Damping of a superconducting current in tunnel junctions

A. I. Larkin and Yu. N. Ovchinnikov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
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The lifetime of a metastable current state of a superconducting junction is determined by classical fluctuations at temperatures above a certain temperature T_0 and by quantum fluctuations at temperatures below T_0. We find the temperature and current dependence of the lifetime close to T_0 for any viscosity and in the two limiting cases of large and small viscosity for the whole temperature range.

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1. INTRODUCTION

The current state of a Josephson junction corresponds to the minimum of the free energy $U(\phi)$ of the junction as function of the phase difference of the two superconductors. These minima are separated by a potential barrier. The lifetime of such a state is finite. At not too low temperatures such a state disintegrates due to thermal fluctuations.[1,2] At low temperatures quantum tunnelling becomes important. For a sufficiently large capacitance of the junction the tunnelling time is large. In that case the adiabatic approximation is applicable for the potential $U(\varphi)$ and the transition probability can be obtained employing the usual quantum mechanical formula.[3] In a number of experiments[4,5] junctions with a small capacitance were used. In such systems, quantum-mechanical tunnelling is determined not only by the change in a single coordinate (the phase difference φ) but also by a large number of electron degrees of freedom. Both real and virtual transitions are then important.

In the present paper we show that after averaging over the electron degrees of freedom the effective potential turns out to be retarded; we find the form of this potential and solve the problem of the quantum mechanical tunnelling through a retarded potential.

At zero temperature the problem of the lifetime of the metastable state of a tunnelling junction was considered phenomenologically by Calderia and Leggett[6] and microscopically in Ref. 7.

In the present paper we find the decay probability of a current state at any temperature.

Ambegaokar *et al.*[8] gave a microscopic derivation of the effective action. In what follows we express the transition probability in terms of the effective action. We study the various limiting cases for the temperature dependence of that probability. We show that for any current below the critical one there exists a critical temperature T_0 above which the transition probability is determined by the thermal fluctuations and described by the classical formula. We find how T_0 depends on the current through the junction and obtain a general expression for the quantum-mechanical decay probability of the metastable state for currents close to the critical one in the limiting cases of small and large values of the shunting resistance at arbitrary temperatures.

2. TRANSITION PROBABILITY AND EFFECTIVE ACTION

For a given current J the energy $U(\varphi)$ of a Josephson junction equals[9]

$$U(\varphi)=-\frac{J}{e}\varphi-\frac{J_c}{2e}\cos 2\varphi, \tag{1}$$

where 2φ is the phase difference of the order parameters and J_c the critical current of the junction. For currents J less than J_c the energy $U(\varphi)$ as a function of φ has local minima separated by a barrier

$$\delta U=\frac{J_c}{e}\left[\left(1-\left(\frac{J}{J_c}\right)^2\right)^{1/2}-\frac{J}{J_c}\arccos\left(\frac{J}{J_c}\right)\right]. \tag{2}$$

At sufficiently high temperatures the lifetime of the metastable state is determined by thermal fluctuations and proportional to $\exp(\delta U/T)$. When the temperature is lowered, quantum fluctuations which destroy the coherent state described by the phase φ become important. The change in the collective variable φ is connected with changing a large number of single-electron states. The probability for quantum mechanical tunnelling is therefore exponentially small. We shall assume below that this probability is sufficiently small so that the system can reach thermal equilibrium as long as the collective coordinate φ is in the classically accessible region on one side of the barrier.

The transition probability W after a time $t=t_f-t_i$ equals

$$W=Z^{-1}\sum_{i,f}\left|\langle\psi^f|\exp\left(-i\int_{t_i}^{t_f}\hat{H}dt\right)|\psi^i\rangle\right|^2\exp\left(-\frac{E_i}{T}\right);$$
$$Z=\sum_i\exp\left(-\frac{E_i}{T}\right), \tag{3}$$

where ψ^f, ψ^i, E_i are the wave functions and energy of the final f and initial i states and $\hat{H}$ is the total Hamiltonian of the system:

$$\hat{H}=\hat{H}_L+\hat{H}_R+\hat{H}_T+\hat{Q}^2/2C+\Phi^2/2L. \tag{4}$$

In equation (4) $\hat{H}_L$ and $\hat{H}_R$ are the Hamiltonians of the superconductors to the left and the right of the junction:

