Two-dimensional electron-hole system in a strong magnetic field: biexcitons and charge-density waves

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A theory of a two-dimensional electron-hole system in a strong magnetic field is developed. Three different models are considered, whose common distinguishing feature is electron-hole asymmetry. It is shown that this asymmetry influences radically the system properties. It leads to an exciton interaction that does not vanish when the momentum of the exciton relative motion tends to zero. As a result, the scattering amplitude diverges at small momenta, and magnetic biexcitons appear at arbitrarily weak asymmetry. The thermodynamics of such a system in the Hartree-Fock approximation is similar in many respects to the thermodynamics of a monopolar system. The high-temperature phase is the homogeneous one, and the low-temperature phase has the charge density waves. The homogeneous phase with exciton pairing is absolutely unstable.

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1. INTRODUCTION

Progress in the production of thin semiconductor and semimetal layers has prompted the development of a theory of quasi-two-dimensional systems. In these, the size quantization connected with the layer thickness d is so strong that the electrons e and (or) holes h occupy a small number of two-dimensional bands (one band in the quantum limit). These studies were reviewed by Ando *et al.*[1]

Among the most important topics in this field are the structure of the ground state with allowance for the e–e interaction and the phase transitions induced by changes of the temperature T, of the electron density ρ (per unit surface), and of the magnetic field $\mathfrak{H}$. The ground state of a monopolar system at low ρ and T is a Wigner crystal. Its stability region broadens when $\mathfrak{H}$ is large and is oriented along the normal ν to the layer plane xy ($\mathfrak{H}\|\nu\|z$). Such a system is quasi-zero-dimensional: the spectrum of the single-particle states in it is discrete and strongly degenerate. Under these conditions the e–e interaction is particularly substantial. Fukuyama *et al.*[2] have shown by the Hartree-Fock (HF) method that in the quantum limit (with respect to d and $\mathfrak{H}$) there exist in the low-temperature phase, in a wide range of ρ, charge-density waves (CDW) with a melting temperature $T_{\text{melt}} \sim \tilde{e}^2/\lambda(\mathfrak{H})$. Here $\lambda(\mathfrak{H}) = (c\hbar/e\mathfrak{H})^{1/2}$ is the magnetic length, $\tilde{e}^2 = e^2/\varkappa$, and $\varkappa$ is the permittivity. We introduce the degree of level filling $\nu = \rho s(\mathfrak{H})$, where $s(\mathfrak{H}) = 2\pi\lambda^2(\mathfrak{H})$ is the area per quantum state. The Landau level is completely filled at $\nu = 1$. It was shown[3] that at $\nu \ll 1$ the CDW go over into a Wigner lattice made up of "Landau circles." Owing to their infinite mass, the lattice is classical.

In a neutral bipolar system there exists, besides the e–e and h–h interactions, also an e–h attraction that can lead to exciton pairing. This complicates the problem but adds to it new interesting aspects. The simplest model contains one electron band and one hole band with an isotropic dispersion law—we call this model symmetric. It was developed by Lerner and Lozovik; for the latest results see Ref. 4. This model leads to the absence of interaction between excitons at rest in the limit $d \to 0$, $\mathfrak{H} \to \infty$. As a result, the HF approximation with exciton pairing is found to be asymptotically exact, and the exciton gas noninteracting. This made it possible to determine the Green functions and the thermodynamics of the exciton gas[4] as well as its wave function.[5]

We consider below nonsymmetric models. This means that some or all of the matrix elements of the e–e, e–h, and h–h interactions do not coincide. It is assumed also that $\mathfrak{H}$ is large and all the cyclotron frequencies $\omega_c \gg \tilde{e}^2/\lambda(\mathfrak{H})$; state mixing exists therefore within one system of Landau levels. It will be shown below that under these conditions the type of asymmetry does not influence the basic qualitative results. It is important that the complete cancellation of the interactions, which made the exciton gas practically ideal, vanishes in the nonsymmetric models. The exciton-exciton scattering amplitude now diverges when their relative velocity decreases; formation of two-dimensional magnetic biexcitons becomes possible. Calculation of the thermodynamic properties by the HF method has shown that when the temperature is lowered the homogeneous state becomes unstable and vanishes, accompanied by formation of CDW. In this respect a nonsymmetric bipolar system is similar to a monopolar one. In particular, at an asymmetry ~ 1 the scale of $T_{\text{melt}} \sim \tilde{e}^2/\lambda(\mathfrak{H})$ is preserved. The differences are connected mainly with the behavior at $\nu \ll 1$: according to the Earnshaw theorem a classical crystal cannot be made up of magnetized electrons and holes that are far from one another. Exciton formation is energywise disadvantageous. It will be shown, however, that at low T a homogeneous exciton phase is absolutely unstable in the HF approximation.

Calculations for several asymmetric models have shown that the basic qualitative picture is practically the same. It appears that the results are quite common and are valid for a large class of asymmetric models. Real systems have considerably asymmetry. One can therefore count on their behavior being correctly describable by the results of our calculations.

To be specific, we consider below three models. In all cases the system is assumed thin enough ($d \ll \lambda(\mathfrak{H})$), therefore all the film levels but the first are excluded from consideration. The electrons are assumed to have zero spin.

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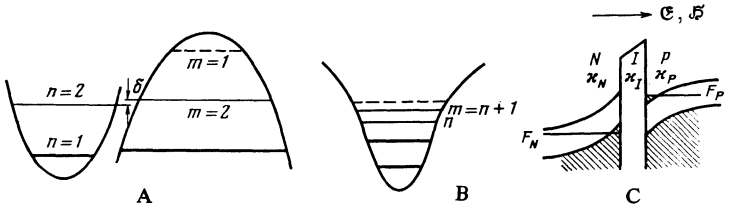

FIG. 1. A) Arrangement of levels in a semimetal plate. Thick lines—filled levels, thin—partially filled, dash-dot—empty. B) Arrangement of levels in a degenerate semiconductor. The level markings are the same as in Fig. A. C) Band scheme in *PIN* structure.

Model A. Homogeneous semimetal film with one electron valley and one hole valley. The dispersion law in the valleys is quadratic, but is isotropic in the (k_x, k_y) plane. To simplify the equations the principal axes of the ellipsoids are assumed parallel (we direct the x and y axes along them). The electron and hole effective masses m_{ex}, m_{ey} and m_{hx}, m_{hy} are different, with $\beta_e \neq \beta_h$; here $\beta_e = (m_{ex}/m_{ey})^{1/4}$ and similarly for β_h. The degree of degeneracy of the Landau levels is $S/s(\mathfrak{H})$ (S is the plate area). This degree is the same for both valleys; therefore, owing to electroneutrality, an equal number of e and h levels is filled. The interelectron interactions are significant only in the presence of partially filled levels. In model A this is possible at $\mathfrak{H}$ such that a pair of resonantly close levels is produced (Fig. 1A). The levels $n = 2$ and $m = 2$ are partially filled by electrons if their mismatch δ is comparable with T and (or) with the e–e interaction.

The asymmetry is introduced into model A by the inequality $\beta_e \neq \beta_h$.

Model B. Homogeneous semiconductor film with a dispersion law that is isotropic in the $(k_x k_y)$ plane but is nonquadratic (narrow-band semiconductor). The electron Hamiltonian is $\hat{H} = \hat{H}(\hat{p}^2)$, where $\hat{p}$ is the kinematic-momentum operator. The fact that $\hat{H}$ is not quadratic does not alter the eigenfunctions, but the energy spectrum becomes nonequidistant. The energy of the s-th level is

$$E_s(\mathfrak{H}) = H((2s+1)\hbar^2/\lambda^2(\mathfrak{H})). \tag{1}$$

It is assumed that the electrons are degenerate and that at equilibrium n lowest Landau levels are fully occupied; the remaining levels are free. By irradiation or by other means, N electrons are transferred from level n to the next level $m = n + 1$ [(m,n) exciton]—see Fig. 1B. The nonequidistance of the spectrum excludes resonance between the energies of two $(n + 1, n)$ excitons and one $(n + 2, n)$ exciton; this fact will subsequently turn out to be important. It is assumed that the time of formation of the spectrum of the interacting system (of excitons, CDW, etc.) is significantly shorter than the time to establish equilibrium between different Landau levels. A modification of this model is a spin exciton—when the spin on level m is opposite to the spin on level n.

In model B the asymmetry is produced by the inequality $n \neq m$.

Model C. Two inversion layers separated by an insulator layer—the *PIN* structure considered in Ref. 6 in connection with an analogous problem. The scheme is shown in Fig. 1C; $\varkappa_N$, $\varkappa_I$ and $\varkappa_P$ are the dielectric constants. The spectrum in the N and P regions is quadratic, isotropic, and with equal effective masses. Such a scheme is conceivable also at equilibrium, if the Fermi level F is so placed that inversion layers are produced in the P and N regions. The applied field $\mathfrak{E}$, however, permits regulation of the carrier density (and of the positions of the Fermi quasilevels F_N and F_P); at low tunnel transparency of the film I the system is quasistationary with arbitrarily large relaxation time.

In model C the asymmetry is produced by the inequality $\varkappa_N \neq \varkappa_P$ and by the fact that the e–h attraction is weakened, compared with the e–e and e–h interactions, because of the presence of a layer with $d_I \gtrsim d$, where d and d_I are the thicknesses of the inversion layers and of the insulator; the derivations become simpler at $d \ll d_I$.

2. HAMILTONIAN AND SCHRÖDINGER EQUATION

We obtain in this section the Hamiltonian and the Schrödinger equation for a system with two partially filled Landau levels, n and $m \geqslant n$; the meaning of the notation is clear from Sec. 1 and Fig. 1. We use the Landau gauge $A_y = \mathfrak{H}x, A_x = A_z = 0$; the electron charge equals hereafter $-e$. To simplify the equations, we transform to dimensionless coordinates x and y and wave vector $p_y \equiv p$:

$$x \to x\lambda(\mathfrak{H}), \quad y \to y\lambda(\mathfrak{H}), \quad p \to p\lambda^{-1}(\mathfrak{H}). \tag{2}$$

We denote the dimensions of the normalization region by $L_x = L_y \equiv L = S^{-1}$. The wave functions ψ_{sp} ($s = m,n$) are then

$$\psi_{sp}(x, y) = L^{-1/2} e^{ipy} \varphi_s(\beta(x+p)), \tag{3}$$

where φ_s are the normalized oscillator functions:

$$\varphi_s(x) = (2^s s! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_s(x). \tag{4}$$

The factor β, equal to β_e or β_h, appears in the model A (Sec. 1); in the remaining models $\beta = 1$.

