Theory of propagation of sound and of the interaction between acoustic and electron waves in metals with complex Fermi surfaces

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An analysis is given of the role of temporal dispersion in angular anomalies of transport coefficients that are due to points (lines) of zero curvature such as parabolic points and points of flatness on the Fermi surface. The presence of such points leads to sharp frequency-angular anomalies and to the temperature dependence of electronic absorption and of the velocity of high-frequency sound in critical directions of propagation. The role of transverse and longitudinal electric fields in such cases is elucidated. The equations of the theory of elasticity and electrodynamics in the collisionless limit as ($l \to \infty$, where l is the electron mean free path) are examined under the conditions where the transport coefficients have singularities.

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1. INTRODUCTION

We shall examine effects due to the temporal dispersion of the contribution of electrons to the elastic moduli of metals.[1)]

The linear response function $\chi(\omega,\mathbf{k})$ of a metal to a stimulus proportional to $\exp[-i(\omega t - \mathbf{kr})]$ is, of course, a function of the frequency ω (temporal dispersion) and of the wave vector $\mathbf{k}$ (spatial dispersion). When the frequency is low in comparison with the Langmuir frequency ω_L and the external magnetic field is absent, a measure of the temporal dispersion is the product $\omega\tau$, where τ is the electron relaxation time, whereas spatial dispersion can be measured by kl and $k\delta$, where $l = v_F\tau$ is the electron mean free path, v_F is the Fermi velocity, and δ is a parameter describing the distribution of the electromagnetic field in the metal (its precise magnitude is determined in each special case). When sound propagation in metals is investigated, it is essential to take into account the accompanying electromagnetic field because the application of any field to a metal is accompanied by the redistribution of its electrons. The resulting electromagnetic field ensures, in particular, that the system remains electrically neutral.

Spatial dispersion is much more important than temporal dispersion during the propagation of sound waves.[2,3] This is so because $kl = (v_F/s)\omega\tau \gg \omega\tau$ (s is the velocity of sound) and, when $kl \gg 1$ (which is possible even when $\omega\tau \ll 1$), we have the transition to the collisionless situation. In particular, the attenuation of sound is determined by the resonance collisionless interaction between electrons and the sound wave (Landau damping[4]). This is why, as $kl \to \infty$, not all the Fermi electrons participate in the formation of the response of the metal: only those that move in phase with the wave do so.[3] Their position on the Fermi surface $\varepsilon(p) = \varepsilon_F$ is determined by the resonance condition $\mathbf{kv} = \omega$ ($\mathbf{p},\varepsilon,\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$ are, respectively, the quasimomentum, energy, and velocity of the electrons), and this defines a curve on the Fermi surface which is often referred to as a "belt." Moreover, if, for $kl \ll 1$, the relative absorption determined by electron viscosity[5] is such that $\Gamma/\omega \sim \omega\tau$, then, in the collisionless limit,[2)] we have[3] $\Gamma/\omega \sim s/v_F$. Thus, it would appear that $\omega\tau$ plays a very minor role and, in particular, it is immaterial whether $\omega\tau < 1$ or $\omega\tau > 1$. However, there is a series of relatively subtle effects that are sensitive to the product $\omega\tau$. These effects will, in fact, be examined in the present paper.

The point is that there are two special situations in which, for $kl \sim 1$, the transition from low-frequency viscous absorption to collisionless absorption does not take place. In particular, any Fermi surface, whatever its complexity, has points at which the curvature is zero.[8,9] These points correspond locally to flat or cylindrical areas[10] and, for $kl \gg 1$, they ensure that the collisionless viscous growth in the relative absorption with frequency will continue provided only the wave vector $\mathbf{k}$ of the sound waves is directed so that electrons in the neighborhood of these particular points interact with the sound wave. When this happens, the wave vector will be referred to as the critical wave vector and will be given the subscript c (k_c).

The experiment reported by Fil' *et al.*[11] confirmed the frequency, temperature,[3)] and angular dependence in this situation. In an earlier paper[12] it was reported that the sound absorption coefficient of copper for $\mathbf{k}\|[100]$ rose to $kl \sim 40$. Suslov[13] has shown that sound then propagates in the critical direction, and the increase in Γ corresponds to the theoretical predictions.[8–10]

The viscous increase in absorption for $kl \gg 1$ in the critical direction (for $\mathbf{k} = \mathbf{k}_c$) is limited by temporal dispersion (the value of the product $\omega\tau$) and, for sufficiently large $\omega\tau$, we find that the main contribution is due to the belt electrons located outside the points of zero curvature (absorption then reaches the usual collisionless limit), or absorption is saturated at some relatively high but finite level. The form of the function $\Gamma = \Gamma(\omega\tau, \mathbf{k} \approx \mathbf{k}_c)$ is closely connected with the local geometry of the Fermi surface and the electrical fields accompanying sound (see below).

2. TRANSPORT COEFFICIENTS AND THEIR PROPERTIES

We shall suppose that we are dealing with a Fermi surface of a general form which has lines of zero curvature and points of flatness, but does not contain finite flat, cylindrical, or conical portions (cf. Ref. 10).

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The propagation of sound in a metal is described by the equations of the theory of elasticity for the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$, the right-hand side of which contains the density of force $\mathbf{F}$ due to the action of electrons on the lattice:[14,15]

$$k^2\hat{s}_0^2\mathbf{u}-\omega^2\mathbf{u}=\mathbf{F}, \quad \mathbf{F}=\mathbf{f}+im\omega\mathbf{j}/e\rho, \tag{1}$$

where ρ is the density of the metal and $\hat{s}_0^2$ is the matrix consisting of the squares of the velocity of sound (without including the contribution of nonequilibrium electrons; in our formulation, it is the well-known matrix consisting of elements $(S_0^2)_{il}$, for given $\varkappa = \mathbf{k}/k$ i.e., given direction of propagation of the sound). The current density $\mathbf{j}$ and the deformation force $\mathbf{f}$ are given by the following expressions that relate them to the electric field $\mathbf{E}$ and the displacement vector $\mathbf{u}$:[15]

$$\begin{gathered}\mathbf{j}=e^2\langle\mathbf{v}Rv_l\rangle E_l+ek\omega\langle\mathbf{v}R\Lambda_l\rangle u_l,\\ \mathbf{f}=\frac{iek}{\rho}\langle\mathbf{\Lambda}Rv_l\rangle E_l+\frac{i\omega k^2}{\rho}\langle\mathbf{\Lambda}R\Lambda_l\rangle u_l.\end{gathered} \tag{2}$$

Here, the "electric field" $\mathbf{E}$ differs from the present electric field (cf. Ref. 15) by the vector

$$\frac{1}{e}\nabla\frac{\langle\lambda_{ik}\rangle}{\langle 1\rangle}u_{ik}+\frac{m}{e}\ddot{\mathbf{u}}.$$

Equations (1) and (2) must be augmented with the Maxwell equations. Angle brackets indicate integration over the Fermi surface:

$$\langle\ldots\rangle=\frac{2}{(2\pi\hbar)^3}\oint_{(F)}\ldots\frac{dS}{v}, \tag{3}$$

where $\mathbf{\Lambda}$ is a vector with components $\Lambda_{il}\varkappa_l$, and $\Lambda_{il} = \lambda_{il} - \langle\lambda_{il}\rangle/\langle 1\rangle$ is the renormalized deformation potential.[3] We note that $\langle 1\rangle$ is identical with the density of states at the Fermi boundary: $\langle 1\rangle = \nu_F$. All the transport coefficients in (2) that appear in front of E_l and u_l contain the quantity

$$R=[i(\mathbf{kv}-\omega)+\nu]^{-1} \tag{4}$$

under the integral sign. This quantity is the Fourier component of the Green function corresponding to the transport equation (in the τ-approximation, $\nu = 1/\tau$) for the electron distribution function.