$$\hat{H}_L=\int d^3\mathbf{r}\psi_{L\sigma}^+(\mathbf{r})\left(-\frac{1}{2m}\frac{\partial^2}{\partial\mathbf{r}^2}-\mu\right)\psi_{L\sigma}(\mathbf{r})-\frac{g_L}{2}\int d^3\mathbf{r}\psi_{L\sigma}^+(\mathbf{r})$$
$$\times\psi_{L\sigma}^+(\mathbf{r})\psi_{L-\sigma}(\mathbf{r})\psi_{L-\sigma}(\mathbf{r}), \tag{5}$$

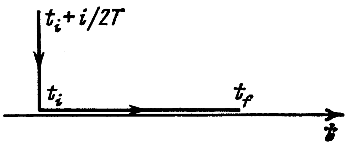

FIG. 1. Integration contour C_1.

$\psi_{L\sigma}(\mathbf{r})$ is the operator annihilating an electron with spin σ,

$$\hat{Q}=\hat{Q}_L-\hat{Q}_R,\quad \hat{Q}_L=e\int d^3r\psi_{L\sigma}^+\psi_{L\sigma},$$

C is the capacitance of the junction, L the inductance of the circuit, $\Phi=\int \mathbf{A}\cdot d\mathbf{l}$ the total magnetic flux in the circuit, $\mathbf{A}$ the vector potential, and $\hat{H}_T$ the tunnelling Hamiltonian:

$$\hat{H}_T=\int\int d^3\mathbf{r}_L d^3\mathbf{r}_R\{\hat{T}(\mathbf{r}_L,\mathbf{r}_R)\psi_{L\sigma}^+(\mathbf{r}_L)\psi_{R\sigma}(\mathbf{r}_R)+\text{c.c.}\}. \quad (6)$$

We can write Eq. (3) for the transition probability W as an integral over the contour C_1 shown in Fig. 1,

$$W=Z^{-1}\sum_{i,f}|\langle\psi^f|S|\psi^i\rangle|^2,\quad S=\exp\left(-i\int_{C_1}\hat{H}dt\right). \quad (7)$$

We use a Hubbard-Stratanovich transformation in order to get rid of the ψ^4 terms in the Hamiltonian. Proceeding as in Ref. 8 we get for the S-matrix the expression

$$S=\int D^2\Delta_L D^2\Delta_R DV\hat{T}\exp\left\{-i\int_{C_1}\hat{H}_{\text{eff}}(t)\,dt\right\}, \quad (8)$$

where $\Delta_{L,R}$ are complex functions, $\hat{T}$ the ordering operator on the contour C_1,

$$\hat{H}_{\text{eff}}=\hat{H}_T+\hat{H}_{L\,\text{eff}}+\hat{H}_{R\,\text{eff}}+\hat{Q}V(t)-\frac{C}{2}V^2(t)+\frac{\Phi^2}{2L}; \quad (9)$$

$$\hat{H}_{L\,\text{eff}}=\int d^3\mathbf{r}\left\{\psi_{L\sigma}^+(\mathbf{r})\left(-\frac{1}{2m}\frac{\partial^2}{\partial\mathbf{r}^2}-\mu\right)\psi_{L\sigma}(\mathbf{r})\right.$$
$$+(\Delta_L^*(\mathbf{r},t)\psi_{L\downarrow}(\mathbf{r})$$
$$\left.\times\psi_{L\uparrow}(\mathbf{r})+\text{c.c.})+\frac{1}{\sigma}|\Delta_L(\mathbf{r},t)|^2\right\}.$$

The quantity $V(t)$ in Eq. (9) has the meaning of the potential in the junction. In zeroth approximation in the transparency of the barrier the path integral over the modulus of Δ can be evaluated by the steepest-descent method. The modulus of Δ is then replaced by its equilibrium value which is independent of coordinates and time. After averaging over the electron states there appear in the effective Hamiltonian terms proportional to the quantities

$$(eV-\partial\varphi/\partial t)^2,\quad (\nabla\varphi(r)-e\mathbf{A})^2, \quad (10)$$

and with coefficients proportional to the volume of the superconductor. The path integral over V can then be evaluated by the steepest descent method. In that approximation