The interelectron interaction operator is

$$\hat{H}_{int} = \frac{1}{2} \iint d\mathbf{r}_1 \, d\mathbf{r}_2 \, \hat{\psi}^+(\mathbf{r}_1) \hat{\psi}^+(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \hat{\psi}(\mathbf{r}_2) \hat{\psi}(\mathbf{r}_1), \tag{5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are two-dimensional vectors, and the annihilation operator is

$$\hat{\psi}(\mathbf{r}) = \sum_p [\hat{a}_{np} \psi_{np}(\mathbf{r}) + \hat{a}_{mp} \psi_{mp}(\mathbf{r})]. \tag{6}$$

The interaction potential $V(\mathbf{r})$ is understood as the effective potential averaged over z with a weight determined by the wave function of the film level. Its detailed form will not be given, and we use only the fact that $V(\mathbf{r}) = V(r)$. Substitution of (6) in (5) yields

$$\hat{H}_{int} = \hat{V}_m + \hat{V}_n + \hat{U}_{mn}, \tag{7}$$

where

$$\hat{V}_m = \frac{1}{2} \sum_{pp_1p_1'p'} V_m(pp_1p_1'p') \hat{a}_{mp}^+ \hat{a}_{mp_1} \hat{a}_{mp_1'} \hat{a}_{mp'}, \tag{8}$$

$\hat{V}_n$ is obtained from $\hat{V}_m$ by the substitution $m \to n$, and

$$\hat{U}_{mn} = \sum_{pp_1p_1'p'} U_{mn}(pp_1p_1'p')\,\hat{a}_{mp}^{+}\hat{a}_{np_1}^{+}\hat{a}_{np_1'}\hat{a}_{mp'}. \tag{9}$$

The matrix elements in (7)–(9) are defined in terms of the functions (3):

$$V_{ss_1s_1's'}(pp_1p_1'p')$$

$$= \iint d\mathbf{r}_1\, d\mathbf{r}_2 \psi_{sp}^{*}(\mathbf{r}_1)\,\psi_{s_1p_1}^{*}(\mathbf{r}_2)\,V(\mathbf{r}_1-\mathbf{r}_2)\,\psi_{s_1'p_1'}(\mathbf{r}_2)\,\psi_{s'p'}(\mathbf{r}_1), \tag{10}$$

$$V_m = V_{mmmm}, \quad V_n = V_{nnnn},$$

$$U_{mn}(pp_1p_1'p') = V_{mnnm}(pp_1p_1'p') - V_{mnmn}(pp_1p_1'p').$$

The condition $p + p_1 = p_1' + p'$, which ensures momentum conservation, is included in the definition of the summation sign. Integration in V_{mnmn}, say with respect to $\mathbf{r}_1$, includes both wave functions $\psi_{mp}(\mathbf{r}_1)$ and $\psi_{np'}(\mathbf{r}_1)$. They are greatly separated in p-space in case A and in $\mathbf{r}$-space in case C. Therefore $V_{mnmn} \approx 0$ in models A and C, while $V_{mnmn} \neq 0$ only in model B (and furthermore only for a non-spin exciton).

The matrix elements in $\hat{H}_{\text{int}}$ can be expressed in terms of the Fourier components of the potential

$$V(\mathbf{r}) = \int_{-\infty}^{\infty} \frac{d\mathbf{q}}{(2\pi)^2} V(\mathbf{q}) e^{i\mathbf{qr}}. \tag{11}$$

The calculation scheme is demonstrated in Appendix 1. The results are given in Eqs. (12)–(17)

$$V_m(pp_1p_1'p') = \frac{1}{2\pi L}\int_{-\infty}^{\infty} dq_x V_{ee}(q)\, w_m^{\,2}(\mathbf{q}_e^{\,2}) \exp\{iq_x(P-P_1)\}; \tag{12}$$

V_n is obtained from V_m by the substitutions $m \to n$, $\mathbf{q}_e^{\,2} \to \mathbf{q}_h^{\,2}$, $V_{ee} = V_{hh}$;

$$V_{mnnm}(pp_1p_1'p') = -\frac{1}{2\pi L}\int dq_x V_{eh}(q)\, w_m(\mathbf{q}_e^{\,2})\, w_n(\mathbf{q}_h^{\,2}) \exp\{iq_x(P-P_1)\}. \tag{13}$$

In these formulas

$$\mathbf{q}_e^{\,2} = \beta_e^{-2}q_x^2 + \beta_e^{\,2}q_y^2, \quad \mathbf{q}_h^{\,2} = \beta_h^{-2}q_x^2 + \beta_h^{\,2}q_y^2, \tag{14}$$

i.e., for models B and C we have $\mathbf{q}_e^{\,2} = \mathbf{q}_h^{\,2} = q^2$;

$$w_m(\mathbf{q}^2) = e^{-\mathbf{q}^2/4}L_m(\mathbf{q}^2/2), \quad P = (p+p')/2, \quad P_1 = (p_1+p_1')/2, \quad q_y = p'-p, \tag{15}$$

and L_m is a Laguerre polynomial; w_n is obtained from w_m by the substitution $m \to n$. The last matrix element (at $\beta_e = \beta_h = 1$) is equal to

$$V_{mnnm}(pp_1p'p_1') = -\frac{1}{2\pi L}\int_{-\infty}^{\infty} dq_x V_{eh}(q)\, w_{mn}(q^2) \exp\{iq_x(P_2-P')\}, \tag{16}$$

where

$$w_{mn}(q^2) = \frac{n!}{m!}\left(\frac{q^2}{2}\right)^{m-n} e^{-q^2/2}\left\{L_n^{m-n}\left(\frac{q^2}{2}\right)\right\}^2 = w_{nm}(q^2)$$

$$P_2 = (p+p_1')/2, \quad P' = (p'+p_1)/2, \quad q_y = p'-p_1, \tag{17}$$

$L_n^{\,s}$ = is an associated Laguerre polynomial.

Introduction of different potentials V_{ee}, V_{hh}, and V_{eh} for the e–e, h–h, and e–h interactions takes into account the possibility of $\varkappa_N = \varkappa_P$ in the *PIN* structure and of weakening of the V_{eh} on account of the I layer. The sign of V_{eh} is reversed, since it corresponds to attraction ($V_{eh} < 0$). Although $V(q)$ has two-dimensional Coulomb behavior at large q, $V(q) \propto q^{-1}$, we shall asume that it becomes regularized at small q (e.g., through screening), and tends to a constant value.

Using the obtained $\hat{H}_{\text{int}}$, we can write down the Schrödinger equation. If the vacuum is defined as a filled n-th level and empty p-th level, i.e., $\hat{a}_{np}^{\,+}|0\rangle = \hat{a}_{mp}|0\rangle = 0$ for all p, the wave function with N excitations takes the form

$$\Psi = \sum_{(p,p')} C(p_1, \ldots, p_N; p_1' \ldots p_N')\,\hat{a}_{mp_1}^{+} \ldots \hat{a}_{mp_N}^{+}\hat{a}_{np_1'} \ldots \hat{a}_{np_N'}|0\rangle; \tag{18}$$

the summation is over all sets of N wave numbers p_i and p_j' ($1 \leqslant i, j \leqslant N$). It is convenient to reckon the energies of the electrons in the states $\hat{a}_{mp}^{\,+}|0\rangle$ and $\hat{a}_{np}|0\rangle$ from the unperturbed values of the energy of the corresponding levels, i.e., their positions in the absence of interelectron interactions. The Schrödinger equation is then

$$\hat{H}_{int}\Psi = \mathscr{E}\Psi, \tag{19}$$

and after rather complicated algebra (see Appendix 2) it takes the form

$$\sum_{i<j}\int_{-\infty}^{+\infty} d\bar{p}_i\, d\bar{p}_j \bar{V}_m(p_ip_j\bar{p}_j\bar{p}_i) C(\ldots \bar{p}_i \ldots \bar{p}_j \ldots;\ p_1' \ldots p_N')$$

$$+ \sum_{i<j}\iint d\bar{p}_i'\, d\bar{p}_j' \bar{V}_n(\bar{p}_j'\bar{p}_i'p_i'p_j') C(p_1 \ldots p_N;\ \ldots \bar{p}_i' \ldots \bar{p}_j' \ldots)$$

$$- \sum_{ij}\iint d\bar{p}_i\, d\bar{p}_j' \bar{U}_{mn}(p_i\bar{p}_j'p_j'\bar{p}_i) C(\ldots \bar{p}_i \ldots;\ \ldots \bar{p}_j' \ldots)$$

$$= (\mathscr{E} - N\mathscr{E}_0) C(p_1 \ldots p_N;\ p_1' \ldots p_N'), \tag{20}$$

where

$$\mathscr{E}_0 = \frac{L}{2\pi}\int_{-\infty}^{\infty} dp' \{V_n(pp'pp') - V_n(pp'p'p) + U_{mn}(pp'p'p)\} \tag{21}$$

or, after transformation with the aid of (12)–(17)

$$\mathscr{E}_0 = \int_{-\infty}^{\infty} \frac{d\mathbf{q}}{(2\pi)^2}\{V_{hh}(q)\, w_n^{\,2}(\mathbf{q}_h^{\,2}) + V_{eh}(q)\, w_{mn}(q^2)\}. \tag{22}$$

Here and below the term with w_{mn} need be retained only in case B, when $V_{mnmn} \neq 0$. In connection with the transition from summation to integration, the matrix elements are redefined in terms of the quantities contained in (12), (13), and (16), in the following manner:

$$\bar{V}(pp_1p_1'p') = \frac{L}{2\pi} V(pp_1p_1'p')\,\delta(p+p_1-p_1'-p'). \tag{23}$$

We have left out of the constant $\mathscr{E}_0$ the overall energy shift due to the electron interaction on the n level when the latter is completely filled. The second term of (21) is cancelled by that part of the third term which stems from (13). The last term of (22) differs from zero only for a zero-spin electron in model B.

We shall hereafter refer to (20) as the Schrödinger equation in the Landau representation (LR). It is the starting point for the solution of the problems considered below.

