Features of current flow in media with strong nonlinearity

A. M. Dykhne, A. M. Volchek, N. I. Gapotchenko, and M. D. Taran

(Submitted 20 April 1983)
Zh. Eksp. Teor. Fiz. **85**, 1465–1469 (October 1983)

We study the flow of current in a conducting medium whose conductivity is a step function of the electric field intensity. In such media the current flow gives rise to regions in which the electric field is constant. Under certain conditions, sharply outlined current jets can appear in an initially homogeneous medium. Two- and three-dimensional problems that cannot be solved exactly are numerically simulated.

PACS numbers: 72.10.Bg

In a number of physical objects (weakly ionized gases, semiconductors) a situation arises in which the system conductivity is a rapidly growing function of the electric field. This rapid growth can be due to the fact that the electron distribution acquires in strong fields an "ionizing tail," as a result of which the number of electrons or of electron-hole pairs can turn out to be a rapidly growing function of the electric field.

At the same time, the only case that lends itself to theoretical analysis is that of small nonlinearity, where the picture does not differ substantially from the linear case. A limiting variant of strong nonlinearity is, for example, a steplike dependence of the conductivity on the electric field.

The purpose of the present communication is an examination of the features of current flow in a medium with nonlinear conductivity in this limiting model, as well as confirmation of the validity of the model.

Let the conductivity as a function of the electric field be of the form

$$\sigma=\begin{cases}\sigma_2, & |E|<E_0\\ \sigma_1, & |E|>E_0\end{cases}\quad(\sigma_2<\sigma_1). \tag{1}$$

On the face of it, this model is inconsistent. Assume that the conductivity as a function of the coordinates takes on only two values, σ_2 and σ_1. We consider the boundary between the highly (I) and poorly (II) conducting regions. The conditions satisfied on this boundary are $E_{\mathrm{I}t}=E_{\mathrm{II}t}$ and $\sigma_1 E_{\mathrm{I}n}=\sigma_2 E_{\mathrm{II}n}$, i.e. continuity of the tangential fields and of the normal currents. It follows then from them and from the inequality $\sigma_1>\sigma_2$ that $E_{\mathrm{I}}^2\leqslant E_{\mathrm{II}}^2$. Thus the condition $|E_{\mathrm{I}}|\geqslant|E_{\mathrm{II}}|$ called for by Eq. (1) cannot be satisfied.

This contradiction can be eliminated by assuming the existence, besides of regions I (where $|E|>E_0$) and II (where $|E|<E_0$), of a certain region III in which $|E|=E_0$ and the conductivity has an intermediate range $\sigma_2\leqslant\sigma_{\mathrm{III}}\leqslant\sigma_1$.

If our model is not inconsistent, this leads to an interesting physical consequence, namely, the existence of a whole region of spaces where the electric field is constant.

Logically admissable is also a second variant in which the model with the jump is physically inconsistent. This would mean that the jump "smears out," and the result would depend on the character of the smearing. In this case and extremely abrupt step would be meaningless, and the result would be sensitive to small changes of the function $\sigma(E)$.

We shall resolve this dilemma by considering a two-dimensional problem, when the transition to the hodograph plane linearizes the equation. This makes it possible to answer the question using physical considerations. The aforementioned dilemma is resolved in favor of the first variant, and thus an abrupt steplike dependence of the conductivity has a physical meaning and can simulate a real nonlinearity.

We consider thus the two-dimensional problem. The system of equations

$$\operatorname{rot}\mathbf{E}=0,\quad \operatorname{div}\sigma(|E|)\mathbf{E}=0 \tag{2}$$

is linearized by introducing, following Ref. 1, a potential such that

$$d\Phi=x\,dE_x+y\,dE_y,$$

and by transforming to the hodograph plane. In the polar variables E and θ, the equation for the potential is

$$\frac{1}{E}\frac{\partial}{\partial E}\left(\sigma E\frac{\partial\Phi}{\partial E}\right)+\frac{d(\sigma E)}{dE}\frac{1}{E^2}\frac{\partial^2\Phi}{\partial\theta^2}=0. \tag{3}$$