Because of the presence of the factor R, the transport coefficients depend on ω and $\mathbf{k}$, and may exhibit singularities as $\nu\to 0$. We emphasize that Eqs. (2)-(4) are valid for real frequencies ω and wave vectors $\mathbf{k}$. Since the transport coefficients are integrals over the Fermi surface, their singularities are connected with the multiple zeros of the denominator in (4). Let $\mathbf{k}\cdot\mathbf{v} - \omega = w(\xi,\eta)$ where ξ and η are orthogonal dimensionless coordinates on the Fermi surface. The necessary condition for the existence of singularities is that the Fermi surface should have points p_c (with coordinates ξ_c, η_c), satisfying the conditions

$$\begin{gathered}w(\xi_c,\eta_c)=\mathbf{kv}_c-\omega=0,\\ \left(\frac{\partial w(\xi,\eta)}{\partial\xi}\right)_{\xi_c,\eta_c}=\left(\frac{\partial w(\xi,\eta)}{\partial\eta}\right)_{\xi_c,\eta_c}=0.\end{gathered} \tag{5}$$

In general, this condition can be satisfied only by imposing certain definite requirements on the vector $\mathbf{k}$ (the frequency ω can be conveniently looked upon as given, or a function of the wave vector that is determined by the solution of the dispersion relation). Let us first show that each point on the Fermi surface will produce a singularity. Consider an arbitrary point on the Fermi surface. The velocity at this point is $\mathbf{v}$. Suppose, further, that the direction of the wave vector $\mathbf{k}$ is the same as that of the velocity $\mathbf{v}$. When $k\approx\omega/v = k_c$, the equation for the belt is then

$$(\xi-\xi_c)^2\pm(\eta-\eta_c)^2=\Delta k/k_c, \quad \Delta k=k_c-k, \tag{6}$$

where the positive and negative signs correspond, respectively, to elliptic and hyperbolic points on the Fermi surface. When $\mathbf{k} = \mathbf{k}_c$, the belt defined by (6) changes its topology, and this gives rise to a singularity in the transport coefficients (Fig. 1). Since $v\gg s$, this singularity lies well away from the mass shell $k = \omega/s$, although it may be reflected in the acoustic properties of metals (see previous section[1,16]). For metals for which the Fermi surface does not have dents or necks, the transport coefficients have no other singularities. However, the Fermi surface of most metals is quite complex, and it has been shown[8,9] that points on the Fermi surface with zero curvature (parabolic points) are sources of angular anomalies (singularities in the direction $\varkappa$). Each parabolic point strictly defines the critical direction of the wave vector $\varkappa = \varkappa_c$, and a line of parabolic points defines a cone of critical directions. For example, surfaces of revolution with lines of parabolic points are shown in Fig. 2a. On real Fermi surfaces, the lines of parabolic points cross at points of flattening (Fig. 2b). We note that a Fermi surface of the form shown in Fig. 2b is encountered in metals belonging to the molybdenum group (hole octahedron). It will be seen below that a doubly critical direction corresponds to an enhancement of singularity.[4)] Since the Fermi surface has a center of inversion, each point p_c has its own "antipode" $\mathbf{p}_{c'}$, and $\mathbf{v}_{c'} = -\mathbf{v}_c$. There are, therefore, closely spaced cones of critical directions whose angular separation is $\approx 2\omega/kv_c$. The detuning, defined as the modulus of the complex quantity $\alpha = \delta\theta - i/kl$, characterizing the departure from the condition given by (5), plays an important role in resonance properties ($\delta\theta$ is the deviation from the resonance direction, i.e., the minimum angle in the direction perpendicular to the line on a unit sphere that corresponds to critical directions). Cones of critical directions corresponding to the antipodal points can, of course, be distinguished only when the detuning is small enough: $|\alpha|\ll\omega/kv_c$, i.e., at any rate, for $\omega\tau\gg 1$. The dependence of the structure of singularities on $\omega\tau$ is a typical example of the role of temporal dispersion.

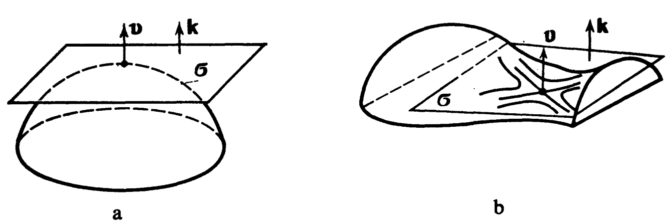

FIG. 1. Structure of critical belts for $k = \omega/v_F$ (σ is the limiting plane perpendicular to $\mathbf{k}$): (a) the critical belt has shrunk to a point at the elliptical point; (b) the critical belt has a self-crossing point at the hyperbolic point.

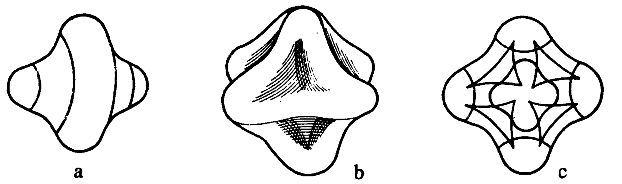

FIG. 2. Lines of parabolic points on the Fermi surface: (a) Fermi surface as a body of revolution; lines of parabolic points do not cross; (b) fragment of the Fermi surface (hole octahedron) in metals belonging to the molybdenum group; (c) crossing of lines of parabolic points on an octahedral cavity of the Fermi surface (cf. Fig. b); points of crossing produce doubly critical directions.

Some of the transport coefficients have the structure $\langle \mathbf{kv}R\Phi \rangle$, where the function Φ satisfies the condition $\langle \Phi \rangle = 0$. This is so in the case of $\langle \Lambda_{ik} R\, \mathbf{kv} \rangle$ and $\langle \mathbf{kv}R\mathbf{v}_\perp \rangle$, where $\mathbf{v}_\perp = \mathbf{v} - \varkappa(\mathbf{v}\varkappa)$ is the projection of the velocity onto the plane perpendicular to the vector $\mathbf{k}$. For these coefficients

$$\langle \mathbf{kv}R\Phi \rangle = \omega^* \langle R\Phi \rangle, \qquad \omega^* = \omega + i\nu, \tag{7}$$

and the symmetry of the function $\Phi = \Phi(\xi,\eta)$ is important for estimates of $\langle R\Phi \rangle$ (Ref. 1). We have assumed that ν is independent of ξ and η. Equation (7) signifies that the singular part of such transport coefficients contains a small factor ω ($\omega^* \to \omega$ as $\nu \to 0$). The following expression is important in the analysis of the singularity in longitudinal conductivity:

$$\langle (\mathbf{kv}) R(\mathbf{kv}) \rangle = -i\omega^* v_F + (\omega^*)^2 \langle R \rangle. \tag{8}$$

3. STRUCTURE OF TRANSPORT COEFFICIENTS FOR NEAR-CRITICAL DIRECTIONS

The singular part of any transport coefficient $\langle \chi R \rangle$, that is due to a parabolic point p_c on the Fermi surface can be written in the following form:

$$\langle \chi R \rangle_{\sin} = -\frac{2i\chi(p_c)}{(2\pi\hbar)^3 k} m_c^2 J, \qquad J = \iint\limits_{(\sin)} \frac{d\xi\, d\eta}{\alpha + \Psi_c(\xi,\eta)}, \tag{9}$$
$$\alpha = \delta\theta - i/kl, \qquad \delta\theta = \theta - \theta_c,$$

where the subscript c refers, as before, to the point p_c (we recall that the curvature is zero at this point). The coefficient m_c is of the order of the electron mass, and is defined by the equation $dS/v_c^2 = m_c^2 d\xi d\eta$. The variables ξ and η are measured from the values ξ_c, η_c [cf. (6)]. The nature of the singularity will, of course, depend on the form of the function $\Psi_c(\xi,\eta)$, which is, in fact, a dimensionless function of w [see (5)]. In the case of an ordinary parabolic point

$$\Psi_c(\xi,\eta) = \xi^2 \pm \eta^2. \tag{10}$$

The positive sign in this expression corresponds to an O-type parabolic point, whereas the negative sign corresponds to a X-type point.[8] The case of a point of flattening will be examined below. When the integral in (9) is evaluated, we must confine our attention to terms having singularities for $kl \to \infty$ and $\delta\theta = 0$ (this is indicated by the subscript "sin" under the integral). The existence of antipodal points must be taken into account in the analysis of the angular singularities.[8,9] It is clear that the structure of the angular dependence due to a pair of antipodal points is fundamentally related to the symmetry of the function $\chi(\mathbf{p})$. When $\chi(-\mathbf{p}) = \chi(\mathbf{p})$, the Re $\langle \chi R \rangle_{\text{oc}}$ add, whereas the Im$\langle \chi R \rangle_{\text{oc}}$ subtract. When $\chi(-\mathbf{p}) = -\chi(\mathbf{p})$, the opposite takes place (this occurs for $\langle \Lambda_i R v_k \rangle$). The symmetry of the Fermi surface may lead to the appearance of multiple pairs of parabolic points. Naturally, when $\langle \chi R \rangle_{\text{oc}}$ is evaluated, the contributions of all the pairs must be summed, and the transformation properties of $\chi(\mathbf{p})$ between one pair and another must be taken into account.

