Concerning the nature of the λ-transition order parameter

Yu. A. Nepomnyashchiĭ

Perm' State University

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It is shown on the basis of an analysis of the equation $\Sigma_{12} = 0$ [A. A. Nepomnyashchiĭ and Yu. A. Nepomnyashchiĭ, JETP Lett. **21**, 1 (1975); Sov. Phys. JETP **48**, 493 (1978)] that the nonzero c-number component $\langle\hat{\psi}\rangle = \psi$ (the condensate wave function) of the initial field operator $\hat{\psi}$, while involved in the criterion for superfluid ordering, is not the quantity that characterizes the field aspects of the macroscopic motion of the superfluid, and can be used as the order parameter in the effective Hamiltonian of the λ-transition theory. The variable characterizing the analogy between the superfluid state (of the superfluid's long-wave coherent subsystem) and a classical field with a nonsingular pair Hamiltonian is the c component $\langle\hat{\tilde{\psi}}\rangle = \tilde{\psi}$ (the "macroscopic wave function") of some "effective" field operator $\hat{\tilde{\psi}}$, a component which corresponds to the choice $|\tilde{\psi}| = (\rho_s/m)^{1/2}$, which is in fact realized in the Ginzburg-Pitaevskiĭ theory in both its original [V. I. Ginzburg and L. P. Pitaevskiĭ, Sov. Phys. JETP **7**, 858 (1958); L. P. Pitaevskiĭ, Sov. Phys. JETP **8**, 282 (1959)] and modified [V. L. Ginzburg and A. A. Sobyanin, Sov. Phys. Usp. **19**, 773 (1976)] form. A field-theoretic description in terms of variables that are not linearly related to $\tilde{\psi}$ is made complicated by the infrared anharmonicity anomalies of the zero-point and thermal oscillations of the field modes: the nonanalyticity of the field Hamiltonian's coefficients, which characterizes specific properties of the condensate (first and foremost the divergence of the longitudinal static susceptibility at all $T \leqslant T_c$, but not at $T \to T_c$) and the excitation gas (the second-sound pole). The infrared anomalies are due to the degeneracy of the superfluid state and the presence at $T > 0$ of an incoherent superfluid component; the equation $\Sigma_{12}(0) = 0$ is a reflection of them in the variables $\hat{\psi}$. It is shown that a similar situation arises in any phase transition connected with spontaneous continuous-symmetry breaking: it is necessary to use in an effective Hamiltonian of the Ginzburg-Landau type some effective order parameter $\langle\hat{\tilde{x}}\rangle = \tilde{x}$ instead of the original one $\langle\hat{x}\rangle = x$.

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1. INTRODUCTION

The vanishing, as a result of the infrared divergence of the diagrams,[1] of the anomalous self-energy part $\Sigma_{12}(0)$ for any interaction potential V_p that is nonsingular at $\mathbf{p}\to 0$ deprives this quantity of its role as the principal characteristic of the superfluid state, a role which is assigned to it in many Bose-liquid investigations based on the use of the field-theoretic method[2] (at $T = 0$ the sound-speed $c \sim [\Sigma_{12}(0)]^{1/2}$ (Refs. 2–5),[1)] for $T \to T_c$ the relaxation time $\tau \sim [\Sigma_{12}(0)]^{-1}$ (Ref. 5), etc.). This role, which is similar to the role played by Δ in the theory of superconductivity,[3] is taken on by $\Sigma_{12}(0)$ (a "vertex" with two ingoing lines) even in the first (Bogolyubov[7]) approximation (BA) of perturbation field theory,[2] in which $\Sigma_{12}(0) = nV_0$ is the coefficient in the anomalous terms in the Bogolyubov Hamiltonian $\hat{H}^B$ responsible for the "superfluid reconstruction" of the spectrum for $p \to 0$.

The equality $\Sigma_{12}(0) = 0$ has cast doubt on the main results of the microscopic theory of superfluidity—the allowance for the arbitrarily strong repulsion (through the summation of the ladder diagrams[2]: $\Sigma_{12}(0) \approx 4\pi m^{-1} n_0 f_0$, $c = [\Sigma_{12}(0)/m]^{1/2}$) and the proof of the strictly hydrodynamic nature of the long-wave excitations of a Bose system with a condensate in the arbitrary case of strong interactions (by summing all the diagrams in the $p \to 0$, $\varepsilon = cp$, $c = (dP/d\rho)^{1/2}$ long-wave asymptotic form[8]). In the alternative field-theoretic method that is free from an infrared divergence, namely, perturbation theory in the hydrodynamic variables $H(\hat{\psi}, \hat{\psi}^+) \to \mathrm{H}(\hat{n}, \hat{\mathbf{v}})$ (Ref. 9):

$$H(\hat{\psi}, \hat{\psi}^+) = \int \hat{\psi}^+ \left(-\frac{\hbar^2\nabla^2}{2m}\right) \hat{\psi}\, d\mathbf{r} + \frac{1}{2}\int V(\mathbf{r}-\mathbf{r}')\, \hat{\psi}^+(\mathbf{r})\, \hat{\psi}^+(\mathbf{r}')\, \hat{\psi}(\mathbf{r}')\, \hat{\psi}(\mathbf{r})\, d\mathbf{r}\, d\mathbf{r}', \quad (1)$$

$$\hat{n} = \hat{\psi}^+\hat{\psi}, \quad \frac{1}{2}(\hat{n}\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{n}) = \hat{\mathbf{j}} = -\frac{i\hbar}{2m}(\hat{\psi}^+\nabla\hat{\psi} - \nabla\hat{\psi}^+\hat{\psi}) \quad (2)$$

(an improved version of which—one with a modified metric in the space of the states of $H_{\mathrm{BZ}}(n_{\mathbf{p}}, \frac{\partial}{\partial n_{\mathbf{p}}})$ (Refs. 10,11)—does not contain ultraviolet divergence as well), while it gives excellent results in the lowest orders of the perturbation theory (thus confirming, as is easy to verify, the Landau quantum ($T = 0$) and two-fluid ($T > 0$) hydrodynamics[12]), raises fundamental difficulties when solving the above-indicated general problems of the microtheory. Namely, the very condition $V_{p\to\infty} < V_{p_0}(p/p_0)^{-[d-1+\delta]}$ ($\delta > 0$ and d is the dimensionality of the space)[13] for the coincidence of the spectra of $H(\hat{\psi}, \hat{\psi}^+)$ and $H(\hat{n}, \hat{\mathbf{v}})$ excludes a potential with a "hard core." The $p \to 0$ hydrodynamic asymptotic form apparently requires limitations of the type $\langle\hat{\psi}\rangle \neq 0$, which are difficult to formulate in terms of[2)] $\hat{n}$ and $\hat{\mathbf{v}}$. Analysis[1] of the role of the infrared divergence, particularly of the limits of applicability of the usual methods of summing the field diagrams, does not change the main results of the "naive" anal-

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ysis performed with $c=(4\pi n_0 f_0)^{1/2}/m$, $c=(dP/d\rho)^{1/2}$. They are merely rederived with allowance for the equation $\Sigma_{12}(0)=0$ and supplemented by results that are at variance with the perturbation-theory approximations: a divergent nonpole correction to $G_{ik}(p\to 0)$, (15), nonanalytic terms in the expression (12) for $\Sigma_{ik}(p\to 0)$, exact formulas for the partial derivatives (8), (10), etc. But the following questions remain open: How does the "non-Bogolyubov" behavior of $\Sigma_{ik}(p\to 0)$, the equality $\Sigma_{12}(0)=0$ itself, manifest itself physically directly? Where is the "physical source" of the violation of the BA for $p\to 0$? Do we have a simple explanation of the equality $\Sigma_{12}(0)=0$ (as in the case of the equation $\mu=\Sigma_{11}(0)-\Sigma_{12}(0)$ (Ref. 16))? It is accidental that the quantities nV_0, $4\pi m^{-1}n_0 f_0$, etc.—incorrect approximations to $\Sigma_{12}(0)$—actually play in the corresponding models the role previously ascribed to $\Sigma_{12}(0)$ (and that the assumption that the $\Sigma_{ik}(p)$ expansion is analytic is in accord with the acoustic character of the spectrum)? And, finally, what can replace the BA as the correct first approximation? These questions are considered in the present paper.

Particular note should be taken of another, perhaps, physically most interesting question connected with the inadequacy of the BA: What is the "microbasis" of the profound physical analogy, known from the phenomenological theory, between the superfluid state and a classical field? What can we use as a model for an effective field-theoretic description of a superfluid at $T=0$ and its "coherent" (entropyless) component at $T>0$, $T\approx T_c$? Without allowance for the infrared divergence, the role of such a model should be played by the BA—the first-order perturbation field theory, which is "quasiclassical," in that it replaces a Bose system by a classical field with the same Hamiltonian $H(\hat{\psi},\hat{\psi}^+)\to H(\psi,\psi^*)$ and quantized modes, and "harmonic," in that it neglects the anharmonicity of the zero-point and thermal oscillations of the normal modes. A similar situation in fact obtains in the description in terms of the hydrodynamic variables: the first-order hydrodynamic perturbation theory,[9] which replaces a Bose system by an ideal fluid with the same Hamiltonian $H(\hat{n},\hat{\mathbf{v}})\to H(n,\mathbf{v})$ and quantized modes, as given by

$$H(\hat{n},\hat{\mathbf{v}})=H(n,\mathbf{v})+H_2(\hat{n}',\hat{\mathbf{v}}')+\hat{H}_{int},$$
$$\hat{n}=n+\hat{n}',\ \hat{\mathbf{v}}=\mathbf{v}+\hat{\mathbf{v}}',\ \langle\hat{n}\rangle=n,\ \langle\hat{\mathbf{v}}\rangle=\mathbf{v},\tag{3}$$

is in fact the "model" for Landau's effective hydrodynamic description.[12] The Landau Hamiltonian $\tilde{H}(\hat{n},\hat{\mathbf{v}})$ which is the basis of a semiphenomenological long-wave ($p\to 0$) description at $T=0$ in terms of the variables $\hat{n}$ and $\hat{\mathbf{v}}$:

$$\hat{\tilde{H}}=\tilde{H}(\hat{n},\hat{\mathbf{v}})=\int[{}^1/_2 m\hat{\mathbf{v}}\hat{n}\hat{\mathbf{v}}+E(\hat{n})]d\mathbf{r},\tag{4}$$

$$\tilde{H}(\hat{n},\hat{\mathbf{v}})=\tilde{H}(n,\mathbf{v})+\tilde{H}_2(\hat{n}',\hat{\mathbf{v}}')+\hat{\tilde{H}}_{int}\tag{5}$$

(the c numbers describe the macroscopic motion; the "operator corrections," the zero-point oscillations, the exact spectra, and the exact interaction between the excitations), differs from the original $H(\hat{n},\hat{\mathbf{v}})$ only in that the parameters are renormalized as a result of the anharmonicity of the zero-point oscillations.