3. SCHRÖDINGER EQUATION IN THE EXCITON REPRESENTATION

The next problem is the onset and interaction of excitons in a bipolar magnetized system. An exciton in a magnetic field, owing to its electroneutrality, has a momentum $\hat{\mathbf{k}}$ which is an integral of the motion and whose components $\hat{k}_x$ and $\hat{k}_y$ commute with each other and with the Hamiltonian.[7-9] The representation in which the operator k is diagonal will be called the exciton representation (ER). The connection between the LR and the ER is via a Fourier transformation.[10,11] Since we are about to investigate the scattering amplitude, a task possible in practice only in the presence of axial symmetry, we exclude model A from consideration in Secs. 3 and 4.

The transition to the ER is technically simplest if Eq. (20) is written for a system with one electron and one hole. Only the third term of the left-hand side of (2) differs then from zero, and after transformations, with account taken of (10) and (13–17) and with change of variables, Eq. (20) reduces to the form

$$\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,w_m(q^2)\,w_n(q^2)\times\exp\{-iq_x(p-p')\}C(p+q_y,p'+q_y)$$
$$-\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,w_{mn}(q_x^2+(p-p')^2)\times\exp\{-iq_xq_y\}C(p+q_y,p'+q_y)$$
$$=(\mathscr{E}-\mathscr{E}_0)C(p,p'),\qquad(24)$$

where we must put $\mathbf{q}_h=\mathbf{q}$ in (22). It can be seen from (24) that in the integral terms both arguments of the function C are shifted relative to the right-hand side by the same quantity—the integration variable q_y. This suggests the transformation used in Ref. 11, namely a transition to the sum and difference variables:

$$p+p'=2u,\quad p-p'=k_y,\quad C(p,\,p')=\tilde{C}(u,\,k_y)\qquad(25)$$

followed by a Fourier transformation with respect to u:

$$f(\mathbf{k})=\int_{-\infty}^{\infty}du\exp(-ik_xu)\tilde{C}(u,k_y).\qquad(26)$$

The transformation (26) effects the transition to the ER, and $\mathbf{k}$ is the exciton momentum. The variables k_x and k_y will be called the exciton variables.

Following the transformation (26), the function f in the left-hand side of (24) remains outside the integral sign (i.e., enters as a multiplier). Therefore (26) jointly with (20) and (22) defines the energy spectrum of the exciton:

$$E_{nm}^{ex}(k)=\mathscr{E}+\int\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,w_m(q^2)\,w_n(q^2)\times\exp(i\mathbf{qk})-\frac{1}{2\pi}V_{eh}(k)\,w_{mn}(k^2).\qquad(27)$$

The second term[12] in (27) corresponds to the same exciton-motion mechanism as that of the Wannier-Mott exciton, namely correlated band motion of the electron and hole. This motion can be obtained also in the two-particle problem. The third term, on the contrary, corresponds to the Frenkel mechanism, i.e., to virtual recombination of an e–h pair at one point and its production at another. Such a contribution exists also for the Wannier-Mott exciton and is called resonant (or exchange),[13] but in the latter it is smaller than the exciton Rydberg by the parameter $(a/a_{ex})^2$ (a is the lattice constant and a_{ex} is the exciton radius). Here we are faced with a situation in which both contributions are comparable at $k\sim 1$, for the theory has a single length λ ($\hbar$). Since virtual recombination presupposes the possibility of quantum transitions between the electron and hole levels, transitions excluded from models A and C, they have no resonant contribution.

The magnetoexciton momentum is proportional to the distance between the electron and hole circles.[9] Therefore exciton dissociation corresponds to the limit as $k\to\infty$, when the second and third terms in (27) vanish and $E_{nm}^{ex}(\infty)=\mathscr{E}_0$; in other words, $\mathscr{E}_0$ is the energy shift produced in the separated e–h pair by the interelectron interactions. The e–h interaction in the exciton is accounted for by the last two terms of (27). We note that when $n=0$, $m=0$, and $k=0$ the first two terms of (27) cancel out (this follows from the form of the Laguerre polynomials), and the third term is zero; therefore $E_{01}^{ex}(0)=0$. This means that for the quantum numbers $n=0$ and $m=1$ the energy of the exciton at rest, with allowance for all the electron interactions, coincides with the energy of an e–h pair having the same quantum numbers, but in the non-interacting system. Since exciton production in model B can be set in correspondence with cyclotron resonance, a definite analogy exists with Kohn's quasiclassical result,[14] that ω_c is independent of the e–e interaction. Our result, however is valid in the opposite limiting case of the quantum limit ($n=0$).

The next step is to express the N-exciton equation (20) in terms of exciton variables. This calls for applying the transformation (25), (26) to all N pairs of variables (p_i,p_i'). In each term of the sum, the transformation with respect to the variables that do not enter in V and U is carried out in standard manner. The transformation with respect to the remaining variables calls for more cumbersome calculations by a scheme given in Appendix 2. The final result is

$$\sum_{i<j}\int\frac{d\mathbf{q}}{(2\pi)^2}H_{ij}(\mathbf{q})f(\mathbf{k}_1\ldots\mathbf{k}_i-\mathbf{q}\ldots\mathbf{k}_j+\mathbf{q}\ldots\mathbf{k}_N)+\sum_i E_{ex}(k_i)f(\mathbf{k}_1\ldots\mathbf{k}_N)=\mathscr{E}f(\mathbf{k}_1,\ldots,\mathbf{k}_N),\qquad(28)$$

where

$$H_{ij}(\mathbf{q})=V_{ee}w_m^2(\mathbf{q}_e^2)\exp\left\{\frac{i}{2}[\mathbf{q},(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}+V_{hh}w_n^2(\mathbf{q}_h^2)\exp\left\{-\frac{i}{2}[\mathbf{q},(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}+2V_{eh}W_{mn}(\mathbf{q})\cos\left\{\frac{1}{2}[\mathbf{q},(\mathbf{k}_i+\mathbf{k}_j)]\mathbf{v}\right\},\qquad(29)$$

$$V_{eh}(q)W_{mn}(\mathbf{q})=V_{eh}(q)w_m(\mathbf{q}_e^2)w_n(q_h^2)$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\mathbf{q}_1V_{eh}(q_1)w_{mn}(q_1^2)\exp(i\mathbf{q}\mathbf{q}_1). \quad (30)$$

Here w_m, w_n, w_{mn}, $E_{ex}\equiv E_{nm}^{ex}$ and $\mathscr{E}_0$ are defined by (15), (17), (27), and (21), in Secs. 3 and 4 we have $\mathbf{q}_e^2=\mathbf{q}_h^2=q^2$. In the symmetrical model

$$w_n=w_m, \quad w_{mn}=0, \quad V_{ee}=V_{hh}=-V_{eh}\equiv V. \quad (31)$$

Since the sets of variables in C and in f are different, the Pauli antisymmetry conditions, which is standard for C, viz.,

$$C(\dots p_i\dots p_j\dots;\dots p_i'\dots p_j'\dots)$$
$$=-C(\dots p_j\dots p_i\dots;\dots p_i'\dots p_j'\dots),$$

is not trivial for f. Its final expression for each pair of arguments $\mathbf{k}_i$ and $\mathbf{k}_j$ is

$$f(\dots\mathbf{k}_i\dots\mathbf{k}_j\dots)$$
$$=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\mathbf{q}f\left(\dots\frac{\mathbf{k}_i+\mathbf{k}_j}{2}+\mathbf{q},\dots\frac{\mathbf{k}_i+\mathbf{k}_j}{2}-\mathbf{q}\dots\right)$$
$$\times\exp\left\{-\frac{i}{2}[\mathbf{q},(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}; \quad (32)$$

it is equivalent to the interchange $p_i\to p_j$ of the arguments in the function C. Corresponding to the symmetry of C with respect to the double permutation $p_i\rightleftarrows p_j$, $p_i'\rightleftarrows p_j'$ is the condition

$$f(\dots\mathbf{k}_i\dots\mathbf{k}_j\dots)=f(\dots\mathbf{k}_j\dots\mathbf{k}_i\dots). \quad (33)$$

Equations (32) and (33) were first obtained in Ref. 5, but cited there without proof; the latter is outlined in Appendix 3.

The antisymmetry equation (32) must be compatible with the Schrödinger equation (28). To this end, the total Hamiltonian $\hat{H}$ must commute with the antisymmetry operator $\hat{L}$, i.e., $[\hat{H},\hat{L}]=0$. The operator $\hat{L}$ is determined from (32): the right-hand side of (32) is, apart from the sign, $\hat{L}_f$. Since the transformations are cumbersome, we present here only the results. If $\hat{H}_1$ and $\hat{H}_2$ are the operators corresponding to the first two terms in (29), we have $[\hat{H}_1,\hat{L}]=[\hat{H}_2,\hat{L}]=0$, i.e., each of these terms is invariant. On the contrary, the last term of (29) is the reciprocal of that term of (28) which contains E_{ex}; the terms with w_nw_m and w_{mn} in (30) are then transformed into the corresponding terms in (27)—the integral term into a nonintegral and vice versa.

4. TWO-EXCITON PROBLEM—SCATTERING AMPLITUDE AND BIEXCITON

As applied to a system of two excitons with total momentum $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2=0$, $\mathbf{k}_1=-\mathbf{k}_2\equiv\mathbf{k}$, Eqs. (28) and (29) lead to

$$\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}\{V_{ee}(q)w_m^2(q^2)\exp(i[\mathbf{q}\times\mathbf{k}]\mathbf{v})$$
$$+V_{hh}(q)w_n^2(q^2)\exp(-i[\mathbf{q}\times\mathbf{k}]\mathbf{v})$$
$$+2V_{eh}(q)W_{mn}(q)\}f(\mathbf{k}-\mathbf{q})+2E_{ex}(k)f(\mathbf{k})=\mathscr{E}f(\mathbf{k}), \quad (34)$$

where $f(\mathbf{k})\equiv f(\mathbf{k},-\mathbf{k})$. The equation (34) for the biexciton contains the indices of only two Landau levels, m and n. This is a consequence of the spectrum anharmonicity (Sec. 1) that eliminates the resonance between the energies of two $(n,n+1)$ excitons and one $(n,n+2)$ exciton, a resonance that would complicate the equation.

