Nonlinear and stochastic dynamics of rays in regular transversely inhomogeneous media

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(Submitted 21 April 1983)
Zh. Eksp. Teor. Fiz. **85**, 1573–1584 (November 1983)

We propose a new method for analyzing the dynamics of rays in waveguide media with arbitrary transverse cross-section shapes. It is based upon ideas about the existence and destruction of invariant tori in Hamiltonian particle mechanics. The main feature of the method is the absence of a connection with the coordinate system of separable coordinates. We consider the effect of nonlinear resonance between the different degrees of freedom of the ray and the ray stochastization by the interaction of the resonances. We obtain constraints on the transverse dimensions of the waveguide channel and on the distance over which the wave front can propagate without being appreciably distorted.

PACS numbers: 42.20.Cc, 42.80.Lt

1. INTRODUCTION

The problem of extra long range propagation of waves in the ocean[1] or the ionosphere[2] leads to new features in the problem of waves in inhomogeneous media. The most characteristic of these features is the possibility of the build-up of small perturbations when the waves propagate over long distances even when one can neglect the effect of the random inhomogeneities in the medium.

One may mention as typical examples of inhomogeneities of waveguide media the inhomogeneities along the axis of propagation of the wave or inhomogeneities in the transverse cross section of the waveguide channel.

The standard form of studying such problems is usually connected either with perturbation theory in some form or with using adiabatic methods.[2] The central role is then played by the ability to separate the variables of the problem at least in zeroth approximation. The assumption that the problem is close to that with separable variables is very strong and limits in an essential way the region of applicability of the theoretical considerations.

New possibilities for the study of wave propagation in inhomogeneous media appear thanks to the application of contemporary methods of nonlinear dynamics which do not use the separability of the variables. In Ref. 3, in particular, it was shown that a phenomenon such as nonlinear "resonance" between the spatial oscillations of the inhomogeneity and the oscillations of the ray trajectory may occur in a medium with a periodic longitudinal inhomogeneity. This phenomenon leads not only to modulated oscillations of the speed of the wave propagation, but also to the possibility that there occurs a stochastic ray-trajectory instability leading to its emission of the ray from the region of the waveguide channel. It was shown, in particular,[3] that such a phenomenon can lead to limitations on the extra-long-range communication in the ionosphere due to the diurnal temperature oscillations of the waveguide layer.

We consider in the present paper another, not less important, case of transverse inhomogeneity of the waveguide medium, without assuming separability of the variables. Methods from nonlinear particle dynamics which are used in what follows enable us to obtain new results along the following way. Firstly, one can, as in Ref. 3, establish a correspondence between the oscillations of the ray in the medium and the particle oscillations in a certain effective two-dimensional potential well. Secondly, under well defined conditions one can obtain a slow spatial modulation of the ray oscillations by nonlinear resonance between various internal degrees of freedom of the ray. Finally and thirdly, one can indicate such conditions on the shape of the transverse cross section of the waveguide channel that if they are satisfied the ray trajectory becomes diffusive even though there are no random forces or random inhomogeneities. This case turns out to be very important for applications, as its realization is accompanied by a maximally strong defocusing of the ray and by loss of its spatial coherence properties. We note that the occurrence of chaotic ray dynamics also leads to a chaotic deformation of the wave phase front.

2. INTERNAL NONLINEAR RESONANCE OF RAYS

Let the waveguide medium in which a ray propagates along the z axis be characterized by the refractive index $n(\mathbf{r})$, where $\mathbf{r} = (x,y)$. To describe the ray trajectory we use a Hamiltonian formalism.[4] The equation for the ray trajectory has the form

$$\frac{d\mathbf{p}}{dz}=-\frac{\partial H}{\partial \mathbf{r}},\quad \frac{d\mathbf{r}}{dz}=\frac{\partial H}{\partial \mathbf{p}}, \tag{2.1}$$

where the Hamilton function is equal to

$$H=H(\mathbf{r},\ \mathbf{p})=-[n^2(\mathbf{r})-\mathbf{p}^2]^{1/2}, \tag{2.2}$$

while the momentum $\mathbf{p}$ is equal to

$$\mathbf{p}=n(\mathbf{r})\dot{\mathbf{r}}/(1+\dot{\mathbf{r}}^2)^{1/2},\quad \dot{\mathbf{r}}=d\mathbf{r}/dz.$$

The vector $(p_x, p_y, -H)$ is directed along the tangent to the ray trajectory. We draw attention to the fact that Eq. (2.2) does not depend explictly on z (i.e., there is no longitudinal inhomogeneity), and the entire individuality of the problem is determined by the form of the refractive index n in the plane perpendicular to the direction of the ray propagation (i.e., to the z axis).