The semiphenomenological Landau theory establishes the nature of superfluidity, but does not consider specifically the field character of the state. The field-theoretic generalization of this theory in the region around T_c, where it is especially important, is furnished by the Ginzburg-Pitaevskiĭ (GP) theory of the λ transition.[17] This theory is based on the idea, first developed in the Ginzburg-Landau theory of superconductivity,[18] that the c-number field—a distinctive macroscopic analog of the wave function of a particle—can be considered to be the order parameter of the phase transition. The field-theoretic formulation of the microtheory has its origin in London's idea,[19] based on an analogy with an ideal gas, that Bose condensation forms the microscopic basis of superfluidity, an idea which was realized by Bogolyubov,[7] who, unlike London,[19] took account of the important role played in the phenomenon of superfluidity by the interboson interaction by separating out the c-number component of the field operator $\hat{\psi}=\sqrt{n_0}+\hat{\psi}'$ and diagonalizing the quadratic part of the Hamiltonian $\hat{H}^B=N^2V_0/2V+H_2(\hat{\psi}',\hat{\psi}^+)$. Belyaev[2] used the representation $\hat{\psi}=\sqrt{n_0}+\hat{\psi}'$ to construct a field perturbation theory (see also Ref. 20), and Gross[21] and Pitaevskiĭ[22] have generalized Bogolyubov's treatment to the case of an inhomogeneous condensate, replacing $\sqrt{n_0}$ by $\psi(r)$:

$$H(\hat{\psi},\hat{\psi}^+)=H(\psi,\psi^*)+H_2(\hat{\psi}',\hat{\psi}'^+)+\hat{H}_{int},\ \hat{\psi}=\psi+\hat{\psi}',\ \langle\hat{\psi}\rangle=\psi.\tag{6}$$

The equation $\Sigma_{12}(0)=0$ precludes the assumption that $\tilde{H}(\hat{\psi},\hat{\psi}^+)\sim H(\hat{\psi},\hat{\psi}^+)$ (or that $\tilde{H}(\hat{n},\hat{\mathbf{v}})\sim H(\hat{n},\hat{\mathbf{v}})$, i.e., the assumption that the anharmonicity of the zero-point and thermal oscillations leads only to the renormalization of the parameters of $H(\hat{\psi},\hat{\psi}^+)$. Thus, the expansion (6) cannot be used as a model for the construction of an effective c-number field-theoretic description (in the way (3) and (5) are used).

The problem of finding the microbasis of the field-theoretic description is complicated by another difficulty: the nonuniqueness of the choice of the λ-transition order parameter.[23] Whereas in the microscopic field theory the modulus of the order parameter is $|\psi|=\sqrt{n_0}$, in the phenomenological GP theory it is $(\rho_s/m)^{1/2}$ (naturally, the condensate separates out in the description in terms of the field variables, while the superfluid component separates out in the Landau hydrodynamics). The two difficulties are found to be interrelated when an attempt is made to remove them. In the present paper we carry out the modification of the BA that is required for the microscopic justification that must be given in the basic scheme of the effective field-theoretic description. This allows us to uniquely establish the nature of the c-number field characterizing the superfluid state both at $T\to 0$ and $T\sim T_c$. This field turns out to be the c-number component not of the original field operator $\hat{\psi}=\psi+\hat{\psi}'$, $\psi=\langle\hat{\psi}\rangle$, $|\psi|=\sqrt{n_0}$ (the "condensate wave function" (CWF)), but of some effective operator $\hat{\tilde{\psi}}=\tilde{\psi}+\hat{\tilde{\psi}}'$ with $|\tilde{\psi}|=(\rho_s/m)^{1/2}$ at $T=0$ and $T\sim T_c$ ("macroscopic wave function" (MWF)), which supports the choice of the order parameter in the GP theory.

The situation considered has a general character in the theory of phase transitions connected with spontaneous continuous-symmetry breaking: in describing such transitions with the aid of an effective Hamiltonian of the Ginzburg-Landau type, we should, in contrast to the nondegenerate case, replace the original order parameter $x = \langle \hat{x} \rangle$ by some effective parameter $\tilde{x} = \langle \hat{\tilde{x}} \rangle$.

In Sec. 2 we establish a connection between the relation $\Sigma_{12}(p \to 0) \to 0$ and the divergence of the longitudinal static susceptibility $\chi_{\parallel}(\mathbf{p} \to 0)$ to the perturbations of $|\psi|$ or $n_0 = |\psi|^2$; the divergence of $\chi_{\parallel}(0)$ is one of the manifestations of the radical difference between $\widetilde{H}(\psi,\psi^*)$ and $H(\psi,\psi^*)$ and, thus, between $\widetilde{H}(\psi,\psi^*)$ and $\widetilde{H}(n,\mathbf{v})$ ($\widetilde{H}(n,\mathbf{v}) \sim H(n,\mathbf{v}) = H(\psi,\psi^*)$). A qualitative difference between the two effective characteristics of a Bose system (a difference that does not occur in the BA: $H(n,\mathbf{v}) = H(\psi,\psi^*)$) is not inconsistent with the nature of the characteristics: the long-wave limits of the two sets of variables $n,\mathbf{v}$ and ψ,ψ^* separate out different subsystems; the anomaly in $\widetilde{H}(\psi,\psi^*)$ reflects a specific property of the field long-wave subsystem (the condensate). But the infrared anomaly in the Hamiltonian $\widetilde{H}(\psi,\psi^*)$ does not allow us to use this Hamiltonian as a basis for an effective field-theoretic description; to realize such a description we must find another Hamiltonian.

In Sec. 3 we show that the description of a fluctuating system with the aid of an effective nonfluctuating system separates out special "adequate" variables that are linearly connected with the "truly normal" modes. The use of special field variables, $\tilde{\psi}$ and $\tilde{\psi}^+$, which are linearly connected with the variables $\hat{n}$ and $\hat{\mathbf{v}}$ that are adequate in the long-wave region, guarantees at $T = 0$ the elimination of the infrared anomaly—both in the perturbation theory and in the effective description. We indicate the source of the infrared anomaly at $T = 0$: the divergence of the phase fluctuations as a result of the degeneracy of the ground state with spontaneously broken phase gauge symmetry. Thus, the equation $\Sigma_{12}(0) = 0$ has the same symmetry-related origin as the equation[16] $\Sigma_{11}(0) - \Sigma_{12}(0) = \mu$, only it takes account of the anharmonicity. The choice of $\tilde{\psi}$ is dictated by the necessity to "exclude" the divergent phase fluctuations: $|\tilde{\psi}|^2 = \langle |\hat{\psi}|^2 \rangle$.

In Secs. 4 and 5 we investigate the characteristics of the $T > 0$ case. Here we add a new source of infrared anomaly in the anharmonicity, the excitation gas. At low T the adequate field variable $\tilde{\psi}_T$ is found from the asymptotic expressions for the Green functions (Sec. 5; in contrast to the results given in the literature,[14,24] the formulas obtained here for the Green functions by the method described in Ref. 14 agree with the general relations obtained in the field theory and in two-velocity hydrodynamics[25]). The definition of $\tilde{\psi}_T$ for excitation differs somewhat from the definition for stationary motion, in which $|\tilde{\psi}_T| = (\rho_s/m)^{1/2}$. The construction of $\tilde{\psi}_T \equiv \langle \tilde{\psi}_T \rangle$ near t_c is similar to that as at $T = 0$, but with the aid of the "long-wave" field operator $\hat{\psi}_L = \langle \hat{\psi} \rangle_{sh}$ instead of the original $\hat{\psi}$: $|\tilde{\psi}_T|^2 = \langle |\hat{\psi}_L|^2 \rangle = \rho_s/m$ ($\langle \ldots \rangle_L$ and $\langle \ldots \rangle_{sh}$ denote long-wave ($p \leqslant Q$) and short-wave ($p > Q$) averaging); $Q \sim \varkappa(T)$, $\varkappa(T)$ is the wave vector of the longitudinal correlations; the averaging $\langle \hat{\psi} \rangle_{sh}$ actually "separates out" the excitation gas; the use of the modulus $|\hat{\psi}_L|$ excludes the phase fluctuations (which are important, since the region $p \lesssim Q$ includes the region where the infrared anomaly is "formed"). The relation between ψ_L and ρ_s/m is proved (for an indication of the existence of this relation, see Ref. 23). A similar replacement of the original order parameter $x = \langle \hat{x} \rangle$ by an "adequate" parameter $\tilde{x} = \langle \hat{\tilde{x}} \rangle, |\tilde{x}|^2 = \langle |\hat{x}_L|^2 \rangle, \hat{x}_L = \langle \hat{x} \rangle_{sh}$ is necessary when we use a Hamiltonian of the Ginzburg-Landau type to describe any phase transition with a spontaneously broken continuous symmetry; whereas in terms of x the longitudinal susceptibility diverges (i.e., $\chi_{\parallel}^{-1}(0) = 0$) at all $T < T_c$, in terms of $\tilde{x}$ the inverse susceptibility $\chi_{\parallel}^{-1}(0) \to 0$ only as $T \to T_c$, i.e., only as a result of the "softening" of the corresponding mode.