$$eV(t)=\partial\varphi/\partial t. \quad (11)$$

Summation over the initial (i) and final (f) states in Eqs. (3), (7) means also integration over the magnetic field vector potential $\mathbf{A}$. This integration can be performed by the steepest-descent method, which means the substitution

$$e\mathbf{A}=\nabla\varphi(r),\quad e\Phi=\pi N-\varphi, \quad (12)$$

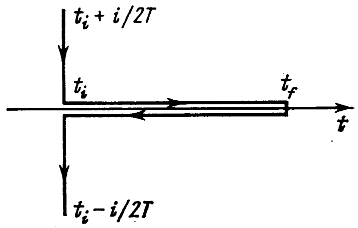

FIG. 2. Integration contour C_2.

where N is the number of flux quanta in the circuit at the initial time. We shall assume that the junction inductance L is sufficiently large and that the magnetic energy is important only when $N\gg 1$. The quantity φ changes by π in the transition from one metastable state to another. Apart from an unimportant constant the quantity $\Phi^2/2L$ can thus be replaced by $-J\varphi/e$, where J is the total current in the circuit. For large inductances we can assume that the total current J is constant.

Taking all the foregoing into account we can write the transition probability W as a trace over electron states and a functional integral over the phase φ:

$$W=Z^{-1}\int D\varphi\,\mathrm{Sp}\,\hat{T}\exp\left\{-i\int_{C_2}dt\left[\hat{H}_{L\,\text{eff}}+\hat{H}_{R\,\text{eff}}+\hat{H}_T\right.\right.$$
$$\left.\left.-\frac{C}{2e^2}\left(\frac{\partial\varphi}{\partial t}\right)^2-\frac{J\varphi}{e}+\hat{Q}\frac{\partial\varphi}{\partial t}\right]\right\}, \quad (13)$$

where the contour C_2 is given in Fig. 2.

As we noted in the Introduction and shall prove in what follows, the averaging over the electron states leads to the arising of a potential barrier for the coordinate φ. We shall assume that the barrier which occurs is quasi-classical. We assume that in the path integral (13) the initial state corresponding to the end points of the contour C_2 lies to the left of the barrier and that the final state corresponding to the point t_f on the contour lies to the right of the barrier. In all other points of the contour the integral is taken over all values of φ. The transition probability was written in Refs. 10, 11 as a path integral.

We change in Eq. (11) to the interaction representation in $\hat{H}_T$. In second order in the barrier transparency the transition probability W takes the form

$$W=Z_1^{-1}\int D\varphi(t)\exp(-A[\varphi]),$$
$$A[\varphi]=-i\int_{C_2}dt\left[\frac{C}{2e^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\frac{1}{e}J\varphi\right]+\mathscr{L}[\varphi], \quad (14)$$

where

$$\mathscr{L}[\varphi]=\frac{1}{2}\left\langle\!\!\left\langle\hat{T}\int_{C_2}dt\int_{C_2}dt_1\hat{H}_T(t)\hat{H}_T(t_1)\right\rangle\!\!\right\rangle. \quad (15)$$

Averaging in Eq. (15) over the electron states leads to the appearance of the Green functions of the left- and right-hand side superconductors for zero transparency of the barrier. Their dependence on the potential $eV=\partial\varphi/\partial t$ reduces thus to a trivial phase factor. Separating these factors, as was done in Ref. 12, we get

$$\mathscr{L}[\varphi]=-\int\limits_{C_2}dt\int\limits_{C_2}dt_1\sum_{\nu,\mu}|T_{\mu\nu}|^2$$
$$\times\{\cos(\varphi(t)-\varphi(t_1))\langle\hat{T}a_\nu(t)a_\nu^+(t_1)\rangle$$
$$\times\langle\hat{T}a_\mu(t_1)a_\mu^+(t)\rangle-\cos(\varphi(t)+\varphi(t_1))$$
$$\times\langle\hat{T}a_\nu(t)a_\nu(t_1)\rangle\langle\hat{T}a_\mu^+(t_1)a_\mu^+(t)\rangle\}. \quad (16)$$