As H does not depend explicitly on the variable z, which plays the role of the time for the equivalent dynamics of the

 0038-5646/83/110915-07$04.00

system (2.1), we can introduce by standard methods[5] the action (I_1,I_2) and angle $(\vartheta_1,\vartheta_2)$ variables, in terms of which

$$H=H(I_1, I_2; \vartheta_1, \vartheta_2),$$

where

$$I_k=\frac{1}{2\pi}\oint_{C_k}\mathbf{p}\,d\mathbf{r},$$

$$\vartheta_k=\frac{\partial}{\partial I_k}\int_{\mathbf{r}_0}^{\mathbf{r}}\mathbf{p}(I_1,I_2;x,y)\,d\mathbf{r}\quad (k=1,2) \tag{2.3}$$

while the contours C_k are the basic contours of the two-dimensional torus in the phase space of the system. In the general case there may not be invariant tori and then the contours C_k do not exist.

We consider the case when H can be written in the form

$$H=H_0(I_1, I_2)+\varepsilon V(I_1, I_2; \vartheta_1, \vartheta_2). \tag{2.4}$$

This means that when there is no perturbation ($\varepsilon=0$) the ray trajectory is characterized by the Hamiltonian H_0 with two independent integrals of motion I_1, and I_2 (either H_0 and I_1 or H_0 and I_2). The variables in the problem can then, in general, not be separated and the only difficulty connected with the determination of the quantities I_1 and I_2 is of a technical nature rather than one of principle. One can thus, notwithstanding the lack of variable separation, state that in phase space the ray is wound around the two-dimensional torus with oscillation frequencies

$$\omega_k=\frac{\partial H_0(I_1,I_2)}{\partial I_k}\quad (k=1,2). \tag{2.5}$$

Equation (2.5) means that the ray trajectory is doubly periodic in the coordinate space of the ray and that it can be written as an expansion in a Fourier series:

$$\begin{pmatrix} x \\ y \end{pmatrix}=\sum_{l_1,l_2=-\infty}^{+\infty}\begin{pmatrix} x_{l_1l_2} \\ y_{l_1l_2} \end{pmatrix}\exp i[l_1\omega_1(I_1,I_2)z+l_2\omega_2(I_1,I_2)z], \tag{2.6}$$

with oscillation periods $2\pi/\omega_1$ and $2\pi/\omega_2$ along the z axis. One finds the amplitudes of the expansion $(x_{l_1l_2}, y_{l_1l_2})$ from the formulae connecting the old variables $(\mathbf{r},\mathbf{p})$ with the new ones $(I_1,I_2;\vartheta_1,\vartheta_2)$.

We now turn to a study of the perturbed layer (2.4). As the variables I and ϑ are canonically conjugate one can write the equation of motion for the ray in the form

$$\dot I_k=-\varepsilon\frac{\partial V}{\partial \vartheta_k},\quad \dot\vartheta_k=\omega_k(I_1,I_2)+\varepsilon\frac{\partial V}{\partial I_k}\quad (k=1,2). \tag{2.7}$$

We write V as a Fourier series expansion in ϑ_1 and ϑ_2:

$$V(I_1,I_2;\vartheta_1,\vartheta_2)=\frac{1}{2}\sum_{m_1,m_2=-\infty}^{+\infty}V_{m_1m_2}(I_1,I_2)\exp[i(m_1\vartheta_1+m_2\vartheta_2)],$$

$$V_{-m_1,-m_2}=V^{*}_{m_1m_2}. \tag{2.8}$$

Equation (2.7) now becomes

$$\dot I_k=-\frac{i}{2}\varepsilon\sum_{m_1,m_2}m_kV_{m_1m_2}(I_1I_2)\exp[i(m_1\vartheta_1+m_2\vartheta_2)],$$

$$\dot\vartheta_k=\omega_k(I_1,I_2)+\frac{\varepsilon}{2}\sum_{m_1,m_2}\frac{\partial V_{m_1m_2}(I_1,I_2)}{\partial I_k}\exp[i(m_1\vartheta_1+m_2\vartheta_2)]\quad (k=1,2). \tag{2.9}$$

It is clear from (2.8) that the strongest effect of the perturbation occurs in the resonance case, i.e., when

$$m_1\omega_1(I_1^0, I_2^0)+m_2\omega_2(I_1^0, I_2^0)=0, \tag{2.10}$$

where I_k^0 is the value of the action at exact resonance. We study the ray trajectory in the vicinity of the nonlinear resonance.

Dropping the nonresonance terms in (2.9) we have

$$\dot I_k=\varepsilon m_kV_{m_1m_2}\sin\Psi,$$

$$\dot\vartheta_k=\omega_k(I_1,I_2)+\varepsilon\frac{\partial V_{m_1m_2}}{\partial I_k}\cos\Psi\quad (k=1,2), \tag{2.11}$$

$$\dot\Psi=m_1\omega_1(I_1, I_2)+m_2\omega_2(I_1, I_2),$$

where m_1 and m_2 are a pair of numbers satisfying the resonance condition (2.10).