The divergence of $\chi_{\parallel} = dm/dh|_{h \to 0} \sim h^{-1/2}$ was first established by Vaks *et al.*[26] for the exchange-ferromagnet model. The general character of this result for $T > 0$ in the case of phase transitions into a degenerate state is pointed out in Ref. 27. As follows from the analysis of the equation $\Sigma_{12}(0) = 0$ in Sec. 2, the divergence of $\chi_{\parallel}$ for a Bose system with a condensate is an exact result of the microtheory for all $T \geqslant 0$; at $T = 0$ we have $\chi_{\parallel} \sim \ln(mc/h)$. According to Ref. 27, the divergence of $\chi_{\parallel}$ implies the inadequacy of the self-consistent theory of phase transitions even at temperatures far from the transition point; the adequacy is restored only for states with a nonequilibrium value of the order parameter

$$(x - x_s)/x_s \gg \mathrm{Gi}/\tau. \tag{7}$$

At the same time, for many problems it is precisely the equilibrium (though slightly inhomogeneous) state that is of primary interest—e.g., in the investigation of boundary effects for superfluid helium.[23] The limitation (7) is particularly "troublesome" in the description of the λ transition, where $\mathrm{Gi} \sim 1$. The use of the adequate variable $\tilde{\psi}_T$ introduced in Secs. 3 and 4 frees us from the limitation (7). This removes an important objection against the ψ theory of the λ transition,[23] and also indicates that the choice of $(\rho_s/m)^{1/2}$ as the modulus of the order parameter is not an accident and is unique.

In Sec. 6 we investigate the behavior of $\Sigma_{12}(p \to 0)$ at $T > 0$. In accord with the complication introduced by the infrared anomaly, the character of the p dependence of Σ_{12} turns out to be significantly more complicated; but, just as at $T = 0$, the equation $\Sigma_{12}(0) = 0$ is rigorously satisfied.

In Appendix I we consider the manifestation of the infrared anomaly of the anharmonicity in the "combined" variables (the field variables at high, and the "hydrodynamic" variables at low, momenta). The manifestation of this anomaly reflects an important difference between the hydrodynamic and field long-wave subsystems (the characteristics of the condensate).

In Appendix 2 we discuss the origin and the specific nature of the manifestation of the quasiparticle inertial properties which determine the quantity $\rho_s = \rho - \rho_n$.

2. DIVERGENCE OF THE LONGITUDINAL SUSCEPTIBILITY. THE EFFECTIVE FIELD HAMILTONIAN

1. In Ref. 1 a connection is noted between the exact multiray vertices with zero external momenta and the "energy" ($E'(n_0,\mu) = \langle \widehat{H} - \mu \hat{n}' \rangle$ (the volume $V = 1$)) derivatives; in particular, it is shown that

$$\Sigma_{12}(0)=n_0\left(\frac{\partial^2 E'}{\partial n_0^2}\right)_\mu=n_0\left(\frac{\partial\mu_0}{\partial n_0}\right)_\mu\qquad\left(\mu_0\equiv\left(\frac{\partial E'}{\partial n_0}\right)_\mu=\mu\right).\tag{8}$$

Since n_0 and μ are in fact related, the indicated derivatives have only a formal meaning. But they acquire a direct thermodynamic meaning if we introduce an external field $\hat{u}_1=-h_1\hat{n}_0$, $\hat{n}_0=\hat{a}_0{}^+\hat{a}_0$ acting on the condensate ($E'(n_0,\mu)$ can then be interpreted as a thermodynamic potential of two subsystems, in one of which—the condensate—the number n_0 of particles is fixed, while in the other—the supercondensate system—the chemical potential μ is fixed[1]). The vanishing of $\Sigma_{12}(0)$ then implies the divergence of the susceptibility to homogeneous perturbations of the condensate density:

$$\chi_{1\parallel}{}^{-1}=\left(\frac{dn_0}{dh_1}\right)^{-1}_{h_1=0}\sim\Sigma_{12}(0)=0.\tag{9}$$

Indeed, let $\hat{H}(h_1)=\hat{H}-h_1\hat{n}_0, h_1\to 0$. We then have

$$E'(h_1)=E'[n_0(h_1),\mu(h_1)]-h_1n_0,\qquad \mu(h_1)=\left(\frac{\partial E'(h_1)}{\partial n_0}\right)_\mu,\qquad n'(h_1)=-\left(\frac{\partial E'(h_1)}{\partial\mu}\right)_{n_0};$$

$$\mu(h_1)-\mu\approx\left(\frac{\partial\mu_0}{\partial n_0}\right)_\mu[n_0(h_1)-n_0]+\left(\frac{\partial\mu_0}{\partial\mu}\right)[\mu(h_1)-\mu]-h_1,$$

$$n'(h_1)-n'\approx\left(\frac{\partial n'}{\partial n_0}\right)_\mu[n_0(h_1)-n_0]+\left(\frac{\partial n'}{\partial\mu}\right)_{n_0}[\mu(h_1)-\mu].$$

Using (8) and the relations (see Ref. 1)

$$\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}=-\left(\frac{\partial n'}{\partial n_0}\right)_\mu=1-\frac{1}{n_0}\Sigma_{12}(0)\frac{dn_0}{d\mu},$$
$$\left(\frac{\partial n'}{\partial\mu}\right)_{n_0}=\frac{dn}{d\mu}-\frac{1}{n_0}\Sigma_{12}(0)\left(\frac{dn_0}{d\mu}\right)^2,\tag{10}$$

and recognizing that $n'(h_1)+n_0(h_1)=n'+n_0$, we find

$$\mu(h_1)-\mu=-\frac{dn_0}{dn}h_1,\qquad n_0(h_1)-n_0=\left[\frac{1}{n_0}\Sigma_{12}(0)\right]^{-1}\left[1+\frac{1}{n_0}\Sigma_{12}(0)\frac{dn_0}{d\mu}\frac{dn_0}{dn}\right]h_1,\tag{11}$$

from which we obtain the relation (9).

The dependence of $\chi_{1\parallel}(h_1)\approx[n_0^{-1}\Sigma_{12}^{(h_1)}(0)]^{-1}$ on $h_1\to 0$ can easily be found from the formula[1]

$$\Sigma_{12}(p\to 0)=\Delta\Sigma+O(\varepsilon^2,p^2)\sim-[n_0/\Pi_0(p)],\tag{12}$$

which is valid at $T=0$, as well as at low $T>0$ if ε and $|p|$ are far from the second-sound pole (see (82));

$$\Pi_0(p)=i\int\frac{d^4p_1}{(2\pi)^4}[G_{11}(p_1+p)G_{11}(p_1)+C_{12}(p_1+p)C_{12}(p_1)]\tag{13}$$

(for $T>0$, $i\int d\varepsilon_1/2\pi\cdots\to -T\,\Sigma_{\varepsilon_{1n}}\cdots$). Taking into account the asymptotic form of the spectrum $\varepsilon(p\to 0)$ for $h_1\neq 0$

$$\varepsilon\approx[c^2(p^2+2mh_1n_0/n)]^{1/2}\qquad(\Sigma_{11}^{(h_1)}(0)-\Sigma_{12}^{(h_1)}(0)-\mu=h_1),$$

we find that

$$\chi_{1\parallel}(h_1)\approx\Pi_0^{(h_1)}(0)\sim\left(\frac{n_0}{n}\right)^2m^2c\ln\frac{mc^2}{h_1}\qquad(T=0);$$
$$\left(\frac{n_0}{n}\right)^2\frac{m^{3/2}T}{\sqrt{h_1}}\qquad\left(T\gg\left[\frac{mc^2h_1n_0}{n}\right]^{1/2}\right).\tag{14}$$

More detailed information—about the anomaly in the longitudinal susceptibility $\chi_\parallel\sim\chi_{1\parallel}$ to inhomogeneous perturbations of the condensate wave function ($|\psi|$)—is given by the divergent nonpole correction to $G_{ik}(p\to 0)$ [Eq. (15) of Ref. 1]. Indeed, if we set $\varphi=0$ in $\langle\hat\psi\rangle=\psi=n_0^{1/2}e^{i\varphi}$, the susceptibility to longitudinal perturbations $\hat{U}=\int(h\hat\psi(\mathbf{r})+h\cdot\hat\psi^+(\mathbf{r}))d\mathbf{r}$ corresponds to the commutator of the Hermitian parts of the operator $\hat\psi'=\hat\psi-\sqrt{n_0}$:

$$\mathrm{Re}\,\hat\psi'=\frac{1}{2}\sum_{\mathbf{p}\neq 0}(\hat a_\mathbf{p}+\hat a_{-\mathbf{p}}{}^+)e^{i\mathbf{pr}},$$

while the susceptibility to transverse perturbations corresponds to the commutator of the anti-Hermitian parts:

$$\mathrm{Im}\,\hat\psi'=\frac{1}{2i}\sum_{\mathbf{p}\neq 0}(\hat a_\mathbf{p}-\hat a_{-\mathbf{p}}{}^+)e^{i\mathbf{pr}};$$

using the expression[1]

$$G_{ik}(p\to 0)=(-1)^{i+k}\left[\frac{n_0}{n}mc^2+O(\varepsilon,p^2)\right](\varepsilon^2-c^2p^2)^{-1}-[4\Delta\Sigma(p)]^{-1}\tag{15}$$

we find for the longitudinal and transverse susceptibilities at $T=0$ and at low $T>0$ (i.e., for ε and $|\mathbf{p}|$ far from the second-sound pole) the expressions[3)]

$$\chi_\parallel(p)=\frac{1}{4}\sum_{i,k=1}^{2}G_{ik}(p)\approx O(\varepsilon^2,p^2)[\varepsilon^2-c^2p^2]^{-1}-[4\Delta\Sigma]^{-1},\tag{16}$$

$$\chi_\perp(p)=-\frac{1}{4}\sum_{i,k=1}^{2}(-1)^{i+k}G_{ik}(p)\approx-\left[\frac{n_0}{n}mc^2+O(\varepsilon^2,p^2)\right](\varepsilon^2-c^2p^2)^{-1}\tag{17}$$