The indexes μ and ν number the states in the left- and right-hand side superconductors. The averaging in Eq. (16) is performed without the potential $V(t)$ for real values of the order parameters $\Delta_{L,R}$. The Green functions depend only on the energy of the states μ and ν and have a steep maximum close to the Fermi surface. The matrix element $|T_{\mu\nu}|^2$ averaged over these states can be expressed in terms of the resistance of the junction above the transition point. Summation over the states μ and ν leads to the appearance of the Green functions $g(t,t')$ and $F(t,t')$ integrated over the energy variable. As a result Eq. (16) becomes

$$\mathscr{L}[\varphi]=-\frac{\pi}{2R_Ne^2}\int\limits_{C_2}dt\int\limits_{C_2}dt_1\{\cos(\varphi(t)-\varphi(t_1))g_L(t,t_1)g_R(t_1,t)$$
$$-\cos(\varphi(t)+\varphi(t_1))F_L(t,t_1)F_R(t_1,t)\}. \quad (17)$$

These Green functions can be expressed in terms of retarded and advanced Green functions in the same way as was done in Ref. 7. Therefore, Eqs. (14) and (16) express the transition probability as a path integral over the variable $\varphi(t)$.

3. EXTREMAL ACTION

With exponential accuracy the transition probability is determined by the extremum of the action $A[\varphi]$. The extremal trajectory corresponding to real t determines with quasi-classical accuracy the motion in the classically accessible region. The extremum of the action then corresponds to values of $\varphi(t)$ which are equal on the two sides of the contour.[11] Such sections of the contour do not contribute to the action. One can thus divide the whole trajectory into three parts: the first is the motion in the classically accessible region up to the instant t_m of tunnelling which we put equal to zero; the second is the tunnelling process (motion along imaginary time along the vertical axis) and, finally, the third is motion in the classically accessible region. We shift the contour of integration so that the vertical section goes through $t=0$ (instant of tunnelling). The value of the path integral does not change when the contour is moved in this way. Up to exponential accuracy the transition probability W equals

$$W\sim\exp\{-A[\varphi_{\text{extr}}]\}. \quad (18)$$

The factor of the exponential in (18) is proportional to $t_f - t_i$ and is determined by the deviation of φ from its extremal value. In this paper we restrict ourselves to calculating the index of the exponent in Eq. (18).

We find the function φ_{extr} from the integral equation

$$\delta A[\varphi]/\delta\varphi=0. \quad (19)$$

One can verify that the Green functions in Eq. (17) have the properties

$$g(t,t_1)=-g(t^*,t_1^*),\quad F(t,t_1)=F(t^*,t_1^*). \quad (20)$$

The solution of Eq. (19) thus satisfies the condition

$$\varphi(t+i\tau)=\varphi(t-i\tau). \quad (21)$$

It also follows from the properties (20), (21) that we can in Eqs. (14), (17) cancel the contributions from the regions whenever one of the arguments t or t_1 lies on the section of the contour parallel to the real axis. The contribution to the extremal action comes therefore only from the vertical section of the contour. Making the substitution $t=-i\tau$ we get

$$A[\varphi]=\int\limits_{-1/2T}^{1/2T}d\tau\Big\{\frac{C}{2e^2}\Big(\frac{\partial\varphi}{\partial\tau}\Big)^2-\frac{1}{e}J\varphi+\frac{\pi}{2R_Ne^2}\int\limits_{-1/2T}^{1/2T}d\tau_1\Big[2\sin^2\Big(\frac{\varphi(\tau)-\varphi(\tau_1)}{2}\Big)g_L(\tau-\tau_1)g_R(\tau-\tau_1)-\cos(\varphi(\tau)+\varphi(\tau_1))$$
$$\times F_R(\tau-\tau_1)F_R(\tau-\tau_1)\Big]+\frac{\pi T^2}{R_{\text{sh}}e^2}\int\limits_{-1/2T}^{1/2T}d\tau_1\frac{\sin^2[(\varphi(\tau)-\varphi(\tau_1))/2]}{\sin^2\pi T(\tau-\tau_1)}\Big\}, \quad (22)$$