It is clear from the foregoing that, when $\delta\theta = 0$, the expression given by (9) does not allow a limiting transition to the collisionless situation ($l \to \infty$). In fact, it diverges, and the nature of the divergence is determined by the function $\Psi_c(\xi,\eta)$. When p_c is an ordinary parabolic point and $\operatorname{Im}\chi = 0$, the divergence is logarithmic: Re $\langle \chi R \rangle$ diverges at an X-type point, whereas Im $\langle \chi R \rangle$ has a finite jump. The opposite occurs at an O-type point [cf. (10)].

The existence of a point of flattening ensures that the expansion of $\Psi_c(\xi,\eta)$ begins with a cubic term in at least one of the variables for a particular direction of the wave vector (doubly critical direction). It may be shown[1] that near doubly critical directions

$$\Psi_c(\xi,\eta) = \delta\varphi\xi^2 + \gamma\xi^3 + \eta^2, \qquad |\delta\varphi| \ll 1, \quad \gamma \sim 1 \tag{11}$$

where the angles φ and θ define the unit vector $\varkappa$ and $\delta\varphi = \varphi - \varphi_c$, where θ_c, φ_c, is the doubly critical direction. The existence of three small parameters ($\delta\varphi$, $\delta\theta$, $1/kl$) complicates the structure of the singularity. On the ($\delta\varphi$, $\delta\theta$) plane, we have the line

$$\delta\theta = (4/27\gamma^2)(\delta\varphi)^3, \tag{12}$$

and the belt changes its shape when this line is crossed. When $|\delta\theta - 4(\delta\varphi)^3/27\gamma^2| \ll \delta\varphi, 1/kl$, the denominator of the integrand in (9) can be given the form

$$\eta^2 - \delta\varphi(\xi - \xi_0)^2 - \left(\delta\theta - \frac{4(\delta\varphi)^3}{27\gamma^2}\right) - \frac{i}{kl},$$
$$\xi_0 = -\frac{2\delta\varphi}{3\gamma}, \qquad |\xi - \xi_0| \ll |\xi_0|, \tag{13}$$

from which it is clear that, when $\delta\varphi < 0$ and $kl \to \infty$, the crossing of the line given by (12) is accompanied by an O-type singularity, whereas, for $\delta\varphi > 0$ and $kl \to \infty$, the singularity is of the X-type. When

$$\delta\theta \to 4(\delta\varphi)^3/27\gamma^2, \qquad |\delta\varphi| \gg (kl)^{-1} \gg |\delta\theta - 4(\delta\varphi)^3/27\gamma^2|$$

we have the estimate

$$|J| \sim \frac{1}{(|\delta\varphi|)^{1/2}} \ln kl. \tag{14}$$

The approximation given by (13) improves with increasing distance from the point $\delta\varphi = \delta\theta = 0$ (although $|\delta\varphi|, |\delta\theta| \ll 1$). Exact directional resonance requires separate analysis. When $\delta\varphi = \delta\theta = 0$, Eqs. (9) and (11) yield

$$J = \frac{\pi}{\gamma^{1/2}} (\gamma kl)^{1/6} 2^{1/2} (1+i) \int_1^\infty \frac{dx}{(x^3-1)^{1/2}} \simeq \frac{10{,}1}{\gamma^{1/2}} (\gamma kl)^{1/6} (1+i). \tag{15}$$

In the general case, when

$$\Psi(\xi,\eta) = \delta\varphi\xi^2 + \gamma\xi^n + \eta^2,$$

the asymptotic behavior of J for $\delta\varphi = 0$ is $J \sim (kl)^{1/2 - 1/n}$.

Well away from the line given by (12), the term $\gamma\xi^3$ (or $\gamma\xi^n$) can be neglected, and the nature of the singularity in $\varphi\theta$

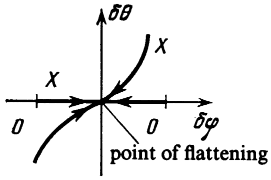

FIG. 3. Portion of the ($\delta\theta$, $\delta\varphi$) plane. The transport coefficients have singularities on the thick line. X- and O-type singularities are indicated.

for $kl\to\infty$ depends on the sign of $\delta\varphi$: when $\delta\varphi>0$, we have an O-type singularity and, when $\delta\varphi<0$, the singularity is of the X type. Since we are then close to the critical direction in φ, the singular part is anomalously high according to (11):

$$J(\delta\varphi)\approx\frac{1}{(|\delta\varphi|)^{1/2}}\iint\frac{d\xi\,d\eta}{\eta^2\pm\xi^2+\alpha},\qquad 1\gg|\delta\varphi|\gg\frac{1}{kl}. \tag{16}$$

Figure 3 shows a portion of the ($\delta\varphi$, $\delta\theta$ plane and the lines of singular points. As can be seen, the presence of the points of flattening (crossing of lines of parabolic points) results in an enhancement of the singularities in the transport coefficients.

4. FIELD RENORMALIZATION

We shall now use the Maxwell equations

$$k^2\left(\mathbf{E}-\frac{m\omega^2}{e}\mathbf{u}\right)-\mathbf{k}\left(\mathbf{kE}-\frac{m\omega^2}{e}\mathbf{ku}\right)=\frac{4\pi i\omega}{c^2}\mathbf{j}, \tag{17}$$

to eliminate the electric field $\mathbf{E}$ from the expression for the deformation force $\mathbf{f}$. We shall divide the procedure into two stages. We begin by eliminating the longitudinal field $\mathbf{E}_\parallel=\varkappa(\varkappa\mathbf{E})$ from $\mathbf{f}$ and from the transverse current $\mathbf{j}_\perp$. Using the condition $\mathbf{j}\varkappa=0$, we have

$$\mathbf{E}_\parallel=-\mathbf{k}\left\{\frac{\langle(\mathbf{vk})R(\mathbf{v}_\perp\mathbf{E}_\perp)\rangle}{\langle(\mathbf{vk})^2R\rangle}+\frac{k\omega}{e}\frac{\langle(\mathbf{vk})R(\Lambda\mathbf{u})\rangle}{\langle(\mathbf{vk})^2R\rangle}\right\}. \tag{18}$$

Substituting this in (2), we obtain[15]

$$\begin{aligned}\mathbf{j}_\perp={}&e^2\left\{\langle\mathbf{v}_\perp Rv_\alpha\rangle-\frac{\langle\mathbf{v}_\perp R(\mathbf{kv})\rangle\langle(\mathbf{kv})Rv_\alpha\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right\}E_\alpha\\&+ek\omega\left\{\langle\mathbf{v}_\perp R\Lambda_l\rangle-\frac{\langle\mathbf{v}_\perp R(\mathbf{kv})\rangle\langle(\mathbf{kv})R\Lambda_l\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right\}u_l,\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{f}={}&\frac{i\omega k^2}{\rho}\left\{\langle\Lambda R\Lambda_l\rangle-\frac{\langle\Lambda R(\mathbf{kv})\rangle\langle(\mathbf{kv})R\Lambda_l\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right\}u_l\\&+\frac{iek}{\rho}\left\{\langle\Lambda Rv_\alpha\rangle-\frac{\langle\Lambda R(\mathbf{kv})\rangle\langle(\mathbf{kv})Rv_\alpha\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right\}E_\alpha.\end{aligned} \tag{20}$$

The subscript l labels the three components of the vectors, and the subscript α labels two components (on the plane perpendicular to the wave vector $\mathbf{k}$).