(in (15)–(17), $O(x)$ is a small quantity of the order of x: cf. (22));

$$\chi_\parallel(\mathbf{p},0)\approx-\frac{n_0m^2}{n^2}\ln\frac{mc}{p}\ (T=0);\qquad -\frac{nm^2T}{n^2p}\ (T\gg cp);$$
$$\chi_\perp(\mathbf{p},0)\approx\frac{n_0m}{np^2},\tag{18}$$

(see (12));

$$\Pi_0(\mathbf{p},0)\sim\left(\frac{n_0}{n}\right)^2m^2c\ln\frac{mc}{p}\ (T=0);\qquad\left(\frac{n_0}{n}\right)^2\frac{m^2T}{p}\ (T\gg cp).\tag{19}$$

On the other hand, within the framework of the BA

$$\chi_\parallel{}^\mathrm{B}(\mathbf{p},0)=-\frac{1}{4mc_\mathrm{B}^2},\qquad\chi_\perp{}^\mathrm{B}(\mathbf{p},0)=\frac{m}{p^2}.\tag{20}$$

It is also not difficult to express $\chi_\parallel$ and $\chi_\perp$ in terms of the Green functions $g_{ab}(p)=-\langle ab\rangle$ ($p<q_0$) of the method of "combined variables" (the "hydrodynamic variables" π and φ in the long-wave region and the field variables in the short-wave region[14]):

$$\psi=\psi_L+\psi_{sh},\qquad\psi_L=\tilde n_L{}^{1/2}e^{i\tilde\varphi_L},\qquad\psi_L=\sum_{|\mathbf{p}|\leq q_0}a_\mathbf{p}e^{i\mathbf{pr}},$$
$$\pi=\tilde n_L-\langle\tilde n_L\rangle,\qquad\varphi=\tilde\varphi_L,\qquad\langle\tilde n_L\rangle=n_0,\tag{21}$$

using the relation between G_{ik} and g_{ab} (Ref. 28):

$$\chi_{\parallel}(p)=\frac{1}{4n_0}g_{\pi\pi}(p)-\frac{n_0}{2}g_{\varphi\varphi*}g_{\varphi\varphi}(p) \qquad \chi_{\perp}(p)\approx -n_0 g_{\varphi\varphi}(p); \tag{22}$$

$$g_{\pi\pi}(p)=\frac{np^2/m}{\varepsilon^2-c^2p^2}, \qquad g_{\pi\varphi}(p)=\frac{i\varepsilon}{\varepsilon^2-c^2p^2}, \tag{23}$$

$$g_{\varphi\varphi}(p)=\frac{d\mu/dn}{\varepsilon^2-c^2p^2}\ (T=0),$$

$$-\frac{n_0}{2}g_{\varphi\varphi*}g_{\varphi\varphi}(p)=\frac{1}{4n_0}\Pi_0^{(p_1\leqslant q_0)}(p) \tag{24}$$

($p_1 \leqslant q_0$ is a limitation on the momentum region of integration (see (13)); the simple form of $g_{ab}(p)$ for $T=0$ in (23) was written under the assumption that

$$p_c \ll q_0 \ll p_h = mc, \tag{25}$$

where p_c is a characteristic infrared-anomaly momentum such that the quantities $\Sigma_{ik}(p_c), \Pi_0(p_c),\ldots$ or the field-diagram integrals $\Sigma_{ik}^{(p_1>p_c)}(0), \Pi_0^{(p_1>p_c)}(0),\ldots$ over the region $p_1 > p_c$ exhibit a marked infrared divergence; for the model (27) $V_0\Pi_0(p_c)\sim 1, V_0\Pi_0^{(p_1>p_c)}(0)\sim 1(\Delta\Sigma(p) \approx n_0V_0(1-V_0\Pi_0(p))^1)$; hence (see (19) and (27))

$$p_c\sim p_0 e^{-1/\alpha}(T\doteq 0); \qquad \sim \alpha mT/p_0 (T\gg cp_c). \tag{26}$$

2. A number of "physical arguments" seem to indicate that the effective and original field Hamiltonians must be similar, i.e., that $\widetilde{H}(\psi,\psi^*)\sim H(\psi,\psi^*)$. If it were the case this would guarantee an effective *BA*-based field-theoretic description. All that can be seen here, however, is an indication of the existence of some adequate modification of the BA, which establishes the true microscopic meaning of the effective field-theoretic description.

The BA is an exact description for an ideal Bose gas—another (simplified) example of the quasiclassical field-theoretic approach

$$H_0(\hat{\psi}, \hat{\psi}^+)=H_0(\psi, \psi^*)+H_2(\hat{\psi}', \hat{\psi}'^+)$$

(the condensate is treated as a classical field with quantized modes). In view of this, one would think that the BA would be a good approximation to the "quasiharmonic" model, namely a "compressed" Bose system with weak interaction

$$\alpha=V(r=0)/p_0^2m^{-1}\sim mp_0|V_{p_0}|\ll 1, \quad \beta=(n/p_0^3)^{1/3}\sim 1/\alpha^{1/3}\gg 1 \tag{27}$$

(p_0 is a characteristic momentum transfer) for which the anharmonicity corrections—the diagrams with "loops" (integrations over internal momenta) are "suppressed" both in the variables $\hat{n}$, $\hat{\mathbf{v}}$ and in $\hat{\psi}$, $\hat{\psi}^+$: each "loop" contributes a factor $\alpha \ll 1$. As is not difficult to verify, for the model (27) we have $\widetilde{H}(n,\mathbf{v})\approx H(n,\mathbf{v})$ $(E(n)\approx 1/2n^2V_0$ (see (4)). We can, through a simple generalization of (27), also take account of an arbitrarily strong short-range repulsion; if we "smooth out" the potential in $H(\psi,\psi^*)$ beforehand, i.e., if we take the ladders (the "corpuscular anharmonicity") into account before allowing for the loops (the "field anharmonicity"), we have here also $\widetilde{H}(n,\mathbf{v})\approx H(n,\mathbf{v}) = H(\psi,\psi^*)$. The validity of the relation $\widetilde{H}(n,\mathbf{v})\sim H(n,\mathbf{v})$ for an arbitrary condensate-containing Bose system follows from the asymptotic form of the field Green functions $G_{ik}(p\to 0)$.[8,1] If we assume that the hydrodynamic description $\widetilde{H}(n,\mathbf{v})$ is simply the part of the field description that remains after subtracting the "strictly field" properties, as obtained in the Ba ($|\psi|^2 = n_0 \approx n$)

$$H(n, \mathbf{v})=H(\psi, \psi^*), \qquad \psi=n^{1/2}e^{i\varphi}, \qquad \mathbf{v}=\hbar m^{-1}\nabla\varphi, \tag{28}$$

it might seem that we could write $\widetilde{H}(\psi,\psi^*)\approx H(\psi,\psi^*)$ for the model (27) and $\widetilde{H}(\psi,\psi^*)\sim H(\psi,\psi^*)$ in the general case.

The relation $\widetilde{H}(\psi,\psi^*)\sim H(\psi,\psi^*)$ is apparently supported also by the analogy between a condensate-containing Bose system and a crystal—the corpuscular version of the quasiclassical approximation ($\hat{\mathbf{r}}_a = \mathbf{r}_a + \hat{\mathbf{r}}'_a$, $\mathbf{r}_a = \langle\hat{\mathbf{r}}_a\rangle$). In both cases it is natural to expect the state to be close to the harmonic approximation ($\widetilde{H}\sim H$) until the zero-point or thermal oscillations destroy the characteristic of the state (the "off-diagonal" or crystalline long-range order), i.e., so long as the condensate or the lattice is preserved ($\langle\hat{\psi}\rangle\neq 0$ or $\langle\mathbf{r}_a\rangle = \mathbf{r}_a$).[4)]

Also supporting the relation $\widetilde{H}(\psi,\psi^*)\sim H(\psi,\psi^*)$ is the fact that the BA (the field $H(\psi,\psi^*)$ with quantized modes) reflects all the most important features of the superfluid state that are postulated or established in the semiphenomenological treatments: the hydrodynamic description for $p\to 0$, with $H(\psi,\psi^*) = H(n,\mathbf{v})$ (for nonstringent conditions on the potential, $V_p (V_0 > 0$, $V_{p_0} < 0)$ is the phonon-roton line); the Goldstone nature of sound; the quantization of the velocity circulation[22]; the similarity to the condensate of noninteracting bosons[5)] with $m\neq 0$.