The set (2.11) is integrable and describes the dynamics of the ray in the vicinity of the resonance $(I_1^0, I_2^0;m_1,m_2)$. We note first of all that from (2.11) follows the existence of an integral of motion:

$$J=m_2I_1-m_1I_2, \tag{2.12}$$

which is the analog of the Manley-Rowe relation in the theory of parametric excitation.

Differentiating in (2.11) the equation for $\dot\Psi$ we get the so-called phase oscillation equation[6] for the ray:

$$\ddot\Psi+\Omega^2\sin\Psi=0,$$

$$\Omega^2=\varepsilon\left|V_{m_1m_2}\sum_{l_1,l_2=1,2}m_{l_1}m_{l_2}\frac{\partial^2H_0}{\partial I_{l_1}\partial I_{l_2}}\right| \tag{2.13}$$

with the small-oscillation frequency Ω.

3. RAY DYNAMICS IN THE VICINITY OF INTERNAL RESONANCE

We introduce the variable

$$P=\dot\Psi \tag{3.1}$$

which is canonically conjugate to Ψ. Equation (2.13) then corresponds to the universal Hamiltonian function

$$\bar H(P,\Psi)=P^2/2-\Omega^2\cos\Psi. \tag{3.2}$$

Each ray trajectory is characterized by the integrals of motion $W=\bar H(P,\Psi)$ and J from (2.12). For values of $W<-\Omega^2$ the ray performs limited oscillations in the variable Ψ, i.e., the ray is trapped by the resonance. When $W>-\Omega^2$ the ray trajectory is unbounded in Ψ. According to (3.2) the maximum width of the internal resonance in the variable P equals

$$\Delta P=2\Omega=\left\{4\varepsilon\left|V_{m_1m_2}\sum_{l_1,l_2=1,2}m_{l_1}m_{l_2}\frac{\partial^2H_0}{\partial I_{l_1}\partial I_{l_2}}\right|\right\}^{1/2}. \tag{3.3}$$

We determine the resonance widths ΔI_1, ΔI_2 from the action variables I_1, I_2. According to Eqs. (2.12), (2.10) the variables

$$\Delta I_1=I_1-I_1^0,\quad \Delta I_2=I_2-I_2^0$$

are related as follows

$$\Delta I_2=-\frac{\omega_1(I_1^0,I_2^0)}{\omega_2(I_1^0,I_2^0)}\Delta I_1. \tag{3.4}$$

On the other hand, according to (2.11) and (3.1) the variable

P is related up to terms of order ε to the quantities ΔI_1 and ΔI_2 through the equation

$$P=\dot{\Psi}=\sum_{l_1,l_2=1,2} m_{l_1}\frac{\partial\omega_{l_1}(I_1^0,I_2^0)}{\partial I_{l_2}}\Delta I_{l_2}. \tag{3.5}$$

Finally, using Eqs. (3.3) to (3.5) we get

$$\Delta I_k=[4\varepsilon\,|V_{m_1m_2}[\omega_k^2(I_1^0,I_2^0)B(I_1^0,I_2^0)]^{-1}|]^{1/2}\quad (k=1,2),$$

$$B(I_1,I_2)=\sum_{l_1,l_2=1,2}(-1)^{l_1+l_2}\frac{1}{\omega_{l_1}\omega_{l_2}}\frac{\partial^2 H_0}{\partial I_{l_1}\partial I_{l_2}}. \tag{3.6}$$

One can show similarly that the width of the resonance in terms of the frequencies ω_1 and ω_2 equals

$$\Delta\omega_k=\max|\omega_k(I_1,I_2)-\omega_k(I_1^0,I_2^0)|$$

$$=[4\varepsilon V_{m_1m_2}/B(I_1^0,I_2^0)]^{1/2}\left|\frac{1}{\omega_1}\frac{\partial\omega_k}{\partial I_1}-\frac{1}{\omega_2}\frac{\partial\omega_k}{\partial I_2}\right|. \tag{3.7}$$

We note that approximation with an isolated resonance described by Eqs. (2.13) to (3.7) is valid when the distance between neighboring resonances of the kind (2.10) in the variables I_1 and I_2 exceeds the width (3.6) of the separate resonances.

We discuss now the physical meaning of the nonlinear resonance of rays. It is well known that when there is no perturbation, rays which satisfy the resonance condition (2.10) correspond to trajectories which are on the whole periodic, i.e., the projection of the ray trajectory on the transverse (x,y) plane is closed. For small perturbations of the profile of the cross section of the waveguide channel of the medium the rays begin to execute near these trajectories additional (phase) oscillations relative to these closed ray trajectories of the unperturbed waveguide. Around the unperturbed resonance ray trajectory which satisfies (2.10) there is thus formed a distinctive waveguide channel with an effective size given in the variables I_k by Eq. (3.6). The trajectories of the rays in that channel are given by the solutions of Eq. (2.13) which are bounded in Ψ:

$$\Psi=2\arcsin[d\,\mathrm{sn}(\Omega z|d)],\quad d=[(\Omega^2-|w|)/2\Omega^2]^{1/2},$$

$$\Delta I_k=\dot{\Psi}/m_k\omega_k^2B(I_1^0,I_2^0)\quad (k=1,2), \tag{3.8}$$

where $\mathrm{sn}(u|d)$ is an elliptical Jacobi function.