where $g(\tau)$ and $F(\tau)$ are Matsubara Green functions. The last term in Eq. (22) has been added because a tunnelling junction often turns out to be shunted by the normal resistance R_{sh}. The contribution to the action arising from this resistance is obtained from Eq. (17) in which the Green function $g(\tau)$ is replaced by its value in the normal metal. This substitution is justified if the main contribution to the shunt resistance arises from a single tunnelling junction inside the normal metal. Such a situation occurs, apparently, in Nb junctions. In these junctions there are regions of the normal phase near the barrier. These regions, in which the gap in the excitation spectrum is small, determine the normal current $I=V/R_{\text{sh}}$ through the junction at low temperatures. The case is also possible when the shunt is a long normal-metal short-circuit. We must then replace $\sin^2\varphi/2$ in the last term in Eq. (22) by a periodic parabola $\varphi^2/4$ (with period 2π). In most limiting cases considered below φ is small and this substitution is unimportant. The resistance of the junction in the normal state R_h equals

$$R_n^{-1}=R_{\text{sh}}^{-1}+R_N^{-1}. \quad (23)$$

The resistance R_N determines the magnitude of the critical current through the junction.[13]

We can look for the solution of the nonlinear integral Eq. (19) with the functional $A[\varphi]$ determined by Eq. (19) in the form of a Fourier series

$$\varphi(\tau)=\varphi_0+\sum_{n=1}^{\infty}a_n\cos(2\pi T\tau n). \quad (24)$$

In a number of limiting cases it is possible to find an analytical expression for the coefficient a_n of the extremal action $A[\varphi_{\text{extr}}]$ and the transition probability W. In the model of quantum tunnelling with friction these cases were studied in Ref. 14.

4. HIGH TEMPERATURES

At sufficiently high temperatures $T>T_0$ the unique solution of Eq. (19) is time-independent:

$$\varphi=\varphi_0=\text{const}. \quad (25)$$

In this case $A = U(\varphi_0)/T$ and $U(\varphi)$ is given by Eq. (1). The point φ_0 corresponds to the maximum of the energy $U(\varphi)$. It is convenient in what follows to take for the energy origin the point where $U(\varphi)$ is a minimum. For that choice of energy origin the partition function Z gives merely a multiplying factor to the exponential. With exponential accuracy the transition probability equals

$$W \sim \exp\{-\delta U/T\}, \tag{26}$$

where the quantity δU is given by Eq. (2). This solution corresponds to the classical transition due to thermal fluctuations. When $T > T_0$ this solution is stable. Expanding the action $A[\varphi]$ in powers of a_n all coefficients of a_n^2 are positive. The temperature T_0 at which quantum tunnelling becomes important is determined by the condition that the coefficient of a_1^2 vanishes. Close to that temperature

$$A[\varphi] = U(\varphi_0)/T + Ba_1^2 + B_1 a_1^4 + Da_1^2 a_2 + Ea_2^2, \tag{27}$$

where

$$\begin{gathered}B = \frac{C}{e^2}\pi^2 T + 0.5\left[K(1) + (L(0)+L(1))\cos 2\varphi_0 + \frac{\pi}{R_{sh}e^2}\right];\\ B_1 = -\frac{1}{8}\left(K(1) - \frac{K(2)}{4}\right) - \frac{\cos 2\varphi_0}{16}[1.5L(0) + 2L(1)\\ +0.5L(2)] - \frac{\pi}{16R_{sh}e^2},\\ D = -{}^1/_4 \sin 2\varphi_0 [L(0) + 2L(1) + L(2)],\\ E = \frac{4\pi^2 CT}{e^2} + \frac{K(2)}{2} + \frac{\cos 2\varphi_0}{2}[L(0) + L(2)] + \frac{\pi}{R_{sh}e^2}.\end{gathered} \tag{28}$$

The quantities $K(m)$ and $L(m)$ in Eqs. (28) can be expressed in terms of the Matsubara Green functions $g(\omega)$ and $F(\omega)$ through the formulae

$$\begin{gathered}K(m) = -\frac{\pi}{4R_N e^2}\sum_{\omega_n} g_L(\omega_n)\\ \times [2g_R(\omega_n) - g_R(\omega_{n+m}) - g_R(\omega_{n-m})],\\ L(m) = \frac{\pi}{4R_N e^2}\sum_{\omega_n} F_L(\omega_n)[F_R(\omega_{n+m}) + F_R(\omega_{n-m})].\end{gathered} \tag{29}$$