Finally, the BA directly reflects the "gauge nature" of the λ transition: ψ satisfies the condition for a minimum

$$H^{(\mu)}(\psi,\psi^*)=H(\psi,\psi^*)-\mu\int \psi^*\psi\, d\mathbf{r};$$

$\min H^{(\mu)}$ is attained at $\psi = a_0e^{i\varphi}$, $a_0 = (\mu/V_0)^{1/2}$; in the long-wave limit $H^{(\mu)}(\psi,\psi^*)$ is analogous to the thermodynamic potential $\Omega(T,\mu,\psi)$ of the Landau theory of phase transitions (i.e., to the Ginzburg-Landau Hamiltonian):

$$H^{(\mu)}=\int dr\left[\frac{|\nabla\psi|^2}{2m}-\mu|\psi|^2+\frac{1}{2}V_0|\psi|^4\right]; \tag{29}$$

thus, the BA admits of a natural generalization to the case $T\approx T_c$, where $H^{(\mu)}\to\widetilde{H}_T^{(\mu)} = \Omega(T,\mu,\psi)$.

The gapless spectrum

$$\varepsilon_p^B=[\varepsilon_p^0(\varepsilon_p^0+2\mu)]^{1/2}, \qquad \varepsilon_p^0=p^2/2m, \qquad \mu=V_0a_0^2=\Sigma_{12}^B(0); \tag{30}$$

which characterizes the Hamiltonian (29), differs essentially from its nondegenerate analog, with $\psi = \mathrm{Re}\psi$ (Goldstone's theorem), but $\chi_{\parallel}(\mathbf{p})$ coincides with $\chi(\mathbf{p})$ in the case when $\psi = \mathrm{Re}\psi$:

$$\chi_{\parallel}=-\frac{1}{2[\varepsilon_p^0+2\Sigma_{12}^B(0)]}=-\frac{m}{p^2+\varkappa^2}, \qquad \varkappa^2=4m\Sigma_{12}^B(0);$$

$$\chi_{\perp}=\frac{m}{n^2}. \tag{31}$$

Notice that the reciprocal correlation length $\varkappa = r_c^{-1}$ at $T=0$ is of the same order of magnitude as the characteristic momentum transfer $p_0(\hbar = 1)$ that determines the effective integration domain in the diagrams and as the characteristic hydrodynamic momentum $p_h = mc_B$ (see Sec. 3).

But it is precisely when the superfluid state is portrayed as the result of spontaneous gauge-symmetry breaking that we most clearly see the inadequacy of the BA: according to (9) and (16), the formula $\chi_{\parallel}(0) = -\frac{1}{4}\Sigma_{12}(0)$ (see (31)) is exact (it is preserved when the anharmonicity is taken into account). As follows from the general picture of phase transitions, the quantity $\varkappa^2 = 4m\Sigma_{12}(0) = -m\chi_{\parallel}^{-1}(0)$ should be nonzero at all $T<T_c$, and should go to zero only when $T\to T_c$. But this contradicts the equation $\Sigma_{12}(0) = 0$. Thus, $\tilde{H}(\psi,\psi^*)$ should differ qualitatively from $H(\psi,\psi^*)$, even for the model (27) (the formal expansion parameter $\alpha \ll 1$ is counterbalanced by the infrared divergence; therefore, the "single-loop" approximation, for example, destroys even the linear character of the spectrum for $\mathbf{p}\to 0$). It is the "masking" of this circumstance that caused its neglect both at $T = 0$[2–4,6] and at $T\approx T_c$—see, for example, Ref. 31, where

$$F\equiv \tilde{H}(\psi,\psi^*)=\int\left[\frac{1}{2}A_{\parallel}|\nabla\psi_{\parallel}|^2+\frac{1}{2}A_{\perp}|\nabla\psi_{\perp}|^2+f(|\psi|^2)\right]d\mathbf{r},\tag{32}$$

or Ref. 5, where $\Sigma_{12}(0)$ determines the proximity to the transition point. The $\Sigma_{\parallel}(\mathbf{p}\to 0)$ singularity is neglected even in Forster's excellent monograph,[32] where, in particular, it is asserted in connection with the relation $n^{0n}(\mathbf{r})\sim \mathrm{Re}\psi$ that "there is no reason to expect... unusual-length correlations (see (10.55e) in Ref. 32).

3. The qualitative difference between $\tilde{H}(\hat{\psi},\hat{\psi}^+)$ and $H(\hat{\psi},\hat{\psi}^+)$ is determined by the nonanalyticity at ε, $|\mathbf{p}|\to 0$ of the exact field vertices—of the coefficients of the expansion of the effective action $\tilde{S}(\psi,\psi^*)$ in terms of the Fourier amplitudes of the field: $\varepsilon_{ik}(p)$ enters into the quadratic part of $\tilde{S}_2$, $\Gamma_{n\geq 3}$ in the higher-order terms (the "zeroth-order" vertices, on the other hand, are analytic), and the relation between $\tilde{S}$ and $\tilde{H}$ is given by the same formula that gives the relation between S and H. The direct source of the nonanalyticities is the contribution of the pair Green functions $\int G_{ik}(p+p_1)G_{lm}(p_1)d^4p_1$. The non-analytic term in $\Sigma_{ik}(p\to 0)$ is $\Delta\Sigma$ (see (12), (13), and (19)); the nonanalyticity of Γ_3 and Γ_4 can be seen, for example, from Fig. 1: in the diagram

$$\Sigma_{12}^a(0)=\Sigma_{12}(0)-\Sigma_{12}^B(0)-\Sigma_{12}^b(0)\approx -n_0V_0\,(\Sigma_{12}^B(0)=n_0V_0,$$

$$\Sigma_{12}^b(0)\sim\alpha^2 n_0V_0);$$

for the model (27) the vertex $\Gamma_3(p)$, although it removes the divergence occurring in the presence of a zeroth-order vertex, has been constructed such that it counterbalances the arbitrarily small factor α characterizing each integration over a 4-momentum; it is clear from the diagram for $\Pi(0) = -n/mc^2$ that Γ_4 cannot be an analytic function of the 4-momenta.

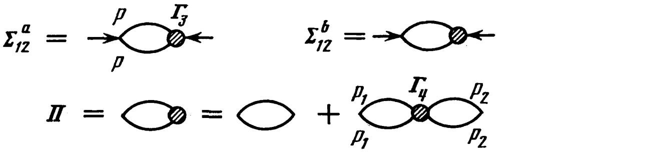

FIG. 1. [In the Σ_{12}^b diagram we have left out the horizontal line between the vertices.]

The existence of a qualitative difference between the two versions of the effective c-number description—the hydrodynamic $\tilde{H}(n,\mathbf{v})$ and field-theoretic $\tilde{H}(\psi,\psi^*)$ descriptions—is not in accord with the BA, in which they strictly agree (see (28)), but it does not itself contain any contradiction: the exact characteristics connected with $\tilde{H}(n,\mathbf{v})$ and $\tilde{H}(\psi,\psi^*)$ pertain to different long-wave subsystems, the "hydrodynamic" and "field" subsystems. They respectively describe the fluctuations and the response to long-wave perturbations of the various "longitudinal" variables: $\hat{n}_L = (\hat{\psi}^+\hat{\psi})_L$, $\mathrm{Re}\hat{\psi}_L$. In contrast to $\chi_{\parallel}$, the exact susceptibility $F_{44}(\mathbf{p}\to 0,0)$ to density perturbations is finite. It follows, for example, from the form (4) of $\tilde{H}(n,\mathbf{v})$ that $F_{44}(p\to 0)$, unlike $G_{ik}(p)$ and $\chi_{\parallel}(p)$, does not contain the nonpole divergent correction of (15) and (16):

$$F_{44}\approx\frac{np^2}{m}(\varepsilon^2-c^2p^2)^{-1},\quad F_{44}(p,0)\approx-\frac{n}{mc^2},$$
$$F_{44}^B(\mathbf{p},0)\approx 4n\chi_{\parallel}^B(\mathbf{p},0)=\frac{n}{mc_B^2}.\tag{33}$$

In the Gross method[21] the "hydrodynamic" characteristics $\tilde{n}_L$, $\tilde{\mathbf{v}}_L$ of a condensate with "adjacent" particles of low momenta ($|\mathbf{p}|\leqslant q_0$) are compared with the field subsystem ψ_L, ψ_L^*. The characteristics of the subsystem manifest themselves in the region $q_0\lesssim p_c$: the susceptibility to $\tilde{n}_L$ perturbations is

$$F_{44}^{(q_0)}(p)\equiv g_{\pi\pi}(p)=F_{44}(p)+\Pi_0^{(p_1>q_0)}(p)\tag{34}$$

(see (22) and (24) with allowance for the nondependence of $\chi_{\parallel}(p)$ on q_0: $4n_0\chi_{\parallel}(p) = F_{44}^{(q_0)}(p) + \Pi_0^{(p_1<q_0)}(p) = F_{44}(p) + \Pi_0(p)$, i.e.,

$$F_{44}^{(q_0)}(\mathbf{p},0)\sim -m^2c\ln\frac{mc}{q_0}\ (T=0);\quad -\frac{m^2T}{q_0}\ (T\gg cq_0).\tag{35}$$

It follows from (18), (35), and the fluctuation-dissipation theorem[33] that at $T>0$ the fluctuations of the Fourier components of $\mathrm{Re}\psi$ (or $|\hat{\psi}|$) and $\hat{n}_L$ diverge respectively like $1/p$ and $1/q_0$; the fluctuations do not diverge at $T=0$. The explicit form of $\tilde{H}_2(\tilde{n}_L,\tilde{\mathbf{v}}_L)$ (which differs substantially from that of $H_2(\tilde{n}_L,\tilde{\mathbf{v}}_L)$ in the region $q_0\lesssim p_c$) is given in Appendix 1.