The antisymmetry equation (32) for the function $f(\mathbf{k})$ is of the form

$$f(\mathbf{k})=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\mathbf{q}f(\mathbf{q})\exp(-i[\mathbf{q}\mathbf{k}]\mathbf{v}). \quad (35)$$

It was shown in Ref. 11 that in the symmetric model Eqs. (34) and (35) have an exact solution $f(\mathbf{k})=\delta(\mathbf{k})-(2\pi)^{-1}$.

The next problem is to determine from (34) the two-exciton scattering amplitude. We solve it for the case of slow excitons, when the system energy is $\mathscr{E}=2E_{ex}(k_0)\to 2E_{ex}(0)$ (k_0 is the momentum of the colliding excitons). It is assumed here that the absolute minimum of (27) is reached at $k_0=0$.

Having no general proof, we advance arguments for claiming that in a number of important situations this assumption is valid. We take $V_{eh}(q)$ to be of constant sign; it is then obvious that $V_{eh}(q)<0$. In models A and C, the third term of (27) is absent, and the section is a minimum at $k=0$ if $n=m$. In model B, the third term predominates at small k if $V(q)$ is regularized at $q\ll 1$ (see Sec. 3); since $w_{mn}>0$ according to (17), this term is positive and increases with k. We put therefore below $E_{ex}(0)=\min E_{ex}(k)$.

The distinguishing feature of (34) is that it is its own Fourier transform, i.e., in contrast to the usual situation this equation does not have two representations, since the coordinate and momentum representations are identical. This is what makes the problem unusual, and a nonstandard method must therefore be used to solve it.

It was noted in Sec. 3 that k is proportional to the distance between the electron and hole Landau circles.[1)] Therefore the region of large k ($k\gg 1$) corresponds to almost free particle. In this asymptotic region the first two terms in (34) are small because of the rapidly oscillating factors $\exp(\pm i\mathbf{q}\times\mathbf{k}\cdot\mathbf{v})$. Using (27), we represent the third term of (34) in the form

$$2\int\frac{d\mathbf{q}}{(2\pi)^2}\int d\mathbf{p}(E_{ex}(p)-\mathscr{E}_0)e^{i\mathbf{p}\mathbf{q}}f(\mathbf{k}-\mathbf{q}),$$

which enables us to expand the slow factor $f(\mathbf{k}-\mathbf{q})$ in terms of the small parameter q/k, since the characteristic scale q, which is determined by the preceding integration with respect to $\mathbf{p}$, is equal to $q\sim 1$. Retaining the terms $\sim(q/k)^2$ inclusive and integrating over the angles, we obtain the following expression for the third term:

$$2\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}\int_{-\infty}^{\infty}d\mathbf{p}(E_{ex}(p)-\mathscr{E}_0)e^{i\mathbf{p}\mathbf{q}}\left\{f(\mathbf{k})+\frac{q^2}{6}\Delta f(\mathbf{k})\right\}$$
$$=2(E_{ex}(0)-\mathscr{E}_0)f(\mathbf{k})-\left[\frac{d^2E_{ex}}{dp^2}\right]_{p=0}\Delta f(\mathbf{k}).$$

Substituting in (34) and using the fact that $E_{ex}(k)\to\mathscr{E}_0$ as $k\to\infty$, we get

$$\Delta f(\mathbf{k})+k_0^2f(\mathbf{k})=0, \quad (36)$$

where

$$k_0^2=M(\mathscr{E}-2E_{ex}(0)), \quad M^{-1}=[d^2E_{ex}/dk^2]_{k=0}. \tag{37}$$

Thus the exciton motion at infinity is asymptotically free and is described by the Bessel equation (36). Its solution are Bessel functions of argument k_0k.

The free motion over large distance permits the scattering problem to be formulated in the usual manner. The asymptotic forms of the incident and scattered waves

$$\psi_{k_0}(k,\varphi)=e^{ik_0k_x}+\frac{F(k_0,\varphi)}{\sqrt{k}}e^{ik_0k}, \tag{38}$$

where φ is the azimuth, F is the scattering amplitude, and k_0 is a parameter that determines the energy $E(k_0)$ of the scattered excitons. A phase-shift analysis leads to a two-dimensional analog of the Faxen-Holtsmark formula.[15] If the potential is symmetric, the angle functions are $\Phi_l=\pi^{-1/2}\cos(l\varphi)$, and the radial functions can be chosen in the form of linear combinations of the Bessel functions $J_l(kk_0)$ and $Y_l(kk_0)$. At $k\gg1$ the function $\psi_{k0}(k)$ is a linear combination of the functions

$$\Phi_l(\varphi)\frac{1}{(kk_0)^{1/2}}\mathrm{Re}\left\{\exp\left[\pm i\left(k_0k-\frac{2l+1}{4}\pi-\delta_l\right)\right]\right\}.$$

Using the expansion of the plane wave

$$\exp(ik_0k_x)=J_0(kk_0)+2\sum_{l=1}^{\infty}i^lJ_l(kk_0)\cos(l\varphi) \tag{39}$$

and canceling the converging wave, we obtain the connection between the partial amplitudes $S_l(k_0)$ and phases $\delta_l(k_0)$:

$$F(k_0)=\frac{\exp(-i\pi/4)}{(2\pi k_0)^{1/2}}(S_l-1), \quad S_l=\exp(2i\delta_l). \tag{40}$$

We shall consider below only the s-wave. In this case, averaging (38) over φ, we obtain

$$\overline{\psi}_{k_0}(k)\equiv\frac{1}{2\pi}\int_0^{2\pi}d\varphi\psi_{k_0}(k,\varphi)=J_0(kk_0)+\frac{F_0(k_0)}{\sqrt{k}}e^{ik_0k}. \tag{41}$$

Also simplified for the s wave is the condition (35) for the function $f(\mathbf{k})=f(k)$:

$$f(k)=-\int_{-\infty}^{\infty}\frac{dq}{2\pi}f(q)e^{iqk}=-\int_0^{\infty}dq\,qJ_0(qk)f(q); \tag{42}$$

we have integrated here over the angle.

We apply now the general relations (38)–(41) to the problem considered. In the spirit of scattering theory, we choose according to (41), at $k\gg1$, the incident wave to be $J_0(kk_0)$; the scattered wave is described by the Hankel function $H_0^{(1)}(kk_0)$. It will turn out later to be most important that these solutions are valid in the entire $k\gtrsim1$ region, and therefore, in the limit as $k_0\to0$, practically in the entire range of the values of the argument kk_0.

The next step is to construct, starting from J_0 and $H_0^{(1)}$, functions that satisfy the condition (42). According to formula 7.14.1(9) of Ref. 16, indefinite integrals of the two Bessel functions are obtained and expressed anew in terms of a product of two Bessel functions. This allows us to calculate the integrals

$$\int_0^{\infty}dq\,qJ_0(kq)J_0(k_0q)=2\delta(k^2-k_0^2), \tag{43}$$

$$\int_0^{\infty}dq\,qJ_0(kq)H_0^{(1)}(k_0q)=-\frac{2i/\pi}{k^2-k_0^2-i0} \tag{44}$$

and verify that the singularities in the right-hand sides of these equations stem from the region $k\gg1$, where the solutions $J_0(k_0k)$ and $H_0^{(1)}(k_0k)$ are asymptotically exact. The singularities were therefore obtained rigorously, while the corrections $\sim k_0$ appear only in the factors preceding them. Equation (42) makes it also possible to reconstruct from the right-hand sides of (43) and (44) the functions J_0 and $H_0^{(1)}$, i.e., the procedure is self-consistent. The solution of (34) can therefore be written in the form

$$f(k)=[J_0(k_0k)-2\delta(k^2-k_0^2)]+c(k_0)\left[H_0^{(1)}(k_0k)+\frac{2i/\pi}{k^2-k_0^2-i0}\right]. \tag{45}$$

It contains one unknown constant c, which is determined by the condition that (45) satisfy (34) at small $k\sim k_0$. After substituting (45) in (34) it is necessary to compare the values of the different terms. The first bracket in (45) makes a contribution ~1. The function J_0 must be substituted in (34) in the terms with single integration; since all the $w(q)$ are large only if $q\lesssim1$, we can put $k=0$ and $k_0=0$ in $J_0(k_0q)$ and in the arguments of the exponentials. The term $2\delta(k^2-k_0^2)$ must be substituted in the resonant term. The situation is similar for the second bracket in (45), except that when $H_0^{(1)}$ is substituted it can be assumed that

$$H_0^{(1)}(k_0q)\approx\frac{2i}{\pi}\ln k_0$$

and that the term obtained by substituting the last term of (45) into the resonant term of (34) also contains a large logarithmic factor. The nonintegral term can be omitted as $k\to0$, since its order of magnitude, on account of the last term in (45), is c, but we shall show presently that $c\sim|\ln k_0|^{-1}\ll1$. The principal terms yield the equation

$$G\left(1+c\frac{2i}{\pi}\ln k_0\right)=0, \tag{46}$$

where

$$G=\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}\{V_{ee}(q)w_m^2(q^2)+V_{hh}(q)w_n^2(q^2)+2V_{eh}(q)[w_m(q^2)w_n(q^2)+w_{mn}(q^2)]\}. \tag{47}$$

It follows from (46) that

$$c(k_0)\approx\frac{\pi i}{2}\ln k_0. \tag{48}$$

Substituting $c(k_0)$ in (45) and using the expansion

$$H_0^{(1)}(k_0k)\approx\left(\frac{2}{\pi k_0k}\right)^{1/2}e^{-i\pi/4}e^{ik_0k} \quad \text{at} \quad k_0k\gg1, \tag{49}$$

we obtain, from a comparison with (41), for the scattering amplitude in the limit $k_0\to0$

$$F_0(k_0)=c\left(\frac{2}{\pi k_0}\right)^{1/2}e^{-i\pi/4}\approx\left(\frac{\pi}{2k_0}\right)^{1/2}\frac{e^{i\pi/4}}{\ln k_0}. \tag{50}$$

According to (40), $F_0(k_0)$ determines the s-scattering phase shift: $\delta_0 = \pi/2 \ln k_0$. Thus, in the nonsymmetric model the amplitude and phase have a universal behavior as $k \to 0$, the amplitude increasing without limit. This result is typical of the two-dimensional problem—$\ln k_0$ in the potential scattering amplitude is replaced by $\ln k_0(a)$, where a is the radius of the potential.