For superconductors without paramagnetic impurities these functions are equal to

$$g(\omega_n) = \frac{-i\omega_n}{(\omega_n^2+\Delta^2)^{1/2}},\ F(\omega_n) = \frac{\Delta}{(\omega_n^2+\Delta^2)^{1/2}},\ \omega_n = \pi T(2n+1). \tag{30}$$

If the capacitance C is sufficiently large, or the shunt resistance is small, or the current density is close to critical, the temperature $T_0 \ll T_c$. In that case we have for superconductors without paramagnetic impurities

$$\begin{gathered}K(m) = m^2 \frac{3\pi^3 \Delta_L^2 \Delta_R^2 T}{2^{3/2} R_N e^2 (\Delta_L^2+\Delta_R^2)^{5/2}}\\ \times F\left(\frac{5}{4}, \frac{7}{4}, 2; \left(\frac{\Delta_L^2-\Delta_R^2}{\Delta_L^2+\Delta_R^2}\right)^2\right),\\ L(m) = \frac{J_c}{2eT} - \frac{\pi^3 T \Delta_L \Delta_R m^2}{2^{5/2} R_N e^2 (\Delta_L^2+\Delta_R^2)^{3/2}}\\ \times F\left(\frac{3}{4}, \frac{5}{4}, 2; \left(\frac{\Delta_L^2-\Delta_R^2}{\Delta_L^2+\Delta_R^2}\right)^2\right),\end{gathered} \tag{31}$$

where F is a hypergeometric function. When the superconductors are identical Eqs. (31) for $K(m)$ and $L(m)$ become

$$K(m) = m^2 \frac{3\pi^3}{16R_N e^2}\frac{T}{\Delta},\quad L(m) = \frac{J_c}{2eT} - \frac{\pi^3 m^2}{16R_N e^2}\frac{T}{\Delta}. \tag{32}$$

The temperature T_0 is determined by the equation $B(T_0) = 0$. If $T_0 \ll T_c$ this equation has the form

$$C^*\pi^2 T_0 + \frac{\pi}{2R_{sh}} = \frac{eJ_c}{2T_0}\left[1 - \left(\frac{J}{J_c}\right)^2\right]^{1/2},\quad C^* = C + \frac{e^2}{2\pi^2 T}K(1). \tag{33}$$

The quantity C^* has the meaning of a renormalized capacitance. The transition probability is determined by the extremal value of $A[\varphi]$ with respect to φ_0, a_1, a_2 and is equal to

$$\begin{gathered}W \sim \exp\left\{-\frac{\delta U}{T} + \frac{1}{eT}J_c\left[1 - \left(\frac{J}{J_c}\right)^2\right]^{1/2}\right.\\ \times B^2\left[\left(\frac{J}{J_c}\right)^2 (L(0)+L(1))^2\right.\\ \left.\left. + \frac{4J_c}{eT}\left(1 - \left(\frac{J}{J_c}\right)^2\right)^{1/2}\left(B_1 - \frac{D^2}{4E}\right)\right]^{-1}\right\}.\end{gathered} \tag{34}$$

The coefficients B, B_1, D, E in Eq. (34) are determined by Eqs. (28) in which we must put

$$\sin 2\varphi_0 = J/J_c,\quad \cos 2\varphi_0 = -[1 - (J/J_c)^2]^{1/2}. \tag{35}$$

Equation (33) is valid for such values of current and temperature that the coefficient $B < 0$ and sufficiently small so that the second term in Eq. (33) is small compared to the first one. For sufficiently large values of the current or the temperature when $B > 0$ we must retain in Eq. (34) only the first term.

When $T_0 \ll T_c$ we find from Eqs. (28), (32), (34)

$$\begin{gathered}W \sim \exp\left\{-\frac{\delta U}{T} + \frac{J_c X_0}{2eT}(X - X_0)^2\right.\\ \left.\times\left[1 + \left(\frac{J}{J_c}\right)^2 \frac{2y}{1+3y} - \frac{X_0^2}{2(1+y)}\right]^{-1}\right\},\end{gathered} \tag{36}$$

where

$$X = [1 - (J/J_c)^2]^{1/2},\ y = 2\pi TC^* R_{sh},\ X_0 = \pi(1+y)T/eJ_c R_{sh}.$$

In the point T_0 there appears thus a jump in the second derivative of the logarithm of the transition probability with respect to the temperature (with respect to the magnitude of the current). This singularity is smeared out by fluctuations near the extremal trajectory. The corresponding calculation neglecting dissipation was performed in Ref. 15.