The change of the nature of the ray trajectories which enter into resonance leads to a change in the velocity v of the signal propagation. We study the features of the change produced in the local propagation speed by with resonance rays. As the latter is equal to the group velocity of the wave, the local velocity of signal propagation satisfies the equation

$$\frac{c}{v}=\frac{d(k_0|H|)}{dk_0}=-H-\sum_{l=1,2}k_0\frac{\partial H}{\partial I_l}\frac{dI_l}{dk_0}, \tag{3.9}$$

where k_0 is the wave number in vacuo and c the wave velocity in the homogeneous medium. According to a wave analysis each mode of the waveguide is characterized by the integers s_1,s_2 which are connected with the actions (quantization rules)

$$k_0I_k\approx\pi s_k,\quad s_k=1,2,\ldots\quad (k=1,2). \tag{3.10}$$

From Eqs. (3.8) and (3.9) it follows that

$$\frac{c}{v}=-H(I_1,I_2;\vartheta_1,\vartheta_2)+\dot{\vartheta}_1I_1+\dot{\vartheta}_2I_2, \tag{3.11}$$

where the expression for H is given by Eq. (2.4).

Differentiating (3.11) with respect to z up to terms of order ε we get an expression for the longitudinal gradient of the reciprocal of the group velocity of the wave:

$$\frac{d}{dz}\left(\frac{c}{v}\right)=\varepsilon m_1\omega_1V_{m_1m_2}\left[\frac{I_1}{\omega_1}\frac{\partial\omega_1}{\partial I_1}-\frac{I_1}{\omega_2}\frac{\partial\omega_1}{\partial I_2}+\frac{I_2}{\omega_1}\frac{\partial\omega_2}{\partial I_1}-\frac{I_2}{\omega_2}\frac{\partial\omega_2}{\partial I_2}\right]\sin\Psi. \tag{3.12}$$

It follows from (3.12) that the group velocity of the wave corresponding to resonance rays is modulated along the z axis. Notwithstanding the longitudinal modulation of the group velocity, the average velocity of signal propagation $\bar{v}$, given by the relation

$$\overline{\left(\frac{c}{v}\right)}=\lim_{z\to\infty}\frac{1}{z}\int_0^z dz'\frac{c}{v(z')}=\frac{c}{v(0)},$$

remains, up to terms of order ε, equal to its initial value when there is no perturbation.

In concluding this section we discuss the condition for the applicability of the results to the case of an isolated resonance. As indicated above the isolated resonance approximation is valid when the width $\Delta\omega_k$ of the resonances is appreciably less than the distance $\delta\omega_k$ between neighboring resonances, i.e.,

$$K=(\Delta\omega_k/\delta\omega_k)^2\ll 1. \tag{3.13}$$

The quantity $\delta\omega_k$ is determined by the actual geometry of the system and can be determined for any given kind of medium with refractive index $n(x,y)$. We give in Sec. 5 an example of the determination of the parameter K for a concrete case.

4. STOCHASTIZATION OF THE RAYS

If condition (3.13) is not satisfied and

$$K\gg 1, \tag{4.1}$$

the invariant tori in the phase space of the rays are destroyed and there occurs a stochastic dynamics of the rays.[6,7] An example of the occurrence of chaotic motion of rays in a waveguide channel with a longitudinal inhomogeneity due to the overlap of resonances was considered in Ref. 3. In the present case the picture of the stochastization of the rays is analogous. The motion of the representative point of the ray in a plane perpendicular to the axis of the waveguide is stochastic. The additional integral of motion J given by Eq. (2.12) is destroyed and the only integral of motion remains the quantity H, i.e., the z-component $(k_0|H|)$ of the wave vector of the wave. Under the conditions of the situation envisaged here ($K\gtrsim 1$) one can describe the dynamics of the representative point of the ray using the diffusion equation. In the next section we give the corresponding results for a concrete example.

5. WAVEGUIDE CHANNEL WITH A "STADIUM" KIND OF TRANSVERSE CROSS-SECTION

In many problems of the propagation of radiowaves in the ionosphere[2] and of soundwaves in the ocean[1] one assumes that the refractive index $n(x,y)$ of the medium depends mainly on the vertical coordinate (say, on x) and can be assumed to be homogeneous along the other transverse coordinate (along y). It is natural to assume that in real situations the waveguide channel has finite dimensions also along the y axis. The nature of the wave propagation will then depend strongly on the shape of the transverse cross section of the waveguide channel. The simplest example of allowance for the finite dimensions of the channel along the y axis leads to a "stadium" type shape for its transverse cross section (Fig. 1).