Let us clarify the result of choosing the constant c of the solution (45) in the form (48). In the significant integration region in (34) we have $k \lesssim 1$, therefore $J_0(k_0 q) + cH_0^{(1)}(k_0 q) \approx 0$, as follows from the expansion at $k_0 q \ll 1$, i.e., the $f(k)$ terms that are leading at $k \sim 1$ are canceled out in the principal order.

In the symmetric model the result is qualitatively different: $F_0(k_0) \to 0$ as $k_0 \to 0$, and Eq. (48) does not hold, for $G = 0$ according to (31). The result for F_0 was obtained in Ref. 11; here $c \propto k_0^2$ and $F(k_0) \propto k_0^{3/2}$ as $k_0 \to 0$. The result is understandable also from (34), where (31) causes the integral terms to vanish as $k_0 \to 0$, and the nonintegral terms describe the free motion of these excitons.

Biexciton formation, just as exciton scattering, is determined by Eq. (34), except that now $\mathscr{E} < 2E_{ex}(0)$. The fact that the scattering could be considered only in the limit $k_0 \ll 1$ means that analytically one can find only weakly bound states with spatial scale $\alpha^{-1} \gg 1$ or with binding energy $E_B = 2E_{ex}(0) - \mathscr{E} \ll 1$. Accordingly the asymmetry inherent in the Hamiltonian should be weak (but finite) for some reason or another, i.e., $|G| \ll 1$.

To solve the problem we return to Eq. (36), but replace in it k_0^2 by $(-\alpha^2)$. Its solution is then the Macdonald function $K_0(\alpha k)$, and the use of condition (42) leads to the function [see Ref. 16, formula 7.14.2(39)]

$$f_\alpha(k) = K_0(\alpha k) - 1/(k^2+\alpha^2). \tag{51}$$

When (51) is substituted in (34), $K_0(\alpha k)$ makes the main contribution to the integral terms and can be replaced in them by $K_0(\alpha q) \approx -\ln \alpha$, since the important values correspond to $q \sim 1$. On the contrary, the second term in (51) is important in the nonintegral terms. Taking (37) into account, we obtain

$$-G \ln \alpha = (E_B + k^2/M)/(k^2+\alpha^2). \tag{52}$$

This equality holds for all $k \lesssim \alpha$ only if

$$\ln(\alpha^{-1}) = (MG)^{-1}, \quad E_B = \alpha^2/M. \tag{53}$$

Thus, at $0 < G < 1$, biexcitons are produced. This criterion can be satisfied, for example, in the model C if $n = m$ and $V_{ee} = V_{hh} \approx -V_{eh}$.

The formation of magnetic biexcitons in the case of weak binding apparently indicates that they exist in a wide range of parameters. Since the bond between them is not valent, it does not saturate and one can expect formation of clusters of several excitons (polyexcitons). Their (planar) structure is entirely different than that of needle-like magnetic polyatomic molecules.[17]

5. HAMILTONIAN OF BIPOLAR SYSTEM IN THE MAGNETIC-SUBLATTICES REPRESENTATION

We transform now the Hamiltonian system into a new basis: we replace the Landau functions by functions that correspond to the magnetic-sublattices representation (MSR).[18] This allows us to consider from a single viewpoint various pairings that occur in a bipolar system in a magnetic field, and the ensuing structures.

The transition from the LR (Sec. 2) into the MSR is by introducing the operators $\hat{T}_{\mathbf{a}_1}$ and $\hat{T}_{\mathbf{a}_2}$ ($\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the xy plane) in accord with the definition

$$\hat{T}_{\mathbf{a}_i}\psi(\mathbf{r}) = \exp(-ia_{ix}y)\psi(\mathbf{r}+\mathbf{a}_i) \tag{54}$$

($\psi(\mathbf{r})$ is an arbitrary function). The operators $\hat{T}_{\mathbf{a}_i}$ commute with the electron Hamiltonian in the LR [it can be verified that they commute with them on the complete system of functions (3)]. They commute also with one another if

$$[\mathbf{a}_1 \times \mathbf{a}_2]\boldsymbol{\nu} = 2\pi t, \quad t - \text{integer}. \tag{55}$$

Choosing $\mathbf{a}_1$ and $\mathbf{a}_2$ that satisfy the condition (55) with $t = 1$, we obtain the basis of the magnetic lattice. The reciprocal lattice vectors $\mathbf{b}_1 = \mathbf{a}_2 \times \boldsymbol{\nu}$ and $\mathbf{b}_2 = \boldsymbol{\nu} \times \mathbf{a}_1$ make up the basis of the reciprocal lattice. We choose the vectors $\mathbf{a}_i$:

$$\mathbf{a}_1 = (a_{1x}, a_{1y}), \quad \mathbf{a}_2 = (0, a_{2y}), \quad a_{1x}a_{2y} = 2\pi. \tag{56a}$$

Then

$$\mathbf{b}_1 = (b_{1x}, b_{1y}) = (a_{2y}, 0); \quad \mathbf{b}_2 = (b_{2x}, b_{2y}) = (-a_{1y}, a_{1x}). \tag{56b}$$

The new basis functions, obtained as linear combinations of the functions (3):

$$\psi_{\mathbf{k}}(\mathbf{r}) = \left(\frac{a_{1x}}{S}\right)^{1/2} \sum_l \exp\left\{iy(k_y + la_{1x}) + ik_x a_{1x} l - \frac{i}{2} a_{1x}a_{1y}l(l+1)\right\} \times \varphi_s(x - k_y - a_{1x}l), \tag{57}$$

are normalized to $[(2\pi)^2/S]\delta(\mathbf{k} - \mathbf{k}')$ and form a complete system if $\mathbf{k}$ is chosen within the limits of the cell $(\mathbf{b}_1, \mathbf{b}_2)$. The functions ψ_k satisfy in the basis (56a) the condition

$$\hat{T}_{\mathbf{a}}\psi_{\mathbf{k}}(\mathbf{r}) = \exp(i\mathbf{k}\mathbf{a})\psi_{\mathbf{k}}(\mathbf{r}).$$

The corresponding new operators

$$\hat{A}_{\mathbf{k}} = \left(\frac{a_{1x}}{L}\right)^{1/2} \sum_l \exp\left\{-ik_x a_{1x} l + \frac{i}{2} a_{1x}a_{1y}l(l+1)\right\} \hat{a}_{m\,k_y + a_{1x}l} \tag{58}$$

anticommute on $[(2\pi)^2/S]\delta(\mathbf{k} - \mathbf{k}')$. The operators $\hat{B}_k$ are obtained from the operators $\hat{a}_n$ also by Eq. (58). Equations (57) and (58) correspond to choosing the field operators in the form

$$\hat{\psi}(\mathbf{r}) = S \int \frac{d\mathbf{k}}{(2\pi)^2} \psi_{\mathbf{k}}(\mathbf{r}) \hat{A}_{\mathbf{k}}, \quad \{\hat{\psi}^+(\mathbf{r}), \hat{\psi}(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}'). \tag{59}$$

The integration here and below, if its limits are not indicated, is over the reciprocal-lattice cell.

The conversion of the Hamiltonian from the LR to the MLR entails unwieldy transformations. We present the final result, but will demonstrate the transformation scheme in Appendix 4. It can be seen from the Appendix that the transformation of $\hat{H}_{\text{int}}$ includes a transition from the quantum numbers p, which vary in the interval $(-\infty, \infty)$, into the quasimomentum $\mathbf{k}$, which varies within the limits of the reciprocal-lattice unit cell. The transformation formula for the

operator $\hat{A}_k$, as follows from (58), is of the form

$$\hat{A}_{\mathbf{k}+\mathbf{Q}}=\hat{A}_{\mathbf{k}}\exp\left\{ik_xQ_y+\frac{i'}{2}Q_xQ_y-\frac{i}{2}a_{1y}Q_y-i\pi l_1l_2\right\}, \quad (60)$$

where $\mathbf{Q}=l_1\mathbf{b}_1+l_2\mathbf{b}_2$ is an arbitrary reciprocal-lattice vector. We transform also from the operators $\hat{B}_k$ to the hole operators $\hat{C}_{\mathbf{k}}\equiv\hat{B}_{\mathbf{k}}^{+}$. Then, apart from an insignificant additive constant,

$$\begin{aligned}\hat{H}_{int}=&\frac{S^2}{2}\iint\frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{ee}(q)\,w_m^{\,2}(\mathbf{q}_e^{\,2})\\&\times\exp\{i(q_y+k_{1y}-k_{2y})q_x\}\hat{A}^{+}_{\mathbf{k}_1+\mathbf{q}}\hat{A}^{+}_{\mathbf{k}_2-\mathbf{q}}\hat{A}_{\mathbf{k}_2}\hat{A}_{\mathbf{k}_1}\\&+\frac{S^2}{2}\iint\frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{hh}(q)\,w_n^{\,2}(\mathbf{q}_h^{\,2})\\&\times\exp\{i(k_{2y}-k_{1y}-q_y)q_x\}\hat{C}^{+}_{\mathbf{k}_1+\mathbf{q}}\hat{C}^{+}_{\mathbf{k}_2-\mathbf{q}}\hat{C}_{\mathbf{k}_2}\hat{C}_{\mathbf{k}_1}\\&+S^2\iint\frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,W_{mn}(\mathbf{q})\\&\times\exp\{i(k_{1y}-k_{2y})q_x\}\hat{A}^{+}_{\mathbf{k}_1+\mathbf{q}}\hat{C}^{+}_{\mathbf{k}_2+\mathbf{q}}\hat{C}_{\mathbf{k}_2}\hat{A}_{\mathbf{k}_1}. \quad (61)\end{aligned}$$

The transformations carried out above were exact. Now, however, we change to the Hartree-Fock (HF) approximation. This is conveniently done by introducing the vacuum mean values

$$\begin{aligned}&\langle\hat{A}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}'}\rangle=\frac{(2\pi)^2}{S}\delta(\mathbf{k}-\mathbf{k}')\psi_e(\mathbf{k}),\\&\langle\hat{C}_{\mathbf{k}}^{+}\hat{C}_{\mathbf{k}'}\rangle=\frac{(2\pi)^2}{S}\delta(\mathbf{k}-\mathbf{k}')\psi_h(\mathbf{k}), \quad (62)\\&\langle\hat{A}_{\mathbf{k}}\hat{C}_{\mathbf{k}'}\rangle=\frac{(2\pi)^2}{S}\chi(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}').\end{aligned}$$