In conclusion, let us note that, in contrast to the "technical" manifestations of the infrared divergence (in the computation of the anharmonicity corrections and in the derivation of the asymptotic form $G_{ik}(p\to 0)$), such manifestations of this divergence as the existence of a qualitative difference between $\tilde{H}(\psi,\psi^*)$ and $H(\psi,\psi^*)$, the specific nature of the condensate (a special subsystem in a Bose liquid), and the impossibility of the use of the condensate wave function $\psi = \langle\hat{\psi}\rangle$ for an effective c-number description of the superfluid sate turn out in a sense to be "physical" (see also Ref. 33a, where the role played by the equation $\Sigma_{12}(0) = 0$ in the establishment of the character of the response function for density perturbations is investigated).

3. "ADEQUATE" VARIABLES. THE EFFECTIVE *c*-NUMBER FIELD OF A SUPERFLUID. REASONS FOR THE "DISTINCTNESS" OF THE HYDRODYNAMIC VARIABLES

1. The important differences between $\widetilde{H}(\psi,\psi^*)$ and $H(\psi,\psi^*)$, which are preserved even in the case (27) of an arbitrarily small α (i.e., the qualitative deviations from the BA), appear in a condensate-containing Bose system not only at $p\to 0$. (The model (27) is convenient for revealing all effects such as the changes that occurs in the spectral curve in the presence of anomalous dispersion,[34] breaking of this curve and the appearance of a "plateau" of finite dimensions,[35,36] the formation of quasiparticle bound states,[37,36] the fact that the homogeneous phase becomes unstable when the roton minimum is lowered,[38] the "drag effect" in a mixture of superfluids,[39,40] the production of second sound at $T>0$, or the "critical behavior" of the system in the vicinity of the λ point; similar effects occur also in a crystal.) But the situation at $p\to 0$ possesses a distinctive feature—the infrared-anharmonicity anomaly is "noninvariant" under a change of variables: appearing when $\hat{\psi}$ and $\hat{\psi}^+$ are used, it disappears on going over to the variables $\hat{n}$ and $\hat{\mathbf{v}}$. It is noteworthy that the variables switch roles in the opposite limiting case (i.e., $p\to\infty$): $\hat{n}$ and $\hat{\mathbf{v}}$ correspond to an ultraviolet divergence, which is absent in the description in which the variables $\hat{\psi}$ and $\hat{\psi}^+$ are used.

The formal origin of the "mutual complementarity" of the two sets of variables $\hat{n}$, $\hat{\mathbf{v}}$ and $\hat{\psi},\hat{\psi}^+$ can be seen even from the "quadratic" form of the original Hamiltonian

$$\hat{H}=\sum_{\mathbf{p}}\left[\frac{p^2}{2m}a_{\mathbf{p}}^+a_{\mathbf{p}}+\frac{1}{2}V_{\mathbf{p}}n_{\mathbf{p}}n_{-\mathbf{p}}\right]\qquad (V_0\neq 0,\ V_{p\to\infty}\to 0);\tag{36}$$

for $\mathbf{p}\to 0$ the second term is "more important;" for $\mathbf{p}\to\infty$, the first term. A more detailed explanation is obtained when we compare the *c*-number expansions of $H(\psi,\psi^*)=H(n,\mathbf{v})$ in terms of the normal modes that diagonalize the quadratic parts of H_2:

$$\begin{aligned}&H(\psi,\psi^*)\\&=E_0+\sum_{\mathbf{p}}\varepsilon_p^B\alpha_{\mathbf{p}}^*\alpha_{\mathbf{p}}+\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0}[\Gamma^{(1)}_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}\alpha_{\mathbf{p}_1}^*\alpha_{\mathbf{p}_2}^*\alpha_{\mathbf{p}_3}^*+\ldots]+\ldots,\\&H(n,\mathbf{v})\\&=E_0+\sum_{\mathbf{p}}\varepsilon_p^BB_{\mathbf{p}}^*B_{\mathbf{p}}+\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0}[\gamma^{(1)}_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}B_{\mathbf{p}_1}^*B_{\mathbf{p}_2}^*B_{\mathbf{p}_3}^*+\ldots]+\ldots.\end{aligned}\tag{37}$$

Both the field amplitudes (α_p) and hydrodynamic amplitudes (B_p) are "normal," but they differ by terms that are nonlinear in the amplitudes, and this makes the anharmonic vertices different. As $\mathbf{p}\to 0$, the γ vertices in $H(B_{\mathbf{p}},B_{\mathbf{p}}^*)$ tend to zero, and do so faster than $\varepsilon_p^B\sim p$ (the contribution from the term with nv^2, i.e., with $\mathbf{p}_2\mathbf{p}_3n_{\mathbf{p}_1}\varphi_{\mathbf{p}_2}\varphi_{\mathbf{p}_3}$, is $\sim(p/p_h)^{3/2}$, that from the term with $\mathbf{p}_2\mathbf{p}_3n_{\mathbf{p}_1}n_{\mathbf{p}_2}n_{\mathbf{p}_3}$ is $\sim(p/p_h)^{7/2}$, that from the term with $\mathbf{p}_2\mathbf{p}_3n_{\mathbf{p}_1}n_{\mathbf{p}_2}n_{\mathbf{p}_3}n_{\mathbf{p}_4}$ is $\sim(p/p_h)^4$, etc. (the terms from the expansion of the quantity $\nabla n\cdot 1/n\Delta n$); $p_h=mc_{\rm B}$ (if $V_{p_h}\sim V_0$)); but the vertices in $H(\psi,\psi^*)=H(a_{\mathbf{p}},a_{\mathbf{p}}^*)$ are finite, while those in $H(\alpha_{\mathbf{p}},\alpha_{\mathbf{p}}^*)$ (Γ) even diverge like $(p/p_h)^{-1/2}$. On the other hand, in the case $\mathbf{p}\to\infty$ the vertices in $H(B_{\mathbf{p}},B_{\mathbf{p}}^*)$ diverge, while those in $H(a_{\mathbf{p}},a_{\mathbf{p}}^*)$ and $H(\alpha_{\mathbf{p}},\alpha_{\mathbf{p}}^*)$ tend to zero (together with $V_{p\to\infty}$).

It can be seen that in a system with fluctuations we have as physically defined variables not just the "normal" variables (diagonalizing H_2), but also the "adequate" variables $\tilde{x}$ "minimizing" the anharmonicity and eliminating the "noninvariant" anomalies (the refinement is in the terms nonlinear in amplitude); $\widetilde{H}$ and H are closest when they are expressed in terms of the variables $\tilde{x}$: the anomalies in $\widetilde{H}(\tilde{x})$ reflect only the invariant effects of the anharmonicity. Evidently, for a condensate-containing Bose system we have

$$\hat{\tilde{x}}=(\hat{n},\ \hat{\mathbf{v}}(\mathbf{p}\to 0);\ \hat{\psi},\ \hat{\psi}^+(\mathbf{p}\to\infty)).\tag{38}$$

If $p_c\ll p_h=mc$ for a given Bose system (p_h is the momentum at which the hydrodynamic vertices γ, which are growing as a result of the ultraviolet divergence, are no longer small), we find, choosing q_0 in accordance with (25),

$$\hat{\tilde{x}}_L=(\hat{n}_L,\ \hat{v}_L),\qquad \hat{\tilde{x}}_{sh}=(\hat{\psi}_{sh},\ \hat{\psi}_{sh}^+)\tag{39}$$

(by definition[6)] $x_L=\Sigma_{|\mathbf{p}|<q_0}x_{\mathbf{p}}e^{i\mathbf{pr}}$, $\chi_{sh}=x-x_L$).

We can now narrow down the adequate-variable concept by requiring that the adequate variables $\tilde{x}$ be linearly related with the exact quasiparticle operators (the "truly normal" modes); the variables $\hat{n}$ and $\hat{\mathbf{v}}$ for $\mathbf{p}\to 0$ (as well as $\hat{\psi}$ and $\hat{\psi}^+$ for $\mathbf{p}\to\infty$ in the case (27) in the lowest approximation in α) satisfy this definition as well. The indicated variables correspond to the most complete description of a fluctuating system with the aid of an effective nonfluctuating system or a system "with independently fluctuating normal modes." In terms of the other variables, $\widetilde{H}$ gives, despite the accuracy of the vertices, only a rough agreement: the normal modes of the nonfluctuating analog do not coincide with the "truly normal" modes. Therefore, for example, the Feynman formula $\varepsilon_p=p^2/2mS(p)$, although it follows strictly from the general form of $\widetilde{H}(n,\mathbf{v})$ (as distinguished from (4) without the requirement that $p\to 0$)[41]:

$$\widetilde{H}(n,\mathbf{v})=\int\frac{mn\mathbf{v}^2}{2}d\mathbf{r}+\frac{1}{2}\int\Phi(\mathbf{r}-\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}\,d\mathbf{r}',$$

is nonetheless approximate (being exact only at $p\to 0$): its derivation essentially presupposes adequacy of n and $\mathbf{v}$. In this connection, let us note that the postulates of quantum hydrodynamics that follow in the microtheory from the asymptotic form[8,1] $G_{ik}(p\to 0)$ are not only the form (4), of $\widetilde{H}=\widetilde{H}(n,\mathbf{v})$, but also the adequacy of n and $\mathbf{v}$.

Thus, if we compare the hydrodynamic and field long-wave subsystems in a condensate-containing Bose system, the first subsystem comes out as the adequate one.