It is useful to pursue the analogy between our problem of ray dynamics and problems of billiard-ball motion of particles. In the present case (Fig. 1) there appears a non-scattering billiard motion for which the possibility of the stochastization of the particle motion was shown by Bunimovich.[8] The study given in what follows will mainly follow Ref. 9. We show, however, that apart from the case of total stochastization of the ray trajectory in phase space there may also exist a sub-region of stochastization corresponding to the overlap of resonances with large numbers.

Let the refractive index $n(x,y)$ have a constant value n_0 within the waveguide channel and n_∞ ($n_\infty < n_0$) outside it. The sides AB and CD of the transverse cross section are deformed and described by the function $\varepsilon f(x)$. We can write the analytical dependence of $n(x,y)$ on the transverse coordinates x,y in the form

$$n(x,y)=\begin{cases} n_1(y+\varepsilon f(x)), & \text{when } |x|<a; \\ n_\infty, & \text{when } |x|>a, \end{cases} \tag{5.1}$$

where

$$n_1(y)=\begin{cases} n_0, & \text{when } |y|<b; \\ n_\infty, & \text{when } |y|>b. \end{cases}$$

When $\varepsilon \ll 1$ one can write the refractive index in the form

$$n^2(x,y)=n_0^2(x,y)+\varepsilon n_2(x,y), \tag{5.2}$$

where $n_0(x,y)$ is the refractive index of the waveguide medium in the unperturbed case, i.e.,

$$n_0(x,y)=\begin{cases} n_0, & \text{when } |x|<a,\ |y|<b; \\ n_\infty, & \text{elsewhere.} \end{cases} \tag{5.3}$$

The quantity $\varepsilon n_2(x,y)$ is the perturbing part of the refractive index and has the form

$$\varepsilon n_2(x,y)=\begin{cases} \varepsilon f(x)\,\partial n_1^2(y)/\partial y, & \text{when } |x|<a; \\ 0, & \text{when } |x|>a. \end{cases} \tag{5.4}$$

The Hamiltonian of the system takes the form

$$H(\mathbf{r},\mathbf{p})=H_0(\mathbf{r},\mathbf{p})+\varepsilon V(\mathbf{r},\mathbf{p}), \tag{5.5}$$

$$H_0(\mathbf{r},\mathbf{p})=-[n_0^2(x,y)-\mathbf{p}^2]^{1/2},$$

$$\varepsilon V(\mathbf{r},\mathbf{p})=\frac{\varepsilon n_2(x,y)}{2H_0(\mathbf{r},\mathbf{p})}=-\varepsilon f(x)\frac{d\dot{p}_y}{dz}.$$

The solution of the unperturbed problem with Hamiltonian $H_0(\mathbf{r},\mathbf{p})$ has the following form:

$$\begin{gathered} H_0(I_1,I_2)=-[n_0^2-(I_1/I_{10})^2-(I_2/I_{20})^2]^{1/2}, \quad I_{k0}=2a_k/\pi; \\ x_k=\begin{cases} a_k[4\{\vartheta_k/2\pi\}-1], & \text{when } 0<\{\vartheta_k/2\pi\}<1/2; \\ a_k[3-4\{\vartheta_k/2\pi\}], & \text{when } 1/2<\{\vartheta_k/2\pi\}<1; \end{cases} \\ p_k=\begin{cases} I_k/I_{k0}, & \text{when } 0<\{\vartheta_k/2\pi\}<1/2; \\ -I_k/I_{k0}, & \text{when } 1/2<\{\vartheta_k/2\pi\}<1; \end{cases} \\ \vartheta_k=\omega_k(I_1,I_2)z+\vartheta_{k0}, \quad \omega_k(I_1,I_2)=I_k/(|H_0|I_{k0}^2); \\ (x_1,x_2)=(x,y), \quad (a_1,a_2)=(a,b) \quad (k=1,2). \end{gathered} \tag{5.6}$$

Here $\{\mu\}$ denotes the fractional part of the quantity μ. For waveguide rays the quantities I_1, I_2, and E take values in the ranges

$$\begin{gathered} 0<I_k/I_{k0}<n_0\Delta^{1/2}, \quad \Delta=(n_0^2-n_\infty^2)/n_0^2, \\ -n_0<E<-n_\infty \quad (E=H_0(I_1,I_2)). \end{gathered} \tag{5.7}$$

The Fourier expansion for the momenta p_k has the form

$$p_k=-i\frac{2I_k}{\pi I_{k0}}\sum_{m=-\infty}^{+\infty}\frac{1}{(2m+1)}\exp\{i(2m+1)\vartheta_k\} \quad (k=1,2). \tag{5.8}$$

As the perturbing function $f(x)$ we choose

$$\varepsilon f(x)=l\cos(\pi x/2a) \quad (l\ll a,b;\ |x|<a). \tag{5.9}$$

Using Eqs. (5.5) to (5.9) we write the perturbation of the Hamiltonian $\varepsilon V(\mathbf{r},\mathbf{p})$ with the action and angle as the variables:

$$\begin{gathered} \varepsilon V(\mathbf{r},\mathbf{p})=\frac{1}{2}\sum_{\substack{m=-\infty \\ s=-1,1}}^{+\infty} V_m(I_1,I_2)\exp\{i[s\vartheta_1+(2m+1)\vartheta_2]\}, \\ V_m(I_1,I_2)=-(l/b\,|H_0(I_1,I_2)|)\,(I_2/I_{20})^2. \end{gathered} \tag{5.10}$$