This equation defines the continuity of a certain current in the E plane, with the corresponding conductivity anisotropic. The conductivity tensor has the components

$$\sigma_{EE}=\sigma(E),\quad \sigma_{\theta\theta}=\sigma(E)+E\,d\sigma/dE. \tag{4}$$

These equations solve right away the initially posed problem. Equations (3) and (4) have a simple physical meaning. They pertain to the problem of current flow in an isotropically conducting plane in which a thin ring of increased conductivity is placed. In the limit when $\sigma(E)$ is a step, this ring becomes infinitely thin and superconducting, but its total resistance remains finite. We have thus a purely physical corroboration of the limiting transition.

At $|\mathbf{E}|=E_0$, the superconducting ring C can be replaced by boundary conditions. One of them follows from the continuity of the tangential fields. The second is easily obtained by integrating (5) with respect to E over an infinitely small interval from E_0-0 to E_0+0. As a result we get

$$\left\{\frac{\partial\Phi}{\partial\theta}\right\}=0, \tag{5}$$

$$\left\{\sigma\frac{\partial\Phi}{\partial E}\right\}=\frac{\sigma_2-\sigma_1}{E_0}\frac{\partial^2\Phi}{\partial\theta^2}, \tag{6}$$

where $\{\dots\}$ denotes a jump of the corresponding quantity on going through the ring.

To obtain a complete solution of the problem we must

 0038-5646/83/100849-03$04.00

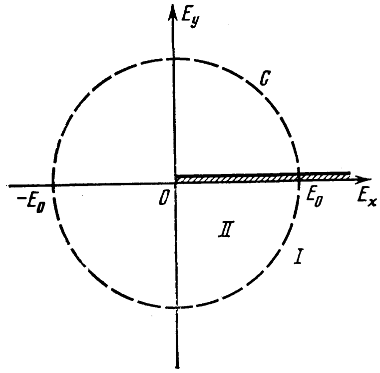

FIG. 1. Mapping of a point dipole in the E plane.

take into account, besides the universal boundary conditions (5) and (6), also the boundary conditions on the physical electrodes that are transferred to the E-plane. These boundary conditions can be formulated as the condition of a direct boundary-value problem only for the simplest electrode geometry in physical space, viz., pointlike electrodes and electrodes in the form of straight-line segments. Boundary-value problems of this kind were investigated in the literature by the methods of complex-variable theory (see, e.g., Ref. 2). In particular, an existence and uniqueness theorem was proved for their solution.

In a number of cases an exact analytic solution can be found. One of the simplest is the case of a point dipole. We note right away the features that follow from the assumed model.

The geometry of this problem is such that the image in the E plane is represented by one electrode—a half-plane. The superconducting ring separates near its boundary the highly and poorly conducting regions (Fig. 1).

As $E \to \infty$ the current does not feel the inhomogeneous region and is determined only by the dipole moment P, therefore at large E we have

$$\Phi = 2P^{1/2}E^{1/2}\sin(\theta/2).$$

We seek a solution in the form $\Phi = f(E)\sin(\theta/2)$. The function Φ is harmonic if f contains terms proportional to $E^{\pm 1/2}$. For the inner region we should confine ourselves only to the term $\sim E^{1/2}$, inasmuch as at $f \sim E^{-1/2}$ the total current di-

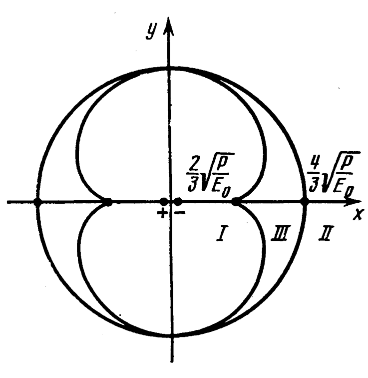

FIG. 2. The three field regions at $\varkappa = 0$.