2. The adequacy of the hydrodynamic variables for $p\to 0$ is not accidental. Their use in the microtheory $H(\hat{\psi},\hat{\psi}^+)\to H(\hat{n},\hat{\mathbf{v}})$ actually implies the replacement of a discrete-density particle system

$$n(\mathbf{r})=\sum_a\delta(\mathbf{r}-\mathbf{r}_a)\tag{40}$$

by a "quantized" continuous medium" with continuous $n(\mathbf{r})$ (i.e., with independent $n_{\mathbf{p}}$: the method of Ref. 9 is equivalent to that of Ref. 10 (see Ref. 11)). The fact that we can make such a substitution without changing the spectrum and the important properties of the macroscopic motion (which are described by the equations of the hydrodynamics of an ideal fluid) reflects important characteristics of the superfluid state: 1) In contrast to what obtains in a classical fluid, sound with normal dispersion at $T=0$ is not damped—departures from local equilibrium that lead to dissipation in the classical approach are energetically forbidden ("frozen out"). 2) In contrast to the case of a normal Fermi liquid, the acoustic mode for $p\to 0$ does not have "competitors" of the pair-continuum type. 3) The sound is collisionless at $T=0$ (therefore, for example, for a tenuous gas ($\beta=(na^3)^{1/2}\ll 1$) the minimum phonon wavelength λ_{ph} is significantly shorter than the "mean free path" $l\sim(na^2)^{-1}$: $\lambda_{ph}\sim\hbar/mc_{\rm B}\sim\beta l\ll l$). 4) The continuation of the acoustic branch is not limited by the condition $\lambda\gg n^{-1/3}$ (the phonon-roton spectrum $\varepsilon_p^{\rm B}=[\varepsilon_p^0(\varepsilon_p^0+2nV_p)]^{1/2}$ of the Hamiltonian $H=H(n,\mathbf{v})$ ($V_0>0$, $V_{p_0}<0$, $V_{p\to\infty}\to 0$) goes over into ε_p^0 as $\mathbf{p}\to\infty$).

At the same time, in the superfluid state, too, the discreteness of the density (40) finds its distinctive manifestation in the "field nature" of the velocity $\mathbf{v}=\hbar m^{-1}\nabla\varphi$, (28): the condition (40) appears in the quantization of $H=H(n,\mathbf{v})$ as a consequence of (28) (the boundedness of the velocity potential $0\leqslant\varphi(r)\leqslant 2\pi$ leads to the discreteness of its "canonical partner" $n(r)$). The question then does not reduce to one of adequacy of the field variables for $\mathbf{p}\to\infty$; the field nature of the velocity leads to remarkable distinctions of the state's long-wave properties, which are lost in the hydrodynamic formulation: 1) A situation in which the phase transition assumes the nature of a spontaneous breaking of the gauge symmetry. 2) A "Goldstone" origin of the sound—a situation in which $\dot{v}$ is determined not by the momentum conservation law (the elasticity of the medium), but by the long-range phase correlations; at $T=0$ the two mechanisms "duplicate" each other; the Goldstone (field) mechanism at $T=0$ represents another departure from sound in a classical fluid; at $T>0$ only the field mechanism is important in the "kinetic-equation" ($\Gamma_T\ll\varepsilon\ll\varepsilon_T$) and quantum ($\varepsilon_T\ll\varepsilon$) regions, but the two mechanisms operate together in the "collisional-hydrodynamic" ($\varepsilon\ll\Gamma_T$) region, the field mechanism predominating in first sound at low T and in second sound at $T\sim T_c$ (ε_T and Γ_T are the energy and damping constant of the thermal excitations). 3) Quantization of the velocity circulation ($\oint\mathbf{v}\cdot d\mathbf{r}=nh/m$)—by the vortex filaments. 4) "Interference phenomena" at the fluid boundary, which are important in connection with the problem of the $\mathbf{v}_s$ jump in the vicinity of a solid boundary.[23]

It is worth noting that at $T>0$ the superfluidity itself is a consequence of the "field nature" of the velocity (the "freezing out" of its "transverse" component $\mathbf{v}_\perp$)[7]: a "continuous medium," in which the velocity $\mathbf{v}$ is not limited by the condition (28), loses its superfluidity on going from $T=0$ to any $T>0$ (since the transverse component $\mathbf{v}_{\mathbf{p}\perp}\perp\mathbf{p}$ is not connected with $n_{\mathbf{p}}$, and enters into $H(n,\mathbf{v})$ quadratically ($\frac{1}{2}nmv_{\mathbf{p}\perp}^2$), we have $\langle v_{\mathbf{p}\perp}^2\rangle=T/mn$, i.e., the transverse momentum density correlator $\langle g_{\mathbf{p}\perp}^2\rangle=Tmn$; hence, using the relation between ρ_s and $\langle g_{\mathbf{p}\perp}^2\rangle$ (Ref. 32), we find $\rho_s=mn-T^{-1}\langle g_{\mathbf{p}\perp}^2\rangle=0$.

3. The fact that the velocity $\mathbf{v}=\langle\hat{\mathbf{v}}\rangle$ has a field ("phase") nature can be highlighted by giving the hydrodynamic variables $\hat{n}=n+\hat{n}'$, $\hat{\mathbf{v}}=\mathbf{v}+\hat{\mathbf{v}}'$ ($n=\langle\hat{n}\rangle$, $\mathbf{v}=\langle\hat{\mathbf{v}}\rangle$) a "field form:" we represent the c-number term in the expansion (5)

$$\tilde{H}(\hat{n},\ \hat{\mathbf{v}})=\tilde{H}(n,\ \mathbf{v})+\tilde{H}_2(\hat{n}',\ \hat{\mathbf{v}}')+\hat{\tilde{H}}_{int}\ ,$$

as a Hamiltonian of some classical field

$$\tilde{H}(n,\mathbf{v})=\tilde{H}(\tilde{\psi},\tilde{\psi}^*)=\int\left[\frac{\hbar^2}{2m}|\nabla\tilde{\psi}|^2+E(|\tilde{\psi}|^2)\right]d\mathbf{r},\qquad \tilde{\psi}=n^{1/2}e^{i\varphi},\quad \mathbf{v}=\hbar m^{-1}\nabla\varphi;\tag{41}$$

further, we identify $\tilde{H}_2(\hat{n}',\hat{\mathbf{v}}')$ with $H_2(\tilde{\psi}',\tilde{\psi}'^+)$ from the expansion (41) of $\tilde{H}(\tilde{\psi}+\tilde{\psi}',\tilde{\psi}^*+\tilde{\psi}'+)$. This gives $\tilde{\psi}'$ as a linear function of $\hat{n}'$ and $\hat{\mathbf{v}}'$, in terms of the c numbers $\tilde{\psi}'=O(n+n')^{\frac{1}{2}}e^{i(\varphi+\varphi')}-\tilde{\psi}$), i.e.,

$$\hat{\tilde{\psi}}(\mathbf{r})=\tilde{\psi}(\mathbf{r})+\hat{\tilde{\psi}}'(\mathbf{r})=\sqrt{n(\mathbf{r})}\,e^{i\varphi(\mathbf{r})}\left[1+\frac{\hat{n}'(\mathbf{r})}{2n(\mathbf{r})}+i\hat{\varphi}'(\mathbf{r})\right];\tag{42}$$

in the homogeneous case

$$\hat{\tilde{\psi}}(\mathbf{r})=\sqrt{n}+\frac{\hat{n}'(\mathbf{r})}{2\sqrt{n}}+i\hat{\varphi}'(\mathbf{r}).\tag{43}$$

Since we use only $\tilde{\psi}_L$ below (see (45)), we need not consider the difficulty (encountered when $\hat{\mathbf{v}}(\varphi)$ is determined from (2)) this definition to take account of the discreteness of the density (40) of the Bose system[43] (see footnote No. 6). We do not use the terms $\tilde{H}_{int}(\hat{\psi}',\hat{\psi}'^+)$ of higher order in $\tilde{\psi}'$ from $\tilde{H}(\tilde{\psi}+\tilde{\psi}',\tilde{\psi}^*+\tilde{\psi}'^+)$: they do not coincide with $\tilde{H}_{int}(\hat{n}',\hat{\mathbf{v}}')$ and, consequently, do not have any physical meaning; setting $\tilde{H}_{int}(\hat{n}'=,\hat{\mathbf{v}}')\equiv\tilde{H}'_{int}(\hat{\psi}',\hat{\psi}'^+)$, we find

$$\tilde{H}(\hat{n},\hat{\mathbf{v}})=\tilde{H}'(\hat{\tilde{\psi}},\hat{\tilde{\psi}}^+)=\tilde{H}(\tilde{\psi},\tilde{\psi}^*)+\tilde{H}_2(\hat{\tilde{\psi}}',\hat{\tilde{\psi}}'^+)+\hat{\tilde{H}}'_{int};\tag{44}$$

the $\tilde{H}'_{int}$ vertices (in contrast to the $\tilde{H}_{int}$ ones) tend to zero as $\mathbf{p}\to 0$. The relations (42) and (43) allow us to introduce the "adequate field variable"

$$\hat{\tilde{\psi}}=(\hat{\tilde{\psi}}_L,\ \hat{\psi}_{sh}).\tag{45}$$

The expression of the original Hamiltonian in terms of this variable, i.e., the transformation

$$\tilde{H}=H'(\hat{\tilde{\psi}},\ \hat{\tilde{\psi}}^+),\qquad H(\hat{n}_L,\ \hat{\mathbf{v}}_L)=H(\hat{\tilde{\psi}}_L,\ \hat{\tilde{\psi}}_L{}^+)=H(\tilde{\psi}_L,\ \tilde{\psi}_L{}^*)+\tilde{H}_2+\tilde{H}_{int}\tag{46}$$

gives us a field perturbation theory that does not contain an infrared divergence. The Hamiltonian $H_2(\tilde{\psi}_L',\tilde{\psi}_L'^+)$ is similar to $H_2(\hat{\psi}_L',\hat{\psi}_L'^+)$, (6), but its coefficients contain $\tilde{\psi}=\sqrt{n}$ instead of $\psi=\sqrt{n_0}$ and the "Bogolyubov operators," $\hat{\tilde{\alpha}}_{\mathbf{p}}$ and $\hat{\tilde{\alpha}}_{\mathbf{p}}^+$, constructed for the diagonalization of $H_2(\tilde{\psi}_L',\tilde{\psi}_L'^+)$ coincide exactly with the hydrodynamic operators $B_{\mathbf{p}}$ and $B_{\mathbf{p}}^+$ (see (37)), and not just in the first-order approximation in the amplitude as in the case of the operators $\alpha_{\mathbf{p}}$ and $\alpha_{\mathbf{p}}^+$ for $H_2(\hat{\psi}_L',\hat{\psi}_L'^+)$ (therefore, $H'_{\rm int}$ does not give rise to an infrared anomaly). Consequently, the wave function in the "correct harmonic" approximation is determined by the condition $\hat{B}_{\mathbf{p}}\neq|\Psi_0\rangle=0$ instead of $\hat{\alpha}_{\mathbf{p}\neq 0}|\Psi_0^{\rm B}\rangle=0$; the state corresponds to the minimum of $H^{(\mu)}\tilde{\psi}_L,\tilde{\psi}_L^*)$ and not of $H^{(\mu)}(\psi_L,\psi_L^*)$, (29).