The structure of the expressions in braces in (19) and (20) shows that they will not become infinite as $kl\to\infty$ even when $\varkappa$ coincides with one of the critical directions $\varkappa_c$. In fact, when $\varkappa=\varkappa_c$, the singular part of any of the coefficients $\langle URV\rangle$, (U, V are arbitrary functions on the Fermi surface) will be equal to $U(\mathbf{p}_c)V(\mathbf{p}_c)\langle R\rangle$, and the singular terms in (19) and (20) will cancel out. The above statement ensues from this. An analogous phenomenon takes place in the tilt effect.[19] The cancelation of singularities is the result of the fact that one point on the Fermi surface ($\mathbf{p}=\mathbf{p}_c$) provides a contribution to each singularity since, otherwise (several nonequivalent points, flat segment), the infinities will not, in general, cancel out [see (19) and 20)].

The cancelation of divergent terms does not mean that the singularities in the transport coefficients vanish. In fact, the singularities remain in the derivatives with respect to $\mathbf{k}$ and ω.

As sound propagates, the renormalization due exclusively to the longitudinal field is practically unimportant because the singular part of the longitudinal conductivity (8) contains the factor ω^2, so that the second terms in the curly brackets are always small.[5)] In fact, they are small if $|J|\ll kv_F/\omega$ [see (8) and (9)], i.e.,

$$\begin{aligned}&\omega\tau\ll(s/v_F)\exp(v_F/s)\ \text{for parabolic point}\\&\omega\tau\ll(v_F/s)^5\ \text{for a point of flattening.}\end{aligned} \tag{21}$$

It is clear that these conditions impose practically no restriction on $\omega\tau$ and, even if we ignore the fact that inequalities opposite to those given in (21) cannot be satisfied, as we shall see below, a self-consistent solution of the dispersion relation (allowance for resonance) leads to a "self-limitation" of the magnitude of attenuation to the effect of renormalization due exclusively to the longitudinal field.[10,12]

We shall therefore neglect renormalization due to the longitudinal field alone. The neglect of the second term in (19) and (20) produces the simpler expressions

$$\mathbf{j}_\perp=e^2\langle\mathbf{v}_\perp Rv_\alpha\rangle E_\alpha+ek\omega\langle\mathbf{v}_\perp R\Lambda_l\rangle u_l, \tag{22}$$

$$\mathbf{f}=\frac{ik^2\omega}{\rho}\langle\Lambda R\Lambda_l\rangle u_l+\frac{iek}{\rho}\langle\Lambda Rv_\alpha\rangle E_\alpha. \tag{23}$$

To eliminate the transverse field $\mathbf{E}_\perp$ from (23), we must use the Maxwell equation (17). Omitting the Tolman-Stewart terms, we have

$$k^2\mathbf{E}_\perp=\frac{4\pi i\omega}{c^2}\mathbf{j}_\perp. \tag{24}$$

It is convenient to take axes 1 and 2 (3$\|\mathbf{k}$), so that the off-diagonal component of the conductivity tensor $\sigma_{12}=e^2\langle v_1RV_2\rangle$ is equal to zero. For $kl\to\infty$ and $\varkappa=\varkappa_c$, this requires that one of the axes (to be specific, axis 2) must be directed at right-angles to the plane containing the vectors $\varkappa_c$ and $\mathbf{v}_c$ We then have[15]

$$E_\alpha=\frac{4\pi i\omega^2ek}{c^2}\frac{\langle v_\alpha R\Lambda_l\rangle u_l}{k^2-(4\pi i\omega e^2/c^2)\langle v_\alpha{}^2R\rangle},\qquad\alpha=1,2, \tag{25}$$

and

$$\mathbf{f}=\frac{ik^2\omega}{\rho}\left\{\langle\Lambda R\Lambda_l\rangle+\frac{4\pi i\omega e^2}{c^2}\times\sum_{\alpha=1}^{2}\frac{\langle\Lambda Rv_\alpha\rangle\langle v_\alpha R\Lambda_l\rangle}{k^2-(4\pi i\omega e^2/c^2)\langle v_\alpha{}^2R\rangle}\right\}u_l. \tag{26}$$

It is clear that these expressions do not diverge as $l\to\infty$ (even for $\varkappa=\varkappa_c$). In fact, as $l\to\infty$, only one term in the sum over α (the one with $\alpha=1, v_2{}^c=0$) is found to diverge, but its limiting value is equal to minus the first term in braces in (26), and this ensures the possibilitiy of a transition to the "collisionless" limit.

It is important to note that this limiting transition essentially corresponds to the intermediate asymptotic behavior $v_F/s \gg |J| \gg (k\delta_L)^2 v_F/s$, where $\delta_L = c/\omega_L$, because (26) is valid when the conditions in (21) are satisfied. It is precisely for this reason that the designation "collisionless" has been placed in quotation marks (see, however, Sec. 6).

Renormalization has produced a substantial complication of the dependence of the transport coefficient on the wave vector. It is essential to take into account the relationship between k^2 and

$$(4\pi e^2\omega/c^2)\,|\langle v_\alpha^2 R\rangle|\sim s|J|/\delta_L^2 v_F$$

(for $\varkappa \neq \varkappa_c$, the factor $|J|$ must be replaced with unity).

When $\omega \gg (s/\delta_L[(s/v_F)|J|]^{1/2}$, the "field terms" can be neglected and we obtain the following expression for the force:[8,9]

$$\mathbf{f}=\frac{i\omega k^2}{\rho}\langle \Lambda R \Lambda_l\rangle u_l. \tag{27}$$

The necessary condition for this expression to be valid for $\varkappa \approx \varkappa_c$ is that the following two inequalities be satisfied:

$$|J|\ll v_F/s, \qquad |J|\ll \omega^2 v_F \delta_L^2/s^3. \tag{28}$$

When $\omega < s/\delta_L \sim 10^{10}\ \mathrm{s}^{-1}$, the second condition is stronger (if it does not vanish by symmetry) whereas, for $\omega > s/\delta_L$, the first condition, which is identical with (21), is the stronger.

5. RENORMALIZATION OF THE VELOCITY OF SOUND. ATTENUATION COEFFICIENT

When the detuning from resonance is large enough, we can use perturbation theory to evaluate the renormalized velocities of sound. As a rule, for a nonsymmetric direction of the vector $\mathbf{k}$, the matrix of the squares of velocities $\hat{s}_0^2$ is nondegenerate, and the correction to each of the three velocites s_j ($j = 1,2,3$) can be calculated separately. In the opposite case, degeneracy can be taken into account in a standard fashion. Projecting the force given by (27) in the direction of the j-th eigenvector $\mathbf{e}_j u_j$ of the unperturbed problem ($\mathbf{e}_j$ is the polarization unit vector), we obtain the following expression from (1):

$$\Delta s_j=-\frac{2}{\rho(2\pi\hbar)^3}\oint_{(F)}\frac{dS}{v^2}\frac{(\Lambda \mathbf{e}_j)^2}{\varkappa\mathbf{n}-s_j/v-i/kl}$$
$$=-\frac{2}{\rho(2\pi\hbar)^3}\left\{i\pi\oint_{(F)}(\Lambda\mathbf{e}_j)^2\delta(\varkappa\mathbf{n}-s_j/v)\frac{dS}{v^2}+\fint_{(F)}\frac{(\Lambda\mathbf{e}_j)^2}{\varkappa\mathbf{n}-s_j/v}\frac{dS}{v^2}\right\}. \tag{29}$$

This is valid for relatively large detuning (cf. Ref. 8). When the detuning is small, the presence of the resonance denominator in the integrand of (27) requires special examination. When the velocities of sound corresponding to different polarizations are appreciably different from one another, the resonance interaction involves only one branch and, in the resonance approximation, the dispersion relation governing the renormalized velocity of sound $s = \omega/k$, assumes the form[10,20]

$$x-x_0=\frac{2m_c^2(\mathbf{e}_j\Lambda_c)^2}{(2\pi\hbar)^3\rho v_c}\iint_{(\sin)}\frac{d\xi\,d\eta}{x-x_0+\Psi_c(\xi,\eta)-i/kl}, \tag{30}$$

where $x = s/v_c$, $x_0 = s_j/v_c = \cos\theta_c$, and the subscript on the integral signs has the same significance as before [see (9)]. The form of the function $\Psi_c(\xi,\eta)$, which determines the nature of the resonance, depends on the nature of the point p_c [see (10) and (11)]. It is assumed in (30) that $\varkappa = \varkappa_c$ or $\theta = \theta_c$, and that the detuning from resonance is determined only by the mean free path.