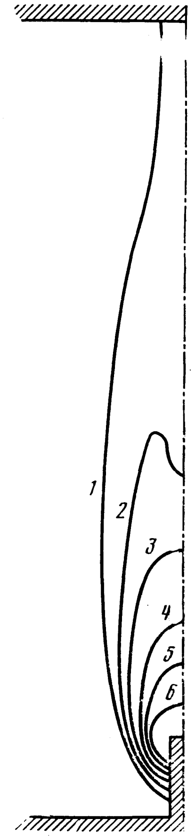

FIG. 3. Level lines of the conductivity σ: 1 — 0.01; 2 — 0.2; 3 — 0.4; 4 — 0.6; 5 — 0.8; 6 — 1,0.

verges. Finding the constant coefficients from the conditions (5) and (6), we get

$$\Phi = \begin{cases} 2P^{1/2}\left(E^{1/2} + \dfrac{1-\varkappa}{3+\varkappa}E_0E^{-1/2}\right)\sin\dfrac{\theta}{2}, & E > E_0 \\ 2P^{1/2}\dfrac{4}{\varkappa+3}E^{1/2}\sin\dfrac{\theta}{2}, & E < E_0 \end{cases},$$

where $\varkappa = \sigma_2/\sigma_1 < 1$.

Figure 2 shows three characteristic regions of the field as $\varkappa \to 0$. There exists an intermediate region III with a constant field but with variable conductivity that reaches in direct space its maximum dimensions. It remains bounded and does not tend to approach the electrodes. The current is concentrated in the highly conducting region, and the distribu-

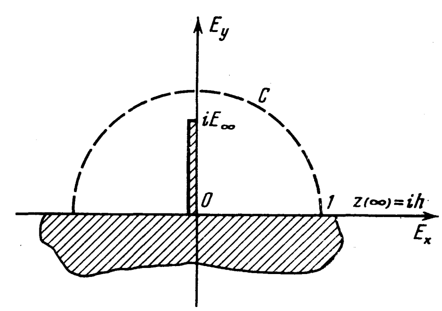

FIG. 4. Mapping of flat electrodes with a post in the E plane.

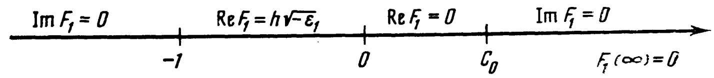

FIG. 5. Corner condition for function $F_1(\varepsilon_1)$; $C_0 = \frac{1}{4}(E_\infty - 1/E_\infty)^2$.

tion of the electric field deviates relatively little from the linear case.

It must be noted that in the intermediate region in the physical plane the potential φ satisfies the eikonal equation

$$\mathrm{grad}^2\,\varphi = E_0^2$$

and the force lines are straight-line segments. The location and shape of this section depend in the general both on the poorly and on the highly conducting regions. Given the geometry of the physical conductors and the value of the current, the decisive role is assumed by the field values E_0 at which the conductivity jump takes place.

Strong-nonlinearity effects appear in the case of an "infinitely" high step: $\sigma_2 \ll \sigma_1$. The presence of a small parameter simplifies considerably the problem and reduces it to an analysis of one of the regions (I or II). We have then the following possibilities:

1. The entire current flows in the highly conducting region I, therefore its boundary is both a line of equal absolute value of the field and a force line. The region I and the E plane impose on the contour C the condition

$$\partial^2\Phi/\partial E^2 = 0. \tag{7}$$

Using such an approximation for the analysis of the problem of two unlike electrodes at a finite distance from each other, we can note that as the field E_0 on the jump tends to a certain critical value the current contracts to the axis between the electrodes.

2. Current flows from I into II through the finite region III. In this case the region with high conductivity contracts to a certain point A, in which the field tends to infinity and straight force lines emerge from a single point.

This approach makes it possible to investigate analytically the perturbations of a homogeneous field of a conducting plane, due to the inhomogeneity of the dimension h of its boundary. We consider in the Appendix an inhomogeneity in the form of a thin projection. When the homogeneous field in the plane E_∞ is close to the critical field E_0, a long narrow channel is produced and the current density in it is much higher than in the surrounding medium. In contrast to a linear medium, its width always remains restricted to a value on the order of h, and the current varies slowly over the length of the channel.