The condition for nonlinear resonance has the form

$$\omega_1=(2m+1)\omega_2 \quad \text{or} \quad \frac{I_1^{(m)}}{I_{10}}=(2m+1)\frac{a}{b}\frac{I_2^{(m)}}{I_{20}}, \tag{5.11}$$

where the $I_k^{(m)}$ are the resonance values of the actions ($k=1,2$).

We show in Fig. 2 the curves of constant $H_0(I_1,I_2)$ in the

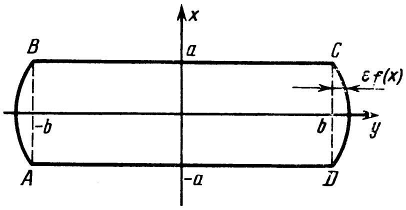

FIG. 1. Shape of a "stadium" type transverse cross section.

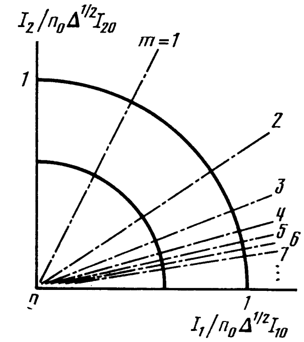

FIG. 2. Curves of constant values of $H_0(I_1,I_2)$ (solid) and the resonance lines (dash-dot) in the (I_1,I_2) action plane.

(I_1,I_2) plane and the resonance lines described by Eq. (5.11). The points where the curves of constant $E=H_0(I_1,I_2)$ intersect with the resonance lines give the resonance values $I_1^{(m)}, I_2^{(m)}$.

One sees from Fig. 2 that the distance between neighboring resonances decreases with increasing number m. One can write this distance in the variables I_1 and I_2 and in the spatial frequencies ω_1 and ω_2 in the form

$$\frac{\delta I_1}{I_{10}}=\left|\frac{I_1^{(m+1)}-I_1^{(m)}}{I_{10}}\right|\approx C\left(\frac{I_2}{I_{20}}\right)^3,\quad \frac{\delta I_2}{I_{20}}=C\left(\frac{I_1}{I_{10}}\right)\left(\frac{I_2}{I_{20}}\right)^2,$$

$$\frac{\delta\omega_1}{\omega_1}=\left|\frac{\omega_1(I_1^{(m+1)},I_2^{(m+1)})-\omega_1(I_1^{(m)},I_2^{(m)})}{\omega_1}\right|$$

$$\approx C\left(\frac{I_1}{I_{10}}\right)^{-1}\left(\frac{I_2}{I_{20}}\right)^3, \tag{5.12}$$

$$\frac{\delta\omega_2}{\omega_2}=C\left(\frac{I_1}{I_{10}}\right)\left(\frac{I_2}{I_{20}}\right),\quad C=\frac{2a}{b(n_0^2-E^2)}.$$

Using Eqs. (3.6), (3.7), and (5.6), and (5.10) one can find the following expressions for the widths of the separate resonances:

$$\frac{\Delta I_1}{I_{10}}=C_0\left(\frac{I_2}{I_{20}}\right)^2,\quad \frac{\Delta I_2}{I_{20}}=C_0\left(\frac{I_1}{I_{10}}\right)\left(\frac{I_2}{I_{20}}\right),$$

$$\frac{\Delta\omega_1}{\omega_1}=C_0\left(\frac{I_2}{I_{20}}\right)\left(\frac{I_1}{I_{10}}\right)^{-1}, \tag{5.13}$$

$$\frac{\Delta\omega_2}{\omega_2}=C_0\frac{I_1}{I_{10}},\quad C_0=\left[\frac{4l}{b(n_0^2-E^2)}\right]^{1/2}.$$

From (5.13) it follows that as $|E|\to n_0$

$$\frac{\Delta I_k}{I_{k0}}\sim(n_0-|E|)^{-1/2},\quad \frac{\Delta\omega_k}{\omega_k}\sim(n_0-|E|)^{-1/2}\quad (k=1,2),$$

i.e., the width of the nonlinear resonances of the rays corresponding to the lowest modes of the waveguide increases.