For an X-type point

$$x-x_0=\frac{2i\pi m_c^2(\Lambda_c\mathbf{e}_j)^2}{(2\pi\hbar)^3\rho v_c}\ln\left(x-x_0-\frac{i}{kl}\right). \tag{31}$$

The coefficient in front of the logarithm is of the order of $(s/v_F)^2$. As $kl\to\infty$, we have with logarithmic precision

$$\mathrm{Im}\,x\approx-\frac{2\pi}{(2\pi\hbar)^3}\frac{m_c^2(\Lambda_c\mathbf{e}_j)^2}{\rho v_c}\ln\frac{v_F}{s_j} \tag{32}$$

and

$$\frac{\Gamma}{\omega}\sim\frac{s_j}{v_F}\ln\frac{v_F}{s_j}, \tag{33}$$

i.e., the resonance has resulted in an increase in the attenuation coefficient by a factor of $\ln(v_F/s)$. Equation (32) is valid if $1/kl \ll |\mathrm{Im} x|$, i.e., if

$$\omega\tau \gg (v_F/s_j)\,[\ln(v_F/s_j)]^{-1}. \tag{34}$$

In view of the foregoing, this condition is less stringent than (21).

In the case of an O-type point, the large logarithmic factor at resonance (as $kl\to\infty$) contains only

$$\mathrm{Re}\frac{\Delta s}{s_j}\approx\frac{s_j}{v_F}\ln\frac{v_F}{s_j}, \qquad \Delta s=s-s_j, \tag{35}$$

and the singular part of the attenuation coefficient due to the appearance of the O-type belt for $\varkappa = \varkappa_c$ is nonzero only because of collisions, and $\Gamma/\omega \sim (\omega\tau)^{-1}$ for $\omega\tau \gg v_F/s$. We thus see that the resonance interaction may result in a fall in attenuation.[6)] The fall in Γ/ω with increasing $\omega\tau$ does not involve the nonresonance part of the absorption coefficient, which reaches its usual level as $\omega\tau$ increases.

For a point of flattening [see (30), (11), and (15)], an analogous analysis leads to the restriction of the absorption coefficient and velocity to

$$\left(\frac{\Gamma}{\omega}\right)_{max},\quad \left(\frac{\Delta s}{s_j}\right)_{max}\sim\frac{s_j}{v_F}\left(\frac{v_F}{s_j}\right)^{2/7}, \tag{36}$$

which occurs for $\omega\tau \gg (v_F/s_j)^{5/7}$.

Strictly speaking, the conditions given by (28) must be satisfied for (33)–(36) to be valid. When these conditions are not satisfied, we must use (26) rather than (27) for the force, so that the resonance terms are annulled. It is important to remember, however, that the terms in the expression for the force that are due to the transverse field may vanish identically, for example, by symmetry, as in the case of sound propagation in an "easy" direction. One should then be definitely able to observe the above self-limitation due to the resonance interaction between electrons and the sound wave (see also the remarks introduced in the Conclusion).

Let us now consider long-wave sound for which

$$k^2\ll\frac{4\pi\omega e^2}{c^2}|\langle v_\alpha^2 R\rangle|, \tag{37}$$

and the force **f** is given (for all $\varkappa$) by the following expression that is simpler than (26):

$$\mathbf{f}=\frac{ik^2\omega}{\rho}\left\{\langle\Lambda R\Lambda_l\rangle-\sum_{\alpha=1}^{2}\frac{\langle\Lambda Rv_\alpha\rangle\langle v_\alpha R\Lambda_l\rangle}{\langle v_\alpha^2R\rangle}\right\}u_l. \qquad (38)$$

Each of the transport coefficients in (38) has a complex angular dependence due to the local structure of the Fermi surface. The formula can be used to calculate the dispersion of sound and the absorption coefficient for a specific Fermi surface and specific directions of the wave vector **k**. We now make only one observation. When the equations given by (1) are solved for any correct expression for the force **f**, this should, of course, yield a complex frequency $\omega=\omega'+i\omega''$ corresponding to the attenuation of the wave (for our chosen dependence on time, $\omega''<0$). This is not at all easy to demonstrate in the general case. If we use perturbation theory [this is valid because the resonant terms in (38) cancel out], we can readily show that in the nondegenerate case,

$$\omega_j''=-\frac{\pi k}{\rho(2\pi\hbar)^3}\left\{\oint_{(F)}(\Lambda\mathbf{e}_j)^2\delta\left(\varkappa\mathbf{n}-\frac{s_j}{v}\right)\frac{dS}{v^2}\right.$$
$$-\sum_{\alpha=1}^{2}\left[\oint_{(F)}(\Lambda\mathbf{e}_j)v_\alpha\delta\left(\varkappa\mathbf{n}-\frac{s_j}{v}\right)\frac{dS}{v^2}\right]^2 \qquad (39)$$
$$\left.\times\left[\oint_{(F)}v_\alpha^2\delta\left(\varkappa\mathbf{n}-\frac{s_j}{v}\right)\frac{dS}{v^2}\right]^{-1}\right\}.$$

Hence, it is clear that $\omega''<0$, as should be the case. The first term in braces is always greater than the sum of the other two because the first integral may be looked upon as the square of the modulus of the function $(\Lambda\mathbf{e}_j)$ specified on the belt, and the second and third may be regarded as the squares of projections along the "unit vectors"

$$\xi_\alpha=v_\alpha\left[\oint_{(F)}v_\alpha^2\delta\left(\varkappa\mathbf{n}-\frac{s_j}{v}\right)\frac{dS}{v^2}\right]^{-1/2}.$$

The orthogonality of ξ_1 and ξ_2 follows from the fact that σ_{12} is zero (see below). Comparison of (39) and (29) will show that, as k increases, the attenuation coefficient will increase because of Joule losses.

6. COLLISIONLESS LIMIT

When $\varkappa\neq\varkappa_c$, the formulas describing the collisionless situation ($l\to\infty$), can be obtained by substituting $\nu=+0$, and do not require separate analysis. We note that the attenuation coefficient does not then vanish or become infinite. As we have already pointed out, the attenuation is determined by collisionless Landau damping, and the solution of the dispersion relation obtained by equating to zero the determinant of the system defined by (1)–(4) and (17) shows that $\omega''\sim-(s/v_F)\omega'$ and $|\omega''|\ll\omega$. Since (2)–(4) are valid for real **k** and ω, the solution of the dispersion equation requires that the functions that appear in it and are given in the form of integrals over the Fermi surface [see (3) and (4)] must be analytically continued into the lower half-plane of the complex variable $\omega=\omega'+i\omega''$. The usual procedure is as follows. The first step is to perform the limiting substitution $\nu\to+0$, and to evaluate the corresponding integrals. The dispersion equation is then solved, assuming that the values of the coefficients in this equation are given by their analytic continuation into the lower half-plane. Analysis shows that, for $\varkappa\neq\varkappa_c$ and $|\omega''|\ll\omega$, this procedure does not lead to error. We shall now demonstrate this by considering a simple example. If we take the force in the form given by (27), and introduce the simplifying assumption that the electron spectrum is isotropic, we find that the integral defining the imaginary part of the force assumes the form

$$I(x)=\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-x-i0},\quad y=\varkappa\mathbf{n};\quad x=\frac{\omega}{\mathbf{k}\mathbf{v}}<1; \qquad (40)$$
$$\operatorname{Im}\varphi(y)=0.$$

If ω is real, we have

$$\operatorname{Im}I(x)=i\pi\varphi(x). \qquad (41)$$

The analytic continuation of the function $I(x)$ to the lower half-plane of the complex quantity $x=\omega/\mathbf{kv}$ requries a deformation of the contour in y (see Fig. 4), and this gives

$$I(x)=\int_{-1}^{1}\frac{\varphi(y)\,dy}{y-x}+2\pi i\varphi(x),\quad \operatorname{Im}x<0.$$

When $x''\ll x'$, this again yields (41).