In the three-dimensional case the foregoing approach does not lead to linearization of the system (2). We solved such problems by numerical methods. The numerical algorithm was based on the use of conservative difference schemes[4] and of a modified Newton's method.[5] The algorithm was tested against the analytic solutions constructed above. The obtained numerical solutions agree well with the corresponding analytic ones. By way of one example of solving three-dimensional problems we present the results of numerical simulation of the problem of a thin cylindrical post on an anode, with anode potential $\varphi_a = 0.4$ and a cathode potential $\varphi_c = 0$ distance between electrodes $d = 2$, post length $l = 0.2$, post radius $R = 0.05$, $E_0 = 0.201$, $\sigma_1 = 1$, and $\sigma_2 = 0.01$.

Figure 3 shows the conductivity level lines obtained in the calculation. It can be seen from the figure that the channel has a transverse dimension of the order of the length of the post.

The numerical experiments have shown that the conclusions drawn above on the basis of exact solutions for the two-dimensional case can be extend to a three-dimensional one.

In conclusion, we wish to thank A. P. Napartovich for interest in the work and for helpful discussions.

APPENDIX

The theory of functions of complex variable permits substantial simplification of the mathematical aspect of the problem. Regarding the radius vector and the electric field as complex quantities, $z = x + iy$ and $\varepsilon = E_x + iE_y$, we introduce in the E plane a complex potential $F(\varepsilon)$ such that $\Phi = \mathrm{Re}\,F$, $\bar z = dF/d\varepsilon$.

The plane $y = 0$ with homogeneous field iE_∞ and having a thin post from $z = 0$ to $z = ih$ is equivalently mapped into the E plane (Fig. 4).

In the solution with respect to the small parameter $\varkappa$ in the second variant we have on the contour C

$$\mathrm{Im}\,F\varepsilon = -h\,\mathrm{Re}\,\varepsilon.$$

Without loss of generality, we put $E_0 = 1$. Carrying out the conformal transformation $\varepsilon_1 = -\frac{1}{4}(\varepsilon + 1/\varepsilon)^2$ on the real axis of the ε_1 plane, we obtain the conditions for the imaginary and real parts of the function $F_1 = iF\varepsilon$ (see Fig. 5).

The solution is given by the Keldysh-Sedov formula[3]

$$\bar z = -\frac{h}{\pi i}[(-\varepsilon_1)^{1/2} + (-1-\varepsilon_1)^{1/2}]\left(\frac{\varepsilon_1+1}{\varepsilon_1-C}\right)^{1/2}\int_{-1}^{0}\frac{(\tau-C_0)^{1/2}(-\tau)^{1/2}}{(\tau-1)^{1/2}(\tau-\varepsilon_1)}d\tau.$$

We consider now real values of ε_1 on the boundary of the region, such that $C_0 \ll |\varepsilon_1| \ll 1$ (as $C_0 \to 0$). The integral can then be easily estimated: its imaginary part is equal to 2 to $o(\varepsilon_1)$. As $\varepsilon_1 \to 0$ the real part of z on the boundary has as its limit $2h/\pi$. Since the curvature of the boundary does not reverse sign, the same quantity determines the half-width of the channel.

[1]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics) Nauka, p. 523, 1954 [Pergamon, 1959].
[2]N. I. Muskhelishvili, Singulyarnye integral'nye uravneniya (Singular Integral Equations), Nauka 1977.
[3]M. A. Lavrent'ev and B. V. Shabat, Metody teorii funktsii kompleksnogo peremennoge (Methods of the Theory of Functions of Complex Variable), Nauka, p. 306, 1973.
[4]A. A. Samarskii, Teoriya raznostnykh skhem (Theory of Difference Schemes) Nauka, Chap. VII, 1977.
[5]L. V. Kantorovich and G. P. Akilov, Finktsional'nyi analiz (Functional Analysis): Nauka, Chap. XVIII, 1977.

Translated by J. G. Adashko