For rays corresponding to higher modes the resonance width has finite values:

$$\Delta\omega_k/\omega_k\sim\Delta I_k/I_k\sim[n_0^2\Delta l/b]^{1/2}\quad (k=1,2).$$

We note that for a given value of the constant of motion $E=H_0(I_1,I_2)$ the actions I_1 and I_2 are interrelated as follows

$$I_2/I_{20}=[n_0^2-E^2-(I_1/I_{10})^2]^{1/2}. \tag{5.14}$$

As $I_2\to 0$ the distance between neighboring resonances (5.12) decreases faster than the resonance width (5.13). Therefore, overlap of resonances sets in for values of I_2 less than some critical value I_2^c. According to (4.1), (5.12), (5.13) the condition for overlap has the form

$$K=\left(\frac{\Delta\omega_k}{\delta\omega_k}\right)^2=\frac{lb}{a^2}\frac{(n_0^2-E^2)}{(I_2/I_{20})^2}\gtrsim 1. \tag{5.15}$$

From (5.15) we get the following value for the critical value I_2^c:

$$\frac{I_2^c}{I_{20}}=\left[\frac{lb}{a^2}(n_0^2-E^2)\right]^{1/2}. \tag{5.16}$$

For a given value of the constant $E=H_0(I_1,I_2)$, the ray trajectories with values of the action

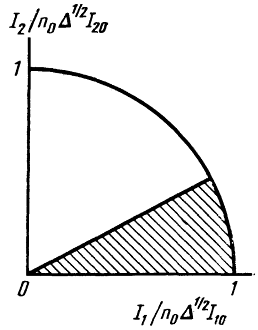

FIG. 3. Region of ray stochastization (hatched region) in the (I_1,I_2) action plane.

$$I_2<I_2^c \tag{5.17}$$

are thus stochastic (Fig. 3).

Let θ be the angle between the tangential vector to the ray trajectory and the x axis. We then have

$$p_x=(n_0^2-E^2)^{1/2}\cos\theta,\quad p_y=(n_0^2-E^2)^{1/2}\sin\theta. \tag{5.18}$$

According to (5.6) and (5.18), in the unperturbed case the actions are related to the angle θ through the following formula:

$$\frac{I_1}{I_{10}}=(n_0^2-E^2)^{1/2}|\cos\theta|,\quad \frac{I_2}{I_{20}}=(n_0^2-E^2)^{1/2}|\sin\theta|. \tag{5.19}$$

Condition (5.17) then means that all rays propagating at an angle $\theta<\theta_c$, where

$$\theta_c=\arcsin\frac{I_2^c}{I_{20}(n_0^2-E^2)^{1/2}}=\arcsin\left(\frac{lb}{a^2}\right)^{1/2}, \tag{5.20}$$

are stochastic.

One sees from (5.16) that the size of the region of stochasticity depends on the waveguide parameters only through the combination lb/a^2. The region of stochasticity of the ray trajectories corresponding to the lower modes of the waveguide ($|E|\to n_0$) tends to zero in proportion to

$$(n_0^2-E^2)^{1/2}\approx(2n_0)^{1/2}(n_0-|E|)^{1/2}.$$

The maximum value of the stochasticity region is reached for rays corresponding to the higher modes ($|E|\to n_0$):

$$\max\left(\frac{I_2^c}{I_{20}}\right)=(n_0^2-n_\infty^2)^{1/2}\left(\frac{lb}{a^2}\right)^{1/2} \tag{5.21}$$

For values of the waveguide parameters for which the condition

$$lb/a^2\gtrsim 1 \tag{5.22}$$

is satisfied all waveguide rays become stochastic. Condition (5.22) is the same as the condition for complete stochastization of the motion of particles in a "stadium" shaped potential well with elastic walls.[9]

6. DIFFUSION OF RAYS IN THE STOCHASTICITY REGION

The most adequate description of the rays in the stochasticity region can be given by means of a kinetic equation. One can show (see Ref. 10) that for a system with the Hamiltonian (2.4), (2.8) the kinetic equation has in the region of stochastization of the motion the form

$$\frac{\partial f}{\partial z}=\sum_{i,j=1,2}\frac{\partial}{\partial I_i}D_{ij}(I_1,I_2)\frac{\partial f}{\partial I_j},\tag{6.1}$$

$$D_{ij}(I_1,I_2)=2\pi\varepsilon^2\sum_{m_1,m_2=0}^{\infty}m_im_j|V_{m_1m_2}|^2\delta(m_1\omega_1-m_2\omega_2).$$

Here $f=f(I_1,I_2;z)$ is the density of the distribution of rays in the (I_1, I_2) action plane.

Using (5.10) one can obtain the following estimates for the diffusion coefficients D_{ij} for the problem considered in Sec. 5:

$$D_{11}=\frac{2l^2}{b|E|}\frac{I_2}{I_{20}},\quad D_{12}=D_{21}=\frac{2l^2}{a|E|}\left(\frac{I_1}{I_{10}}\right),$$
$$D_{22}=\frac{2l^2b}{a^2|E|}\left(\frac{I_1}{I_{10}}\right)^2\left(\frac{I_2}{I_{20}}\right)^{-1}.\tag{6.2}$$

The diffusion of rays leads to an equilibrium distribution of rays (independent of z) in the transverse (x,y) plane. The characteristic distance z_D over which such a distribution is established is determined by the diffusion coefficients D_{ij} and is of the order $z_D^{(k)}\sim I_k^2/D_{kk}$, so that

$$z_D^{(1)}\sim I_1^2I_{20}b|E|/(I_2l^2);\tag{6.3}$$
$$z_D^{(2)}\sim I_2^3a^2|E|(I_1/I_{10})^2/l^2bI_{20},$$

where the diffusion lengths $z_D^{(k)}$ describes the diffusion of rays along the directions $k=x,y$.