Let us now consider the limiting transition to infinite mean free path for $\varkappa=\varkappa_c$. Each of the integrals in (19) and (20) will diverge as $\varkappa\to\varkappa_c$ and $l\to\infty$, and the expressions for the conductivities and elastic moduli will tend to the finite limits

$$\mathbf{j}_\perp=\left\{-\frac{ie^2\nu_F}{\omega}\mathbf{v}_\perp^c v_\alpha^c+e^2\langle(\mathbf{v}_\perp-\mathbf{v}_\perp^c)R(v_\alpha-v_\alpha^c)\rangle\right\}E_\alpha$$
$$+ek\omega\left\{-\frac{i\nu_F}{\omega}\mathbf{v}_\perp^c\Lambda_l^c+\langle(\mathbf{v}_\perp-\mathbf{v}_\perp^c)R(\Lambda_l-\Lambda_l^c)\rangle\right\}u_l, \qquad (42)$$
$$\mathbf{f}=\frac{\omega k^2}{\rho}\left\{\frac{\nu_F}{\omega}\Lambda^c\Lambda_l^c+i\langle(\Lambda-\Lambda^c)R(\Lambda_l-\Lambda_l^c)\rangle\right\}u_l$$
$$+ek\left\{\frac{\nu_F}{\omega}\Lambda^c v_\alpha^c+i\langle(\Lambda-\Lambda^c)R(v_\alpha-v_\alpha^c)\rangle\right\}E_\alpha. \qquad (43)$$

As before, the subscript α labels the components perpendicular to the wave vector **k** and the subscript l labels all three components. The letter c labels quantities taken at points on the Fermi surface that correspond to zero curvature (for $\mathbf{p}=\mathbf{p}_c$). The integrals in (42) and (43) are evaluated over the entire Fermi surface. They have the usual (as for $\varkappa\neq\varkappa_c$) order of magnitude, since the numerators of all the integrands are $\infty(\mathbf{p}-\mathbf{p}_c)^2$ and vanish for $\mathbf{p}=\mathbf{p}_c$. Comparison of the second terms in the curly brackets with the first shows that the former (integral) terms are smaller by a factor of $\mathbf{kv}_F/\omega$ than the latter. However, they are still quite important. The

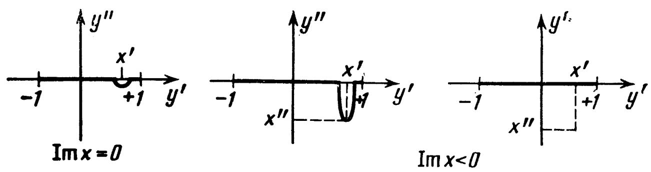

FIG. 4. Contour of integration in the analytic continuation of (40). On the last figure on the right, the point (x', x'') belongs to the contour.

point is that (42) and (43) are valid only for $\operatorname{Im}\omega = 0$. If, on the other hand, they are used to solve the dispersion equation and to calculate the frequency renormalization, it is readily verified that the integral terms lead to the appearance of attenuation, and $|\operatorname{Im}\omega| \sim \omega s/v_F \neq 0$. Rigorous analytic continuation to the lower half-plane of $\omega = \omega' + i\omega''$ for functions of the form of (9), which appear in (19) and (20) and diverge for $\omega'' = 0$, shows that[7] these functions are finite for $J \sim \omega'' \neq 0$: when p_c is an ordinary parabolic point, we have $J \sim \ln|\omega'/\omega''| \sim \ln(v_F/s)$, and when p_c is a point of flattening, we have $J \sim |\omega'/\omega''|^{1/6} \sim (v_F/s)^{1/6}$ [cf. (14) and (15)]. Moreover, it is readily seen from (8) that it is impossible to "overcome" the small factor in front of the singular part of longitudinal conductivity and thus perform the above limiting transition from (19) and (20) to (42) and (43). In other words, when the dispersion of the velocity of sound and of the attenuation coefficient is investigated, the longitudinal field is unimportant even for $\varkappa = \varkappa_c$ and $l \to \infty$, and the transition to the collisionless limit can be performed by using (26). In the case of long waves [see (37) and (38)], we have for $\varkappa = \varkappa_c$ and $\nu \to 0$

$$\mathbf{f} = \frac{ik^2\omega}{\rho}\left\{\left\langle\left(\Lambda - \frac{v_1}{v_c}\Lambda_c\right) R \left(\Lambda_l - \Lambda_l^c \frac{v_1}{v_c}\right)\right\rangle - \frac{\langle \Lambda R v_2\rangle \langle v_2 R \Lambda_l\rangle}{\langle v_2^2 R\rangle}\right\} u_l. \tag{44}$$

Since (26) does not contain the longitudinal conductivity, even a logarithmic increase in the singular parts of the integrals in (26) is sufficient to give us the limiting formula given by (44) ($J \sim \ln(v_F/s)$ as $\nu \to 0$), and $\omega'' \sim (s/v_F)\omega'$ (see above).

Whether the formally valid (for $\omega'' = 0$) formulas given by (42) and (43) can be used remains an open question. If there is reason to neglect the integral terms in these expressions (for example, as a result of symmetry considerations), the sound waves will "intermingle" with the electromagnetic waves. In fact, the Maxwell and elasticity equations yield

$$(\hat{s}_0^2 - s^2\hat{I})\,\mathbf{u} = \frac{\nu_F k^2 \Lambda^c (\Lambda^c \mathbf{u})}{\rho (k^2 - k_0^2)}, \tag{45}$$

$$\mathbf{k}\mathbf{v}_c = \omega, \quad s = \omega/k, \quad k_0^2 = (4\pi e^2 \nu_F/c^2)(v_\perp^c)^2,$$

where $\hat{I}$ is the unit matrix and the Tolman-Stewart terms have, of course, been omitted. The parameter governing the change in the velocity of sound is

$$\nu_F(\Lambda^c)^2/\rho \sim s_0^2, \quad k_0 \sim \omega_L/c.$$

The properties of the coupled waves are more simply demonstrated if we neglect the elastic anisotropy of the crystal. If we divide the vectors Λ^c and $\mathbf{u}$ into longitudinal and transverse parts ($\Lambda^c = \Lambda_l^c + \Lambda_t^c$; $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$), we obtain a set of two equations and, by equating to zero their determinant, we obtain the convenient expression

$$\frac{k^2}{k^2 - k_0^2} = \frac{(s^2 - s_t^2)(s^2 - s_l^2)}{s^2 - \tilde{s}^2}\,\frac{1}{a_l^2 + a_t^2},$$

$$k^2 = k_z^2 + \frac{\omega^2}{v_c^2} = k_z^2\left(1 + \frac{s^2}{v_c^2}\right) \approx k_z^2, \tag{46}$$

$$a_{l,t}^2 = \frac{\nu_F(\Lambda_{l,t})^2}{\rho}, \quad \tilde{s}^2 = \frac{s_l^2 a_t^2 + s_t^2 a_l^2}{a_l^2 + a_t^2},$$