For the first two mean values we have, according to (62), $\psi^{*}_{e(h)}(\mathbf{k})=\psi_{e(h)}(\mathbf{k})$, and since their Fourier transforms (the e-CDW and h-CDW are real), we have also $U_{e(h)}(-\mathbf{k})=\psi_{e(h)}(\mathbf{k})$. Reality is ensured by choosing the operator phases. When substituting (62) in (61) it must be recognized that since the range of variation of $\mathbf{q}$ is infinite, mean values of the type $\langle\hat{A}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}+\mathbf{Q}}\rangle$ occur, and can be expressed with the aid of (60) in terms of the mean values (62). After these transformations, $\hat{H}_{\rm int}$ takes the form

$$\hat{H}_{int}=\hat{H}'_{int}+h_1+h_2, \quad (63)$$

$$\begin{aligned}\hat{H}'_{int}=&-S\int\frac{d\mathbf{k}}{(2\pi)^2}\{\varepsilon_m'(\mathbf{k})\hat{A}_{\mathbf{k}}^{+}A_{\mathbf{k}}+\varepsilon_n'(\mathbf{k})\hat{C}_{\mathbf{k}}^{+}\hat{C}_{\mathbf{k}}-\Delta(\mathbf{k})\\&\times(\hat{C}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}}^{+}+A_{\mathbf{k}}C_{\mathbf{k}})\}, \quad (64)\end{aligned}$$

$$\begin{aligned}h_1=&-\frac{S}{2}\sum_{\mathbf{Q}}\{V_{ee}(Q)\,w_m^{\,2}(\mathbf{Q}_e^{\,2})\,|\tilde{\psi}_e(\mathbf{Q})|^2\\&+V_{hh}(Q)\,w_n^{\,2}(\mathbf{Q}_h^{\,2})\,|\tilde{\psi}_h(\mathbf{Q})|^2\\&+2V_{eh}(Q)\,W_{mn}(\mathbf{Q})\tilde{\psi}_e(\mathbf{Q})\tilde{\psi}_h^{*}(\mathbf{Q})\}, \quad (65)\end{aligned}$$

$$h_2=\frac{S}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\{\varepsilon_m(\mathbf{k})\psi_e(\mathbf{k})+\varepsilon_n(\mathbf{k})\psi_h(\mathbf{k})-2\Delta(\mathbf{k})\chi(\mathbf{k})\}. \quad (66)$$

We have used here the notation

$$\tilde{\psi}_{e(h)}(\mathbf{Q})=\int\frac{d\mathbf{k}}{(2\pi)^2}\exp\{i[\mathbf{kQ}]\mathbf{v}\}\psi_{e(h)}(\mathbf{k}), \quad (67)$$

$$\varepsilon_{m(n)}(\mathbf{k})=\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{ee(hh)}(q)\,w^2_{m(n)}(\mathbf{q}^2_{e(h)})\,\psi_{e(h)}(\mathbf{k}+\mathbf{q}), \quad (68)$$

$$\begin{aligned}\varepsilon'_{m(n)}(\mathbf{k})=&\varepsilon_{m(n)}(\mathbf{k})-\sum_{\mathbf{Q}}V_{ee(hh)}(Q)\,w^2_{m(n)}(\mathbf{Q}^2_{e(h)})\\&\times\exp\{-i[\mathbf{kQ}]\mathbf{v}\}\tilde{\psi}_{e(h)}(\mathbf{Q})\\&-\sum_{\mathbf{Q}}V_{eh}(Q)\,W_{mn}(\mathbf{Q})\exp\{-i[\mathbf{kQ}]\mathbf{v}\}\tilde{\psi}_{h(e)}(\mathbf{Q}), \quad (69)\end{aligned}$$

$$\Delta(\mathbf{k})=\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,W_{mn}(\mathbf{q})\chi(\mathbf{k}+\mathbf{q}). \quad (70)$$

The Fourier components $\psi_e(\mathbf{Q})$ are connected with the electron density $\rho_e(\mathbf{r})=\langle\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$ by the equation [cf. (60)]

$$\rho_e(\mathbf{r})=\sum_{\mathbf{Q}}\tilde{\psi}_e(\mathbf{Q})\,w_m(\mathbf{Q}_e^{\,2})\exp\left\{i\pi l_1l_2-\frac{i}{2}a_{1x}a_{1y}l_2+i\mathbf{Qr}\right\}. \quad (71)$$

In (64), $\varepsilon'_m(\mathbf{k})$ and $\varepsilon'_n(\mathbf{k})$ have the meaning of the energies of the electron and hole elementary excitations, while $\Delta(\mathbf{k})$ stands for the gap; $\hat{H}'_{\rm int}$ should be diagonalized, while ψ_e, ψ_h, and χ must be self-consistent with the solution obtained. The terms $h_{1(2)}$ should be calculated for each of the solutions and taken into account when the thermodynamic functions of the different phases are compared.

It is convenient to carry out the diagonalization in a representation with a fixed chemical potential; we introduce therefore

$$\hat{H}=\hat{H}_{int}-\mu_mS\int\frac{d\mathbf{k}}{(2\pi)^2}\hat{A}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}}-\mu_nS\int\frac{d\mathbf{k}}{(2\pi)^2}\hat{C}_{\mathbf{k}}^{+}\hat{C}_{\mathbf{k}}; \quad (72)$$

μ_m and μ_n, as well as the electron and hole energies, are reckoned from the corresponding unperturbed levels (Sec. 2). Putting

$$\xi_m(\mathbf{k})=\mu_m+\varepsilon_m'(\mathbf{k}),\quad \xi_n(\mathbf{k})=\mu_n+\varepsilon_n'(\mathbf{k}), \quad (73)$$

we represent the operator part of $\hat{H}$ in the form

$$\begin{aligned}\hat{H}_{op}=&-S\int\frac{d\mathbf{k}}{(2\pi)^2}\{\xi_m(\mathbf{k})\hat{A}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}}+\xi_n(\mathbf{k})\hat{C}_{\mathbf{k}}^{+}\hat{C}_{\mathbf{k}}\\&-\Delta(\mathbf{k})(\hat{A}_{\mathbf{k}}\hat{C}_{\mathbf{k}}+\hat{C}_{\mathbf{k}}^{+}\hat{A}_{\mathbf{k}}^{+})\}.\end{aligned} \quad (74)$$

It is diagonalized by the Bogolyubov uv transformation for fermions. In terms of the new variables, $\hat{H}_{\rm op}$ takes the form

$$\hat{H}_{op}=S\int\frac{d\mathbf{k}}{(2\pi)^2}\{\varepsilon_1(\mathbf{k})\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}+\varepsilon_2(\mathbf{k})\hat{c}_{\mathbf{k}}^{+}\hat{c}_{\mathbf{k}}\}+h_3, \quad (75)$$

where $\varepsilon_1(\mathbf{k})$ and $\varepsilon_2(\mathbf{k})$ is the spectrum of the quasiparticles

$$\varepsilon_{1(2)}(\mathbf{k})=\pm\frac{1}{2}[\xi_n(\mathbf{k})-\xi_m(\mathbf{k})]-\eta(\mathbf{k}),\ \eta=\xi\left[1+\left(\frac{\Delta}{\xi}\right)^{2}\right]^{1/2},$$

$$\xi=\frac{1}{2}(\xi_n+\xi_m), \quad (76)$$

while

$$h_3=S\int\frac{d\mathbf{k}}{(2\pi)^2}(\eta(\mathbf{k})-\xi(\mathbf{k})). \quad (77)$$

The diagonalized Hamiltonian consists thus of a non-operator term

$$h=h_1+h_2+h_3 \tag{78}$$

and two operator terms in (75), corresponding to Fermi quasiparticles with average occupation numbers

$$\langle \hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}\rangle = n(\varepsilon_1(\mathbf{k})) = [1+\exp(\varepsilon_1(\mathbf{k})/T)]^{-1}, \tag{79}$$

and analogously for $\langle \hat{c}_{\mathbf{k}}^{+}\hat{c}_{\mathbf{k}}\rangle$ with the replacement $\varepsilon_1 \to \varepsilon_2$.

6. THERMODYNAMICS OF A BIPOLAR SYSTEM

We consider first the general calculation procedure. According to (76), (73), (69), and (70) the spectrum of the quasiparticles is expressed in terms of the functions $\psi_{e(h)}$ and χ. To close the system, we must express in terms of $\varepsilon_{1(2)}$ both the functions $\psi_{e(h)}$, which have according to (62) and (71) the meaning of the electron and hole distribution functions, and $\chi(\mathbf{k})$. Calculation of the mean values using the uv transformation employed on going from (74) to (75) yields

$$\psi_{e(h)} = \frac{1}{2}[1+n(\varepsilon_{1(2)})-n(\varepsilon_{2(1)})] - \frac{\xi}{2\eta}[1-n(\varepsilon_1)-n(\varepsilon_2)], \tag{80}$$

$$\chi(\mathbf{k}) = \frac{\Delta}{2\eta}[1-n(\varepsilon_1)-n(\varepsilon_2)]. \tag{81}$$

It follows from (71) that $\tilde{\psi}_{e(h)}(Q=0)$ are the mean densities of the electrons and holes, which are equal by virtue of the electroneutrality:

$$\rho = \tilde{\psi}_e(0) = \tilde{\psi}_h(0) = \nu/2\pi. \tag{82}$$

Equations (80) and (81) form jointly with (67)–(71) a closed system that makes it possible in principle, given the chemical potentials $\mu_{m(n)}$, to obtain $\psi_{e(h)}$ and χ as well as the energy spectrum, and formulate the thermodynamics (it assumed that the quasi-equilibrium assumptions made in Sec. 1 are valid). Thus, the thermodynamic potential Ω is equal to

$$\frac{\Omega}{S} = \frac{h}{S} - T\int \frac{d\mathbf{k}}{(2\pi)^2}\sum_{i=1,2} \ln\left\{1+\exp\left(-\frac{\varepsilon_i(\mathbf{k})}{T}\right)\right\}. \tag{83}$$

In the general case, however, this program is not realizable; we consider below the most important cases.

The chemical potentials $\mu_{m(n)}$ are fixed by the condition (82). For example, in a spatially homogeneous case it follows from (67), (71), and (82) that $\psi_e = \psi_h = \nu$, $\chi = \text{const}$. According to (73) and (76), these equations are satisfied at

$$\varepsilon_1 = \varepsilon_2 = \text{const}, \quad \xi_m = \xi_n = \xi = \text{const}. \tag{84}$$

We consider now particular cases.