5. CURRENT CLOSE TO CRITICAL

For currents close to critical the potential barrier is concentrated in a narrow region near $\varphi = \pi/4$. We therefore look for $\varphi(\tau)$ in the form

$$\varphi(\tau) = {}^1/_2 \arcsin(J/J_c) + \tilde\varphi(\tau),\quad |\tilde\varphi(\tau)| \ll 1. \tag{37}$$

In this case the temperature T_0 is small compared to T_c. The effective tunnelling time $\tau \gg \Delta^{-1}$. We can thus use for the action (22) the adiabatic approximation

$$A_0[\varphi]=\int_{-1/2T}^{1/2T} d\tau\left\{\frac{C^*}{2e^2}\left(\frac{\partial\tilde{\varphi}}{\partial\tau}\right)^2\right.$$

$$\left.+\frac{J_c}{c}\tilde{\varphi}^2\left(X-\frac{2}{3}\tilde{\varphi}\right)+\frac{\pi T^2}{4R_{sh}e^2}\int_{-1/2T}^{1/2T} d\tau_1\frac{(\tilde{\varphi}(\tau)-\tilde{\varphi}(\tau_1))^2}{\sin^2(\pi T(\tau-\tau_1))}\right\}. \quad (38)$$

Writing $\tilde{\varphi}(\tau)$ in the form

$$\tilde{\varphi}(\tau)=\sum_{n=-\infty}^{\infty} b_n\exp(i\cdot 2\pi T\tau n)$$

and substituting this expression into Eq. (38) we get

$$A_0[\varphi]=\sum_{n=-\infty}^{\infty} b_n^2\left[\frac{C^*}{2e^2T}(2\pi nT)^2\right.$$

$$\left.+\frac{\pi}{R_{sh}e^2}|n|+\frac{J_cX}{eT}\right]-\frac{2J_c}{3eT}\sum_{n,m=-\infty}^{\infty} b_nb_mb_{n+m}. \quad (39)$$

Minimizing the functional $A_0[\varphi]$ with respect to the Fourier coefficients b_n we get a set of equations for these coefficients:

$$b_n\left[\frac{4\pi^2TC^*}{e^2}n^2+\frac{2\pi|n|}{R_{sh}e^2}+\frac{2J_cX}{eT}\right]$$

$$-\frac{2J_c}{eT}\sum_{m=-\infty}^{\infty} b_mb_{m+n}=0. \quad (40)$$

One can solve Eq. (40) analytically for any temperature in two limiting cases: small and large shunting resistance.

A. Small shunting resistance ($2\pi C^*R_{sh}\ T_0\ll 1$)

For sufficiently small values of the shunting resistance when the quantity $R_{sh}C^*$ is small compared to the characteristic tunnelling time we can neglect the first term in Eq. (40). The resulting equation has the exact solution:

$$b_n=B\exp(-b|n|),\quad B=\pi T/J_cR_{sh}e,$$
$$\mathrm{th}\, b=B/X. \quad (41)$$

Up to terms of second order in the capacitance C^* we can obtain the action A by substituting the solution (41) into Eq. (39):

$$A=\frac{J_c(1-(J/J_c)^2)^{3/2}}{2eT_0}\left\{1-\frac{1}{3}\left(\frac{T}{T_0}\right)^2\right\}, \quad (42)$$

where T_0 is given by Eq. (33). To first order in the parameter $2\pi C^*R_{sh}\ T_0\ll 1$ the effect of the capacitance has been reduced to a renormalization of the temperature T_0.