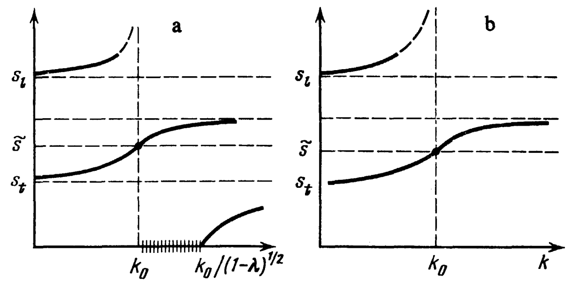

FIG. 5. Wave vector $\mathbf{k}$ as a function of the velocity of the "intermingled" waves according to (46): (a) $\lambda < 1$; (b) $\lambda > 1$. The interval of values of k for which one of the waves is attenuated is shown shaded.

where s_l and s_t are the velocity of longitudinal and transverse sound, respectively, and the directions of the axes are discussed above. The solution of (46) depends on the size of the dimensionless parameter

$$\lambda = a_l^2/s_l^2 + a_t^2/s_t^2. \tag{47}$$

Figure 5 shows the renormalized wave velocities $s = s_\pm(k)$ for $\lambda < 1$ and $\lambda > 1$. It was assumed in the derivation of (45) that $s_\pm \ll v_c$, so that $s = s_+(k)$ for $k \lesssim k_0$ is represented by the dashed curve (see Ref. 1 for further details).

7. ELECTROMAGNETIC AND ELECTRON-VIBRATIONAL SPECTRA OF A METAL

In our analysis of the propagation of sound in metals, we used the condition for slow waves ($s \ll v_F$). When we examine the complete spectrum of a metal, we cannot, of course, use this condition, and the analysis becomes very laborious. In particular, the original (unsimplified) expressions given by (19) and (20) have to be used for the density of force and current. Introducing new notation, these can be written in the form

$$j_\alpha = \sigma_{\alpha\beta}^\perp E_\beta + ek\omega D_{\alpha l}^\perp u_l,$$
$$f_i = \frac{i\omega k^2}{\rho} B_{il}^\perp u_l + \frac{iek}{\rho} D_{i\alpha}^\perp E_\alpha. \tag{48}$$

The tensors that appear in these expressions are readily interpreted by comparing them with (19) and (20) (in Ref. 15, the corresponding matrices are indicated by asterisks—for example, $\sigma_{\alpha\beta}^\perp \equiv \sigma_{\alpha\beta}^*$).

We shall make only one simplification (which we have already used): we shall assume that $\sigma_{\alpha\beta}^\perp = 0$ for $\alpha \neq \beta$. Usually, this is not even a simplification because the matrix $\sigma_{\alpha\beta}^\perp$ can be diagonalized by a suitable choice of coordinates in the plane perpendicular to $\mathbf{k}$. The only exception is provided by rare cases (which we shall not examine) in which $\operatorname{Re} \times \sigma_{\alpha\beta}^\perp$ and $\operatorname{Im}\sigma_{\alpha\beta}^\perp$ are diagonalized by a different choice of axes. When $\sigma_{12} = 0$, we have

$$f_i = \frac{i\omega k^2}{\rho}\left\{B_{il}^\perp + \frac{4\pi i\omega e^2}{c^2}\sum_{\alpha=1}^{2}\frac{D_{i\alpha}^\perp D_{\alpha l}^\perp}{k^2 - (4\pi i\omega/c^2)\sigma_\alpha^\perp}\right\} u_l, \tag{49}$$

where $\sigma_\alpha^\perp$ is the principal value of the tensor $\sigma_{\alpha\beta}^\perp$. Substituting the above expressions into the equations of the theory of elasticity, and equating the determinant to zero, we obtain the following dispersion relation between ω and $\mathbf{k}$ (the Tolman-Stewart components are omitted):

$$D(\omega,\mathbf{k})\equiv\left\| (s_0^2)_{il}-\frac{\omega^2}{k^2}\delta_{il}\right.$$

$$\left.-\frac{i\omega}{\rho}\left\{B_{il}^{\perp}+\frac{4\pi i\omega e^2}{c^2}\sum_{\alpha=1}^{2}\frac{D_{i\alpha}^{\perp}D_{\alpha l}^{\perp}}{k^2-(4\pi i\omega/c^2)\sigma_\alpha^{\perp}}\right\}\right\|=0. \qquad (50)$$

In principle, this should enable us to determine all the branches of the spectrum, i.e., the dependence of the complex frequencies ω on the wave vector $\mathbf{k}$, for which we must perform the analytical continuation of all the functions on (50) to the lower half-plane of $\omega=\omega'+i\omega''$. Temporal dispersion of the moduli [due to the Green function (4)] increases the degree (in the frequency ω) of the dispersion equation and this, in turn, leads to an increase in the number of roots. The new roots are due to the resonance denominator in (4) and can be treated as the electronic branches of the spectrum. The analysis of all the branches of the spectrum in the general case of an arbitrary Fermi surface has not been carried out. It appears that there are no weakly attenuating spectrum branches other than the acoustic and plasma branches (cf. to be sure, Ref. 8), but the problem arises as to whether the equation $\omega=\mathbf{kv}_c$ can be regarded as the dispersion relation for the waves. Let us turn to the dispersion relation given by (50). Apart from the question as to whether the dispersion function $D(\omega,k)$ vanishes at $\omega=\mathbf{kv}_c$, strictly speaking, there is no unattenuated wave with this type of dispersion law (at any rate, for a general Fermi surface; see the Introduction) because the necessary condition for the absence of the unattenuated wave is $D(\omega,\mathbf{k})\propto(\omega-\mathbf{kv}_c)$., and this is possible only in exceptional cases (see Refs. 10 and 20). On the other hand, *any* singularity of the dispersion function will be reflected in the structure of the fields excited in the metal and, when $\omega=\mathbf{kv}_c$, all the components of the tensors in $D(\omega,\mathbf{k})$ have singularities. The question is—how will these singularities appear? To answer this, we must consider some specific formulation of the problem. Thus, to establish the appearance of the singularity at $k=k_c=\omega/v_F$ (see Sec. 2), we must consider the penetration of the wave of frequency ω into the half-space $z>0$ occupied by the metal. As $l\to\infty$, any singularity of $D(\omega,k)$ as a function of $k=k_z$ (the z axis is perpendicular to the boundary of the metal, $v_F=\max v_z$) will appear in the form of a nonexponentially attenuated wave with the following structure:

$$\left(\frac{\omega z}{v_F}\right)^{-n}\exp\left[i(z-v_Ft)\frac{\omega}{v_F}\right],\qquad n>0,$$

where z is the distance from the boundary and n decreases as the singularity of $D(\omega,\mathbf{k})$ becomes stronger. In particular, the asymptotic behavior of the electric field under the conditions of the anomalous skin effect[21] is $E\propto z^{-2}$, and this is a manifestation of this effect. Nonexponential attenuation of longitudinal and transverse sound is examined in Refs. 1 and 16, respectively. In a metal with a nonspherical Fermi surface, the discussion given in Sec. 2 shows that the amplitude of the nonexponentially attenuated wave will depend on the structure of the Fermi surface at the limiting point at which v_z reaches its maximum value [see (6) and Fig. 1]. When the point of contact coincides with a parabolic point or point of flatness, this should appear as a change in the dependence of the wave amplitude on distance z. Evidently, by studying the nonexponentially attenuated waves (see Ref. 22), we should be able, at least in principle, to establish the singularities in the transport coefficients in the collisionless limit (see Secs. 5 and 6, and Ref. 24).

