontaneous appearance of defects (disclinations). The turbulization of the motion effectively signifies that the system behaves analogously to an isotropic phase (i.e., birefringence, splitting of NMR lines, and similar anisotropic characteristics are no longer present).

The effects suggested above have not, as yet, been verified experimentally in a direct manner.

It may be simpler to investigate these processes by examining the resulting textures, rather than by studying directly the time-transient state. In fact, the assumption of homogeneity is equivalent to the assumption that the above behavior (for example, chaotic behavior) occurs independently at each point in space. The observed texture, on the other hand, depends on the parameters of this motion at each point, and is obtained by averaging the inhomogeneous equations over the periods of fast motion. A detailed examination of these questions will be given elsewhere.

Finally, we note a further and more methodological motivation for the present work. The point is that there is a large number of publications (see the references at the end of Ref. 11) on director waves in liquid crystals. All these papers investigate the linearized equation of motion with allowance for inertial terms. On the other hand, the analysis given above shows that all the nontrivial effects connected with the moment of inertia occur only because of the nonlinearity of the equation of motion for the director in the external field.

All the effects discussed above are, of course, connected with the assumption that $I \neq 0$, and do not occur for $I = 0$.

We also note that observation of the nonlinear state for $\Omega \gg \chi_a H^2/I$ may serve as a way of direct determination of $I$, since $\langle n_i n_j \rangle$ is determined by the instantaneous angular velocity given by (13).

The authors are greatly indebted to B. Ya. Zel'dovich for his rigorous but valid critique of the original version of this paper, and to I. E. Dzyaloshinskiĭ, L. P. Gor'kov, and L. N. Shur for discussions of the revised version.

## APPENDIX

The quantity $\langle n_i n_j \rangle$ can be evaluated by taking the average over the stationary solution. We have (see above)

$$\mathbf{Hn} = \text{const}, \quad \mathbf{\Omega}^2 = \text{const}. \tag{A.1}$$

From this, it follows that $\mathbf{n}$ rotates locally around $\mathbf{H}$ (at constant angle). The expression for $\mathbf{n}$ can therefore be written in the form

$$n_i(t) = R_{ij} n_j^0, \tag{A.2}$$

where $R_{ij}$ is the rotation matrix. Without loss of generality, we can take the $z$ axis in the direction of the field: $\hat{z} = \mathbf{H}/H$.

Next, we use the following simple fact. If $\xi_i$ is a unit vector rotating with constant velocity around $\mathbf{L}$ ($\mathbf{L}^2 = 1$), and $\boldsymbol{\xi}\mathbf{L} = 0$, then

$$\langle \xi_i \xi_j \rangle = {}^1/_2 (\delta_{ij} - L_i L_j). \tag{A.3}$$

In our case,

$$\mathbf{n} = \hat{z} \cos\theta + \mathbf{m} \sin\theta, \quad \mathbf{m}^2 = 1, \tag{A.4}$$

where $\langle \hat{z} \rangle = 1$, $\langle \mathbf{m} \rangle = 0$. From (A.3) and (A.4), we have

$$\langle n_i n_j \rangle = \hat{z}_i \hat{z}_j \cos^2\theta + \langle m_i m_j \rangle \sin^2\theta$$
$$= \delta_{i3}\delta_{j3}\cos^2\theta - {}^1/_2\delta_{i3}\delta_{j3}\sin^2\theta + {}^1/_2\delta_{ij}\sin^2\theta$$

[since $\langle m_i m_j \rangle = 1/2(\delta_{ij} - \hat{z}_i \hat{z}_j)$].

Thus,

$$\langle n_i n_j \rangle = {}^1/_2(\delta_{ij} + \delta_{i3}\delta_{j3}\cos 2\theta). \tag{A.5}$$

The difference between the eigenvalues of the matrix $\varepsilon_{ik} = \varepsilon_0 \delta_{ik} + \varepsilon_a \langle n_i n_k \rangle$ is given by

$$\Delta = {}^1/_2\varepsilon_a \cos 2\theta, \quad 0 \leqslant \theta \leqslant {}^1/_2\pi, \tag{A.6}$$

i.e., $\Delta$ varies between $-\varepsilon_a$ and $+\varepsilon_a$ as the motion takes place away from $P$ and along the stationary mode.


[1] V. L. Golo and A. A. Leman, Zh. Eksp. Teor. Fiz. **83**, 1546 (1982) [Sov. Phys. JETP **56**, 891 (1982)].
[2] I. A. Fomin, Science Rev. Phys. **3**, 275 (1981).
[3] A. J. Leggett and S. Takagi, Ann. Phys. **106**, 79 (1977).
[4] S. A. Pikin, Strukturnye prevrashcheniya v zhidkikh kristallakh (Structural Transformations in Liquid Crystals), Nauka, Moscow, 1981.
[5] Z. Guozhen, Phys. Rev. Lett. **49**, 1332 (1982).
[6] Y. Hendrikx and J. Charvolin, J. Phys. (France) **42**, 1427 (1981).
[7] M. G. Clark and F. M. Leslie, Proc. R. Soc. London Ser. A **361**, 463 (1978).
[8] P. G. De Gennes, The Physics of Liquid Crystals, Oxford University Press, 1974 [Russian translation, Mir, Moscow, 1977, p. 214].
[9] E. I. Kats, Zh. Eksp. Teor. Fiz. **65**, 324 (1973) [Sov. Phys. JETP **38**, 158 (1974)].
[10] L. M. Blinov, Elektro- i magnitooptika zhidkikh kristallov (Electro- and Magnetooptics of Liquid Crystals), Nauka, Moscow, 1978, Chap. 4.
[11] F. M. Leslie, in: Advances in Liquid Crystals, ed. by G. H. Brown, 1979, Vol. 4.


Translated by S. Chomet