7. ON THE STRUCTURE OF THE WAVE FRONT

We study the problem of the wave front of the wave in the case where there is a stochastic instability of the rays. To do this we turn to the quasi-classical representation of the wave field $u(\mathbf{r},z)$ in the waveguide channel. Let in the starting plane $z=0$ of the transverse waveguide cross section the field be given

$$u_0(\mathbf{r})=A_0(\mathbf{r})\exp\{ik_0S_0(\mathbf{r})\},\tag{7.1}$$

with a phase distribution $S_0(\mathbf{r})$ and an amplitude $A_0(\mathbf{r})$. The field $u(\mathbf{r},z)$ in the plane z will then have the form[11]

$$u(\mathbf{r},z)=\sum_{\nu=1}^{N}u_0(\mathbf{r}_{0\nu})[\mathcal{T}_\nu]^{-1/2}\exp\{ik_0S_\nu(\mathbf{r},z)+i\delta\Psi_\nu\}.\tag{7.2}$$

Here $S_\nu(\mathbf{r},z)$ is the action along the ray trajectory

$$S_\nu(\mathbf{r},z)=\int_{\Gamma_\nu}n(\mathbf{r})\,d\tau,\tag{7.3}$$

where $d\tau$ is an element of the ray on the ray trajectory, $r_{0\nu}$ the ray coordinate in the $z=0$ plane. The quantity $\mathcal{T}_\nu=\mathcal{T}_\nu(\mathbf{r},z)$ is the generalized ray dispersion. The phase shifts when the ν-ray passes through the caustics and the focal points are described by the quantity $\delta\Psi_\nu$. The summation in (7.2) is over all N trajectories passing through the observation point $\mathbf{r},z$.

In the case of stochastic ray instability the number of rays contributing to the wave field (7.2) increases exponentially[9] according to

$$N\sim\exp(hz),$$

where $h=\omega_k\ln K$ is the instability growth rate. At the same time the actions $S_\nu(\mathbf{r},z)$ in (7.2) are random functions of the coordinates $\mathbf{r}$ and z because of the stochasticity of the ray trajectories. The total wave field is thus a complex interference pattern of the sum of a large number of quasi-planar waves connected with the separate ray trajectories. The wavefront of the field will thus possess a rather complicated quasi-random relief[9] which has a structure analogous to speckle structure.[12]

Over sufficiently long distances of the order of the diffusion length (6.3) one therefore completely loses information about the initial phase $S_0(\mathbf{r})$ and about the distribution of the field strength $|A_0(\mathbf{r})|^2$ in the $z=0$ plane. One can say that when transmitting any image over distances $z>z_D$ this image is completely distorted. A more detailed analysis of the waveguide field in the case where the rays in waveguides are stochastically unstable will be considered separately.

8. CONCLUSION

The main principal result of the present paper is connected with the possibility to study the propagation of rays in waveguide media without using the method of separation of variables. This possibility arises thanks to the use of methods from Hamiltonian nonlinear dynamics of particles with action and angle as the variables. Contemporary theory of the Kolmogorov-Arnol'd-Moser instability[5] enables us to draw a clear-cut analogy between the existence of invariant tori and the existence, correspondingly, of singular ray channels. In that sense the case of internal resonance between different degrees of freedom of the ray, which is considered in the present paper, is the most effective. The loss of stability of the ray, which in the theory of dynamical systems corresponds to the destruction of the invariant tori, has an important practical application. It is connected with the definition of the boundary of the diffusive smearing out of a transmitted image through the waveguide channel. The image-smearing effect described by us is connected neither with well known aberration effects nor with the existence of some strong inhomogeneities of a regular or random kind. The cause of the ray stochastic instability described above is the slow and very irregular buildup of relatively weak transverse inhomogeneities, the existence of which is necessary as a matter of principle for the possibility to channel the ray. We have thus indicated in this paper an effect which in cases of a general nature always leads to a restriction on the size and the shape of ray propagation.

The results given above are also directly applicable to dielectric waveguides in the optical band.[13,14] The greatest difficulty is the study of eigenwaves in waveguides with complicated cross sections. The calculation of the fields in such waveguides reduces to a study of a rather complicated set of integral equations which can be solved only numerically.[15,16] The adiabatic approach to the ray approximation proposed in Ref. 17 to calculate the characteristic eigenwaves in a waveguide with non-separable variables also has a limited range of application since, as we showed above, the adiabatic invariants I_1 and I_2 are not conserved in the case of stochastic instability and nonlinear resonance of the rays.

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Translated by D. ter Haar.