8. CONCLUSION

Since they are functions of frequency ω and wave vector $\mathbf{k}$, the dynamic moduli of a metal in the collisionless limit ($l\to\infty$) exhibit singularities whose nature is very dependent on the local structure of the Fermi surface near the point p_c at which $\mathbf{kv}-\omega=0$ has a repeated zero. The O- and X-type singularities are standard in the sense that, for an arbitrary Fermi surface, there are always critical directions of propagation of sound $\varkappa_c$, the approach to which produces a logarithmic divergence of $|\langle R\rangle|$. The approach to a critical direction can be measured by the "detuning" $\alpha=\delta\theta-i/kl$. The only exception is provided by the doubly critical directions associated with points of flatness, i.e., the crossing of lines of parabolic points on the Fermi surface (see Fig. 2b), the approach to which produces a divergence of the form (see Sec. 3)

$$|\langle R\rangle|\propto|\delta\theta-i/kl|^{-1/6}.$$

If the points of flatness form a line on the Fermi surface (for example, when the Fermi surface has a cylindrical segment, the cross section of which includes a point of zero curvature, which is encountered in quasi-two-dimensional metals), we have a stronger singularity of the form[9]

$$|\langle R\rangle|\propto|\delta\theta-ikl|^{-1/2}.$$

The equations of the theory of elasticity involve the renormalized transport coefficients.[15] We have shown (Sec. 4) that, in the dispersion relation for sound waves, renormalization due to the elimination of the longitudinal electric field is totally unimportant because $s/v_F\sim10^{-3}$. The contribution of electrons to sound velocity and absorption in metals can be evaluated with the aid of (26). Renormalization due to the elimination of the transverse electric field is significant for relatively long sound waves ($k\ll[(s/v_F)|J|]^{1/2}/\delta_L$) and, for $k\gg[(s/v_F)|J|]^{1/2}/\delta_L$, the expression for the force becomes simpler [see (27)], which makes this case particularly convenient for the experimental study of angular anomalies [the higher frequency, which is necessary for the validity of (27), will rise to sharper angular anomalies as $\omega\tau$ increases; see Secs. 2 and 3].

Transverse renormalization is "sharpened up" (as is longitudinal normalization) in such a way that it removes the divergence in the expression for the force given by (26) as $\tau\to\infty$ and $\theta=\theta_c$. This means that the infinite increase in the electronic part of the force given by (26) due to the resonance interaction between the sound wave $\omega=sk$ and the quasiwave $\omega=\mathbf{kv}_c$ is possible only when the terms in (26) that are due to renormalization are absent, for example, as a result of symmetry, just as for the propagation of longitudinal sound in an "easy" direction. On the other hand, Eqs. (33)–(36) show that the resonance will saturate by itself, and will lead to a self-limitation of the absorption coefficient Γ

and the velocity of sound s. The condition for self-limitation demands high enough frequency [see (34) and (36)]: $\omega\tau \gg (v_F/s)\beta$, where $\beta = 1/\ln(v_F/s)$ for an X-type point, $\beta \sim 1$ for an O-type point, and $\beta \sim (s/v_F)^{2/7}$ for a point of flatness.

Since experiments on the propagation of sound are usually performed at a fixed frequency, and the variable is $\tau = \tau(T)$ or $l = v_F\tau$, it is interesting to consider the dependence of Γ and Δs on τ for $\omega =$ const [this will also enable us to consider the transition to the collisionless limit, $\tau \to \infty$]. Suppose that $\omega \ll (s/\delta_L)(s/v_F)^{1/2}$ [see (37)]. For values of τ that are not too high, we must use (26), which excludes the resonance interaction. As τ increases, the role of k^2 in the denominator of the term containing $\alpha = 1$ [see (26)] becomes smaller and, as shown in Sec. 4, we have the possibility of a transition to the "collisionless" limit in accordance with (38) and (39) (the reason why the word collisionless is given in quotation marks is explained above).

When $\omega \gg (s/\delta_L)(s/v_F)^{1/2}$ and τ is not too high, the terms in (26) due to the elimination of the transverse electric field can be neglected, and we can use (27) which describes the resonance interaction of sound with electrons [see (33)–(36)]. It may turn out that, for sufficiently large values of τ, the divergence in $\langle R \rangle$ will "turn on" the terms due to renormalization. Let us verify this by comparing the two limiting expressions for $k\delta$ at relatively low and relatively high τ. Thus, let us suppose that the terms due to renormalization are turned on for sufficiently high values of τ (including even $\tau \to \infty$). The quantity Γ must then be calculated from (39), and this yields $\Gamma/\omega \sim s/v_F$ or $\omega'' = \mathrm{Im}\,\omega \sim -(s/v_F)\omega'$. This means that the maximum value that $|J|$ can assume during the analytic continuation into the half-plane $\mathrm{Im}\,\omega < 0$ [see the beginning of Sec. 6 and (19)] for $\tau \to \infty$ is of the order of $\ln(v_F/s)$ for O- and X-type points, and $\sim (v_F/s)^{1/3}$ for a point of flatness. However, to compensate the divergent terms, we must ensure that [compare this with (28)]

$$\omega \ll (s/\delta_L)\,[(s/v_F)\,|J|]^{1/2}.$$

This means that the inclusion of terms due to renormalization is possible only in the narrow frequency interval

$$(s/\delta_L)\,(s/v_F)^{1/2} \ll \omega \ll (s/\delta_L)\,[(s/v_F)\,\ln\,(v_F/s)]^{1/2}, \tag{51}$$

for the O- and X-type points and, in the somewhat broader interval,

$$(s/\delta_L)\,(s/v_F)^{1/2} \ll \omega \ll (s/\delta_L)\,(s/v_F)^{1/3}. \tag{52}$$

for points of flatness. After inclusion of terms due to renormalization, the quantities Γ and Δs which become anomalously high due to resonance [see (33), (35), and (36)] fall to their usual values that are characteristic for $\varkappa \neq \varkappa_c$. For frequencies greater than $(s/\delta_L)[(s/v_F)\ln(v_F/s)]^{1/2}$ (for $(s/\delta_L)(s/v_F)^{1/3}$), the terms due to renormalization are excluded altogether, and Eqs. (33), (35), and (36) describe the collisionless propagation and attenuation of sound. The increase in Γ and Δs as compared with the ordinary values is responsible for the structure of the Fermi surface at the point of local flattening.

It is clear from the foregoing that experiments with high-purity specimens at maximum possible sound frequencies are the most convenient for the observation of angular anomalies. We emphasize once again that we have not considered the possibility of finite cylindrical or flat areas on the Fermi surface. The presence of such areas would substantially facilitate the observation of anomalies.[9–11,20]

The singularities in transport coefficients in the collisionless limit ($l \to \infty$) as functions of real quantities, namely, the wave vector $\mathbf{k}$ and frequency ω, can appear not only as field drag by electrons into the body of the conductor (see Sec. 7) but also, for example, as frequency singularities in the cross section of the metal for the scattering of electromagnetic waves.[23]

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[1)]A more detailed discussion is given in Ref. 1.

[2)]This result was obtained in Refs. 6 and 7 for high frequencies ($\omega\tau \gg 1$) (in the jellium model).

[3)]The dependence on temperature in special directions is a consequence of the dependence on the relaxation time τ, whereas, in an arbitrary direction and for $kl \gg 1$, the absorption coefficient and velocity of sound are independent of temperature.

[4)]The Fermi surface may also have isolated cylindrical points and points of flatness at which the curvature (or curvatures) vanishes without changing sign. For example, isolated points of flatness occur on the Fermi surface of metals in the molybdenum group.[17] Moreover, such points may also appear when the metal is subjected to an external influence that results in a generalized topological transformation.[18]

[5)]We note that the terms due to the longitudinal field cannot, as a rule, be neglected when electromagnetic phenomena in metals are considered. In particular, the transverse conductivity appears as a coefficient of E_α in (19) and not in (22) (see below).

[6)]This property was found in Ref. 10 and was analyzed in detail by considering metals with Fermi surfaces containing finite (cylindrical) segments.[20] Such anomalies in absorption and in the dispersion of the velocity of sound have recently been found for gallium.[11]

[7)]In the analytical continuation of functions of the form given by (9), one of the integration variables must be taken to be $\xi = \mathbf{kv}$ and the integral with respect to η must be evaluated first [see Eqs. (12)–(17) in Ref. 1].

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