1. Homogeneous system without pairing ($\Delta = \chi = 0$). From (84) and (82) we have

$$\xi = \mu_m + \varepsilon_m' = \mu_n + \varepsilon_n' = -T\ln(\nu^{-1}-1). \tag{85}$$

In this case [recognizing that $V_{ee}(0) = V_{hh}(0) = -V_{eh}(0)$] the potential Ω is equal to

$$2\pi\Omega/S = {}^1/_2(E_m+E_n+2E_r)\nu^2 + 2T\ln(1-\nu), \tag{86}$$

where

$$E_m = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{ee}(q)\,w_m^2(\mathbf{q}_e^2),$$
$$E_r = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,w_{mn}(\mathbf{q}^2), \tag{87}$$

and analogously for E_n. According to (68) and (69) one can express in terms of (87) also

$$\varepsilon_m' = (E_m+E_r)\nu, \quad \varepsilon_n' = (E_n+E_r)\nu, \tag{88}$$

after which it follows from (85) that:

$$\mu_m+\mu_n = -E\nu - 2T\ln(\nu^{-1}-1),$$
$$E = E_m+E_n+2E_r. \tag{89}$$

We introduce also the pressure $\mathcal{P} = -\Omega/S$ [Ω is defined by Eq. (86)].

The usual criterion for the stability of the system is

$$\left(\frac{\partial\mathcal{P}}{\partial\nu}\right)_T = \frac{2}{1-\nu}[T-T_1(\nu)] > 0, \tag{90}$$

where

$$T_1(\nu) = {}^1/_2E\nu(1-\nu) \tag{91}$$

(Fig. 2). The condition for the existence of the two phases is

$$\mathcal{P}(T_0,\nu) = \mathcal{P}(T_0,\nu'),$$
$$\mu_n(T_0,\nu)+\mu_m(T_0,\nu) = \mu_n(T_0,\nu')+\mu_m(T_0,\nu'). \tag{92}$$

The last equation corresponds to equilibrium relative to the e–h pair transitions. The solution of Eqs. (92) is of the form

$$\nu' = 1-\nu, \quad T_0(\nu) = {}^1/_4E(1-2\nu)/\ln(\nu^{-1}-1), \tag{93}$$

with $T_0(\nu) \geqslant T_1(\nu)$ (Fig. 2). The region $T_1 < T < T_0$ corresponds to a metastable high-temperature phase. We did not investigate the sign of the surface tension.

2. Homogeneous system with pairing ($\Delta \neq 0$). From (70), (81), and (84) follows an equation for the gap:

$$\frac{E_0}{2\eta}[1-2n(-\eta)] = 1, \quad E_0 = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{eh}(q)\,W_{mn}(\mathbf{q}). \tag{94}$$

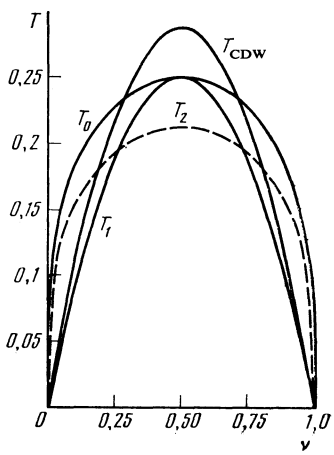

FIG. 2. Phase diagram of bipolar system $T = T(\nu)$, T in arbitrary units: T_0—line of equilibrium between two homogeneous phases, T_1—lower limit of existence of metastable homogeneous phase, T_2—solution for the exciton gap, T_{CDW}—line of absolute instability of homogeneous phase with respect to CDW formation.

At the instant when the gap vanishes we have $\eta = \xi$ [see (76)]; using (85), we obtain for the transition temperature

$$T_2(\nu) = -\frac{E_0}{2}\frac{1-2\nu}{\ln(\nu^{-1}-1)}. \quad (95)$$

The sign of the difference $T_0 - T_1$ is determined by the expression

$$E+2E_0 = E_m+E_n+2E_r+2E_0 \equiv \tilde{E}, \quad (96)$$

which in models A and C, where $E_r = 0$, is equal to

$$\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}[V_{ee}w_m^2(\mathbf{q}_e^2)+V_{hh}w_n^2(\mathbf{q}_h^2)+2V_{eh}w_m(\mathbf{q}_e^2)w_n(\mathbf{q}_h^2)].$$

In model A we have $V_{ee} = V_{hh} = -V_{eh}$ and the integral is positive. In model C we have $n = m$ and an electrostatic calculation such as in Ref. 6 yields $V_{ee}(q) + V_{hh}(q) + 2V_{eh}(q) > 0$. In case B, when $E_r \neq 0$, calculation shows that at $n = 0$, $m = 0$, and at a Coulomb interaction, expression (96) is also positive and $T_0 > T_2$. Thus, in all the cases considered $E > 0$ and $T_0 > T_2$. There remains, however, the region $T_0 < T_2 < T_1$, where in principle anomalous pairing is possible against the background of the metastable state (Fig. 2). We shall prove that such a phase is absolutely unstable. It can be shown that at $\Delta \neq 0$ we have

$$2\pi(\Omega/S) = {}^1/_2\tilde{E}\nu^2+\zeta(T), \quad (97)$$

where $\zeta(T)$ does not depend on ν. The pressure is then $\mathcal{P} = -\Omega/S$, and its derivative is $2\pi(\partial\mathcal{P}/\partial\nu)_T = -\tilde{E}\nu < 0$ on account of $E > 0$.

3. Charge density wave (at $\Delta = 0$). Using the definition of the free energy $F = \langle\hat{H}_{\text{int}}\rangle - \mathcal{S}T$, where $\mathcal{S}$ is the entropy, and Eqs. (63)–(66), we arrive at

$$F/S = -\pi\sum_{\mathbf{Q}}\{X_e(\mathbf{Q})|\tilde{\psi}_e(\mathbf{Q})|^2+X_h(\mathbf{Q})|\tilde{\psi}_h(\mathbf{Q})|^2 + Y(\mathbf{Q})[\tilde{\psi}_e(\mathbf{Q})\tilde{\psi}_h^*(\mathbf{Q})+\tilde{\psi}_e^*(\mathbf{Q})\tilde{\psi}_h(\mathbf{Q})]\}$$
$$+T\int\frac{d\mathbf{k}}{(2\pi)^2}\{(1-\psi_e(\mathbf{k}))\ln(1-\psi_e(\mathbf{k}))+\psi_e(\mathbf{k})\ln\psi_e(\mathbf{k}) + (e\to h)\}, \quad (98)$$

where

$$X_e(\mathbf{Q}) = E_m(\mathbf{Q}) - \frac{1}{2\pi}V_{ee}(Q)w_m^2(\mathbf{Q}_e^2),$$
$$Y(\mathbf{Q}) = -\frac{1}{2\pi}V_{eh}(Q)W_{mn}(\mathbf{Q}), \quad (99)$$
$$E_m(\mathbf{Q}) = \int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}V_{ee}(q)w_m^2(\mathbf{q}_e^2)\exp(i[\mathbf{qQ}]\mathbf{v})$$

and analogously for $X_h(\mathbf{Q})$ and $E_n(\mathbf{Q})$.

We consider below weak perturbations of the electron density:

$$\psi_{e(h)}(\mathbf{k}) = \nu+\psi'_{e(h)}(\mathbf{k}),$$

where $\langle\psi'_{e(h)}(\mathbf{k})\rangle = 0$.

$$\frac{F-F_0}{S} = \pi\sum_{\mathbf{Q}\neq 0}\left\{\left[\frac{T}{\nu(1-\nu)}-X_e(\mathbf{Q})\right]|\tilde{\psi}_e'(\mathbf{Q})|^2+(e\to h) - Y(\mathbf{Q})[\tilde{\psi}_e'(\mathbf{Q})\tilde{\psi}_h'^*(\mathbf{Q})+\tilde{\psi}_e'^*(\mathbf{Q})\tilde{\psi}_h'(\mathbf{Q})]\right\}, \quad (100)$$

where F_0 is the free energy of the homogeneous state. Diagonalizing this form and writing down the condition for the onset of a negative eigenvalue for the momentum $\mathbf{Q}$, we get

$$T_{\text{CDW}}(\nu, \mathbf{Q}) = \frac{\nu(1-\nu)}{2}\{X_e+X_h+[(X_e-X_h)^2+4Y^2]^{1/2}\}. \quad (101)$$

CDW are energywise favored at $T < T_{\text{CDW}}$. Let us compare T_{CDW} with T_1. Omitting in (101) the first term in the radicand, we have

$$T_{\text{CDW}}(\nu, \mathbf{Q}) > \frac{\nu(1-\nu)}{2}\{X_e+X_h+2Y\}\xrightarrow[\mathbf{Q}\to 0]{}\frac{1}{2}E\nu(1-\nu) = T_1. \quad (102)$$

It follows from (102) that in any case $T_{\text{CDW}}(\nu) > T_1(\nu)$ at small Q. It must be emphasized in this connection that since (55) quantizes only the area of the magnetic-lattice cell, one of the basis vector, say $\mathbf{b}_1 = \mathbf{Q}$, can be chosen to be arbitrarily small. The result shows that at low temperatures $T < T_{\text{CDW}}(\nu) = T_{\text{CDW}}(\nu, Q\to 0)$, the state most favored is one with density waves. Since $\psi_e(\mathbf{k}) \neq \psi_h(\mathbf{k})$, these waves are CDW. The other phases are at $T < T_{\text{CDW}}$ absolutely unstable to the onset of a CDW with an infinitely small amplitude. In a monopolar system, CDW are produced via a first-order transition at $T_{\text{CDW}}^{(1)} > T_{\text{CDW}}$ (Ref. 2); it appears that the situation is similar for a bipolar system. The ground state can be obtained only by numerical calculations for concrete models.

7. CONCLUSION

We have shown that the *e–h* asymmetry in the Hamiltonian, regardless of the detailed form of this asymmetry (spectrum, interaction, etc.) alters qualitatively the character of the behavior of a bipolar system. An interaction sets in between the magnetic excitons and can lead to the appearance of magnetic biexcitons and possibly polyexcitons. The phase diagram is just as radically altered. A state that is homogeneous at high temperatures gives way with decreasing temperature to a state with CDW. It is possible that in the region between the T_0 and T_{CDW} curves (Fig. 2) there can exist a homogeneous metastable state. A homogeneous phase with exciton pairing is absolutely unstable. Thus, the thermodynamics of a monopolar system and of an asymmetric bipolar system are similar. The scale of the transition temperature $T_{\text{CDW}} \sim E \sim \tilde{e}^2/\lambda$ ($\mathfrak{H}$) is also the same as in a monopolar system (Sec. 1).