B. Large capacitance

For large values of the capacitance or large values of the shunting resistance the equation for the phase reduces in the main approximation to a differential equation:

$$\frac{C^*}{e^2}\frac{\partial^2\tilde{\varphi}}{\partial\tau^2}=\frac{2J_c}{e}[X\tilde{\varphi}-\tilde{\varphi}^2]. \quad (43)$$

In this approximation the calculation of the lifetime reduces to the quantum mechanical problem of the tunnelling probability through a quasi-classical barrier. The solution of Eq. (43) has the form

$$\varphi(\tau)=\varphi_1-(\varphi_1-\varphi_2)\,\mathrm{sn}^2\left[\tau\left(\frac{eJ_c(\varphi_1-\varphi_3)}{3C^*}\right)^{1/2}\right], \quad (44)$$

where $\varphi_1>\varphi_2>\varphi_3$ are the roots of the cubic equation

$$X\varphi^2-{}^2/_3\varphi^3-E^*=0. \quad (45)$$

The extremal energy E^* for which tunnelling occurs is determined from the condition of periodicity of the function $\varphi(\tau)$ with period $1/T$. From this condition we get

$$\frac{1}{2T}\left(\frac{eJ_c}{3C^*}\right)^{1/2}=\frac{1}{(\varphi_1-\varphi_3)^{1/2}}K(k),\quad k=\left(\frac{\varphi_1-\varphi_2}{\varphi_1-\varphi_3}\right)^{1/2}, \quad (46)$$

where $K(k)$ is a complete elliptical integral.

Expanding the function $\tilde{\varphi}(\tau)$ in a Fourier series we get for the coefficients b_n the equation

$$b_{n\neq 0}=(\varphi_1-\varphi_2)\frac{n\pi^2}{2k^2K^2(k)}\left[\mathrm{sh}\left(\frac{\pi nK(k')}{K(k)}\right)\right]^{-1},$$

$$k'=(1-k^2)^{1/2}. \quad (47)$$

We can obtain the action A up to terms of first order in R_{sh} by substituting the solution (44) into Eq. (39):

$$A=\frac{J_cE^*}{eT}+\frac{\pi}{2}\left(\frac{J_cC^*}{6e^3}\right)^{1/2}(\varphi_1+\varphi_2-2\varphi_3)^{1/2}$$

$$\times(\varphi_1-\varphi_2)^2F\left(-\frac{1}{4},\frac{1}{4},2;\left(\frac{\varphi_1-\varphi_2}{\varphi_1+\varphi_2-2\varphi_3}\right)^2\right)+\frac{2\pi}{R_{sh}e^2}\sum_{n=1}^{\infty} nb_n^2. \quad (48)$$

The temperature dependence of the lifetime was found numerically in Ref. 16, neglecting dissipation. At temperature $T=0$ Eq. (48) is equal to

$$A=\frac{6}{5}X^{5/2}\left(\frac{2C^*J_c}{e^3}\right)^{1/2}\left[1+\frac{45\zeta(3)}{4\pi^4R_{sh}C^*T_0}\right], \quad (49)$$

where the quantities T_0 and C^* are given by Eq. (33). Expression (49) is the same as the result of Ref. 7 and differs from the phenomenological result of Ref. 6 only by the renormalization of the capacitance.

The resistance of the junction at zero temperature was denoted above by R_{sh}. If R_{sh} is infinite the main effect at low temperatures is the renormalization of the capacitance. One needs not take into account the exponentially small corrections to the resistance. If T_0 is of the order of T_c we must use for the effective action the general expression [Eq. (22)]. In that case the strong dispersion both of the potential barrier and of the magnitude of the resistance is important.

6. CONCLUSION

The lifetime of the metastable current state of a superconducting junction is determined with exponential accuracy by the effective action $A(T,J)$. At a temperature T_0 given by Eq. (33) there occurs a change in regime from classical for $T>T_0$ to quantum mechanical for $T<T_0$. In the point T_0 the derivative $\partial A/\partial T$ is continuous but there is a jump in the second derivative $\partial^2A/\partial T^2$. When the temperature is further lowered the quantity $A(T)$ increases to a finite value at $T=0$. The temperature dependence of the effective action is determined by the magnitude of the parameter $2\pi T_0R_{sh}C^*$. For small values of this parameter $A(T)$ has a simple form [Eq.

(42)]. In that case $A(0) = 1.5\,A(T_0)$. In the opposite limiting case (small viscosity)

$$A(0)=\frac{18}{5\pi}A(T_0)=1.146\,A(T_0).$$

Such a behavior was observed experimentally.[4,5]

The detailed behavior of the system in the cases of large and small viscosity is different. However, qualitatively a small shunting resistance has the same effect as a large junction capacitance. Increasing the capacitance and lowering the shunt resistance lower the temperature T_0 and decrease the tunnelling probability at low temperatures.

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