APPENDIX 1

In the simplest case, the determination of the matrix elements reduces to calculation of the integrals

$$I_{mn}(q,\eta) = \int_{-\infty}^{\infty}d\xi e^{iq\xi-\xi^2}H_m\left(\xi+\frac{\eta}{2}\right)H_n\left(\xi-\frac{\eta}{2}\right). \quad (A.1)$$

Using the generating function of Hermite polynomials (Ref. 16, formula 10.13.19), we have

$$\sum_{m,n=0}^{\infty}I_{mn}(q,\eta)\frac{u^m v^n}{m!n!} = \sqrt{\pi}\exp\left\{-\frac{q^2}{4}+2uv+iq(u+v)+\eta(u-v)\right\}; \quad (A.2)$$

after transformation of the exponential and expansion of one of the factors in a series (we assume $m>n$), the right-hand side of (A.2) reduces to

$$\sqrt{\pi}\exp\left\{-\frac{q^2}{4}+(iq-\eta)v\right\}\sum_{m=0}^{\infty}\frac{(iq+\eta)^m}{m!}\left(1+\frac{2v}{iq+\eta}\right)^m u^m,$$

and takes, after simplification with the aid of the generating function for Laguerre polynomials (Ref. 16, formula 10.12.19), the form

$$\sqrt{\pi}\,e^{-q^2/4}\sum_{m,n=0}^{\infty}L_n^{m-n}\left(\frac{q^2+\eta^2}{2}\right)\frac{2^n}{m!}(iq+\eta)^{m-n}u^m v^n.$$

From a comparison with the initial expression (A.1) we obtain

$$I_{mn}(q,\eta)=\sqrt{\pi}\,2^n n!\,e^{-q^2/4}(iq+\eta)^{m-n}L_n^{m-n}\left(\frac{q^2+\eta^2}{2}\right).$$

APPENDIX 2

We demonstrate the transformation of the first term in (20), leaving out the superfluous variables:

$$v_m(p_ip_j;\,p_i'p_j')=\iint_{-\infty}^{\infty}d\bar{p}_i\,d\bar{p}_j\,\bar{V}_m(p_ip_j\bar{p}_j\bar{p}_i)\,C(\bar{p}_i\bar{p}_j;\,p_i'p_j'); \quad \text{(A.3)}$$

after substituting (23), (12), and (15), introducing the notation

$$\Phi(\mathbf{q}^2)=V_{ee}(q)\,w_m^2(\mathbf{q}^2)/(2\pi)^2,$$

integrating with respect to $\bar{p}_j$ and substituting $\bar{p}_i=p_i-q_y$, we get for the right-hand side of (A.3)

$$\int d\mathbf{q}\Phi(\mathbf{q}^2)\exp\{iq_x(p_i-p_j-q_y)\}C(p_i-q_y,\,p_j+q_y;\,p_i',\,p_j').$$

Replacing next C by $\tilde{C}$ in accord with (25) and changing to exciton variables for the pairs of arguments $(p_i,\,p_i')$ and p_j, p_j'), we get

$$\int dq\,\Phi(\mathbf{q}^2)\exp\{iq_x(-q_y+u_i-u_j+{}^1/_2k_{iy}-{}^1/_2k_{jy})\}$$
$$\times\tilde{C}(u_i-{}^1/_2q_y,\,k_{iy}-q_y;\,u_j+{}^1/_2q_y,\,k_{jy}+q_y).$$

The next step is to take the Fourier transform with respect to the variables u_i and u_j:

$$\iint du_i\,du_j\exp\{-i(k_{ix}u_i+k_{jx}u_j)\}v_m(p_ip_j;\,p_i'p_j').$$

After changing the order of the integration, making the substitutions $u_i=v_i+q_y/2$ and $u_j=v_j-q_y/2$, and transforming from $\tilde{C}$ to f in accord with (26), we get

$$\Phi(\mathbf{q}^2)\exp\left\{\frac{i}{2}[\mathbf{q}(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}$$

in agreement with (29).

APPENDIX 3

The antisymmetry condition for the function C:

$$C(p_ip_j;\,p_i'p_j')=-C(p_jp_i;\,p_i'p_j'),$$

rewritten for the function $\tilde{C}$ in the exciton variables (25), is of the form

$$\tilde{C}(u_ik_{iy};\,u_jk_{jy})$$
$$=-\tilde{C}\left(\frac{u_i+u_j}{2}-\frac{k_{iy}-k_{jy}}{4},\,u_j-u_i+\frac{k_{iy}+k_{jy}}{4};\right.$$
$$\left.\frac{u_i+u_j}{2}+\frac{k_{iy}-k_{jy}}{4},\,u_i-u_j+\frac{k_{iy}+k_{jy}}{4}\right).$$

Fourier transformation of the left-hand side yields directly $f(\mathbf{k}_i\cdot\mathbf{k}_j)$. To find the Fourier transform of the right-hand side, it is convenient to use first the inverse transformation (16), and introducing the notation $q_y=u_j-u_i$, $u_i+u_j=2u$, write

$$\tilde{C}(\ldots)$$
$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}ds_i\,ds_j f\left(s_i,\,q_y+\frac{k_{iy}+k_{jy}}{2};\,s_j,\,-q_y+\frac{k_{iy}+k_{jy}}{2}\right)$$
$$\times\exp\left\{iu(s_i+s_j)-i(s_i-s_j)\frac{k_{iy}-k_{jy}}{4}\right\},$$

which reduces, after the substitutions $s_i=q_x+t/2$, $s_j=-q_x+t/2$, to

$$\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}dq_x\,dt f\left(q_x+\frac{t}{2},\,q_y+\frac{k_{iy}+k_{jy}}{2};\right.$$
$$\left.-q_x+\frac{t}{2},\,-q_y+\frac{k_{iy}+k_{jy}}{2}\right)$$
$$\times\exp\left\{iut-\frac{i}{2}q_x(k_{iy}-k_{jy})\right\}.$$

Applying (26) to this expression, rewriting in terms of the variables $(u,\,q_y)$, and integrating with respect to (u,t), we get

$$\iint_{-\infty}^{\infty}du_i\,du_j\exp[-i(u_ik_{ix}+u_jk_{jx})]\tilde{C}(\ldots)$$
$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dq\iint_{-\infty}^{\infty}du\,dt f\left(q_x+\frac{t}{2},\,q_y+\frac{k_{iy}+k_{jy}}{2};\right.$$
$$\left.-q_x+\frac{t}{2},\,-q_y+\frac{k_{iy}+k_{jy}}{2}\right)$$
$$\times\exp\{iu[t-k_{ix}-k_{jx}]\}\exp\left\{-\frac{i}{2}[\mathbf{q}(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dq f\left(\frac{\mathbf{k}_i+\mathbf{k}_j}{2}+\mathbf{q},\,\frac{\mathbf{k}_i+\mathbf{k}_j}{2}-\mathbf{q}\right)$$
$$\times\exp\left\{-\frac{i}{2}[\mathbf{q}(\mathbf{k}_i-\mathbf{k}_j)]\mathbf{v}\right\}$$

in full agreement with (31).

APPENDIX 4

We write down the operator $\hat{V}_m$ [Eq. (8)] with allowance for (12); we change simultaneously from summation over the quasidiscrete values of p to integration:

$$\hat{V}_m=\frac{S}{2}\int_{-\infty}^{\infty}\frac{d\mathbf{q}}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{dp'dp_1'}{(2\pi)^2}V_{ee}(q)\,w_m^2(\mathbf{q}_e^2)$$
$$\times\exp\{iq_x(p'-p_1'+q_y)\}$$
$$\times\hat{a}^{+}_{p'+q_y}\hat{a}^{+}_{p_1'-q_y}\hat{a}_{p_1'}\hat{a}_{p'}.$$

We replace the integration with respect to p' along the entire axis by integration within the cell (b_1,b_2) and summation over the cells; we denote the variable of integration in the cell by k_{1y}:

$$\int_{-\infty}^{\infty} dp' e^{iq_x p'} \hat{a}^{+}_{p'+q_y} \hat{a}_{p'}$$

$$= \int_0^{a_{1x}} dk_{1y} \sum_{l=-\infty}^{\infty} \exp\{i(k_{1y}+a_{1x}l)q_x)\} \hat{a}^{+}_{k_{1y}+a_{1x}l+q_y} \hat{a}_{k_{1y}+a_{1x}l},$$

which reduces after an identity transformation to

$$\frac{1}{a_{2y}} \int_0^{a_{2y}} dk_{1x} \int_0^{a_{1x}} dk_{1y} \sum_{l,l'=-\infty}^{\infty} \exp\{ik_{1x}a_{1x}(l-l')\} \exp\{i(k_{1y}+a_{1x}l)q_x\}$$
$$\times \exp\left\{-\frac{i}{2} a_{1x}a_{1y}l(l+1) + \frac{i}{2} a_{1x}a_{1y}l'(l'+1)\right\} \hat{a}^{+}_{k_{1y}+a_{1x}l+q_y} \hat{a}_{k_{1y}+a_{1x}l'}.$$

This transformation, which introduces the factor $\delta_{ll'}$ under the summation sign, has made it possible to include in the integrand the factors contained in (58). It is therefore possible to "assemble" out of the operators $\hat{a}^+$ and $\hat{a}$ the operators $\hat{A}^+$ and $\hat{A}$, and the factors introduced are mutually cancelled, owing to the $\delta_{ll'}$ symbol, and do not affect the value of the integral. We ultimately get in place of the last expression

$$L \int \frac{d\mathbf{k}_1}{2\pi} \exp(ik_{1y}q_x) \hat{A}^{+}_{\mathbf{k}_1+\mathbf{q}} \hat{A}_{\mathbf{k}_1};$$

the integration with respect to $\mathbf{k}_1$ is over the reciprocal-lattice cell. After substitution in the initial expression we obtain the first term of Eq. (61).

[1)]However, $\mathbf{k}$ retains here the meaning of a momentum, as can be seen from the dispersion law (27).

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