The coefficient $\tilde{\Sigma}_{12}(0)=mc^2$ of the anomalous terms in $\tilde{H}_2(\tilde{\psi}_L',\tilde{\psi}_L'^+)$ (the exact irreducible two-ray vertex as expressed in terms of the variables $\tilde{\psi}_L$) evidently plays precisely the role, heretofore erroneously assigned to $\Sigma_{12}(0)$ (Refs. 1, 3–5), of the principal characteristic of the superfluid reconstruction; for the model (27) we have $\widetilde{\Sigma}_{12}(0) \approx \Sigma^{B}_{12}(0) = nV_0$. Allowance for the anharmonicity leads only to the renormalization of the parameters of $\widehat{H}'(\widetilde{H}' \sim \widehat{H}'$, where $\widetilde{H}'$ and $\widehat{H}'$ are given by (44) and (46) respectively).

Thus, the profound analogy between a condensate-containing Bose system $\widehat{H} = H(\hat{\psi},\hat{\psi}^+)$ and the classical field corresponding to the original approximation for the model (27) and to an effective field-theoretic description in the general case is brought out at $T=0$ not in the original field variables $\hat{\psi} = \psi + \hat{\psi}'$, but in the "adequate" variables $\overline{\psi} = \tilde{\psi} + \overline{\psi}'$ ($\overline{\psi} = \psi_L = \tilde{\psi}_L = \tilde{\psi}$). As before, the field long-wave subsystem (the condensate) figures in the condition $\psi = \langle\hat{\psi}\rangle \neq 0$ for the field to have a quasiclassical character (i.e., for the existence of an analogy between the field and a classical field), but the effective c-number field is determined by the hydrodynamic long-wave subsystem $\tilde{\psi} = \langle\tilde{\psi}\rangle$. Like $\widetilde{H}(\hat{n}_L, \hat{\mathbf{v}}_L)$, (4), the Hamiltonian $\widetilde{H}'(\tilde{\psi}_L, \tilde{\psi}_L^+)$ describes at $T=0$ the macroscopic motion of the system, the behavior in external fields (acting on n and $\mathbf{v}$), the structure, spectrum, and interaction of the long-wave quasiparticles, though with addition of field characteristics.

4. It is not difficult to verify that the immediate cause of the distinctness (adequacy) of the hydrodynamic variables $\hat{n}$ and $\hat{\mathbf{v}}$ (or $\hat{n}$, $\hat{\varphi}$;$\tilde{\psi}$) as $\mathbf{p}\to 0$ and, consequently, the source of the infrared anomaly of the anharmonicity, in any variables, is the infinite degeneracy of the state with broken gauge symmetry with respect to the zeroth ($p=0$) Fourier component of the phase φ, together with the phase fluctuation infrared divergence connected with this degeneracy. Using the notation of the path-integration method, in which a Bose system appears as a fluctuating c-number field, let us compare the variables $\langle\psi\rangle = \langle\sqrt{n}e^{i\varphi}\rangle$ (CWF) and $\langle\tilde{\psi}\rangle = \sqrt{\langle n\rangle}e^{i\langle\varphi\rangle}$ (MWF). Since the distribution law is symmetric with respect to the sign of $(\varphi - \langle\varphi\rangle)$, the phases of the CWF and the MWF coincide, $\langle\psi\rangle = |\langle\psi\rangle|e^{i\langle\varphi\rangle}$, so that the variables ψ and $\tilde{\psi}$ differ only in absolute value. Although the dominant contribution to the difference $\sqrt{\langle n\rangle} - |\langle\psi\rangle|$ is made by the short-wave fluctuations (the long-wave

$$\langle|n(p)|^2\rangle = \frac{np}{2mc}, \quad \langle|\varphi(p)|^2\rangle = \frac{d\mu}{dn}\Big/2cp$$

are suppressed by the smallness of the phase volume d^3p), it is precisely the divergent long-wave phase fluctuations which are responsible for the qualitative difference between the c-number characteristics $\langle\psi\rangle$ and $\langle\tilde{\psi}\rangle$ of the Bose system. The numerically small contribution made to $\langle\psi\rangle$ by the phase fluctuations turns out to be important in the expressions for the correlators $\langle\psi\psi\rangle$, $\langle\psi\psi\psi\rangle$, etc. The "loops" formed by the phase lines diverge as the external momenta tend to zero (at $T=0$ the "loop" formed by two lines diverges logarithmically, the one formed by three lines diverge quadratically, and so on). Accordingly, the Green function $g_{qq} = -\langle q(x)q(x')\rangle$, $q = \langle q\rangle + A\varphi + B\pi + C\varphi^2 + \ldots$ ($\pi = n - \langle n\rangle$) contains, besides the pole term $-\langle\varphi(x)\varphi(x')\rangle = g_{\varphi\varphi}(x-x')$, the divergent correction $(\sim \ln p) \sim \langle\varphi^2(x)\varphi^2(x')\rangle = g^2_{\varphi\varphi}(x-x')$ (the other terms converge); the function $\langle q(x)q(x')q(x'')\rangle$ forms, together with the three-ray diagram with an irreducible vertex from $H(n,\mathbf{v}) = H(\pi,\varphi)$, a diagram with a "loop" of two or three lines, so that there terms $\sim p^3 \ln p$, etc., appear at the irreducible vertex besides the term $\sim p^2$. A particular case of q is the original field

$$\psi = \sqrt{n}\,e^{i\varphi} = (\langle n\rangle + n')^{1/2} e^{i(\langle\varphi\rangle + \varphi')}$$
$$= \sqrt{\langle n\rangle}\,e^{i\langle\varphi\rangle}\left(1 + \frac{n'}{2\langle n\rangle} + \ldots\right)\left(1 + i\varphi' - \frac{1}{2}\varphi'^2 + \ldots\right).$$

It is clear that the transition to the adequate variable $\psi \to \tilde{\psi}$ is possible only when all the terms that are nonlinear in φ are discarded; the requirements of the standard commutation rules again lead to the equalities (42) and (43). Thus, the properties of the variable $\tilde{\psi}$ are unique in regard to the simplicity of the Green functions and the vertices in that in any variables nonlinearly connected with the canonical pair π and φ the Green functions contain divergent nonpole corrections, while the irreducible vertices contain nonanalytic components. A rigorous proof of the assertion that the degeneracy in φ is the sole source of the infrared anomaly of the anharmonicity at $T=0$ follows from the field-theoretic basis[8,1] of the quantum hydrodynamics (4). Notice that the exact vertices in the adequate and inadequate variables differ less by just small nonanalytic corrections, from the zeroth-order vertices (see (37)). Notice also that the infrared anomaly is a distinctive "appendix" to the Goldstone theorem: the conditions that create a gap in the spectrum despite the degeneracy in the phase (the singularity of the potential $V_{p\to 0}$, the external field, the phase symmetry of h_1) also remove the infrared anomaly.

In the foregoing analysis we assumed $\langle\psi\rangle \neq 0$, which can be interpreted as the introduction of a field $h<0$ that suppresses the homogeneous phase fluctuations ("fixes the phase")—the Bogolyubov quasi-averages:

$$H(h) = H + U, \quad U = \frac{1}{2}\int (h\psi + h^*\psi^*)\,dr = \frac{1}{2}h\sqrt{V}(a_0 + a_0^*).$$

Is the infrared anomaly not also removed by a similar "lifting of the degeneracy?" We can, by considering that part of the Hamiltonian which describes the homogeneous fluctuations,

$$H_0^{(\mu)} = -\mu a_0^* a + \frac{1}{2V}V_0(a_0^* a_0)^2 + \frac{h\sqrt{V}}{2}(a_0 + a_0^*)$$
$$\approx -\frac{\mu^2}{2V_0}V + \frac{1}{2}\left(\frac{\mu}{N_0}\Delta N_0^2 + \mu N_0 \delta \Delta\varphi^2\right)$$

($N_0 = |\langle a_0\rangle|^2 = V\mu/V_0, a_0 \equiv \psi\sqrt{\Delta} = (N_0 + \Delta N_0)^{1/2}e^{i(\varphi + \Delta\varphi)}$, $\delta = |h|V_0^{1/2}/\mu^{3/2}$ is a dimensionless external-field parameter), verify that in the thermodynamic limit ($N_0 \to \infty$) the homogeneous fluctuations are negligible (and, thus, the phase is "fixed") when $h \sim 1/N_0 \to 0$ ($N_0\delta^{1/2} \gg 1$ ($T=0$); $N_0\delta \gg T/\mu$ ($T>0$)):

$$\frac{(\Delta N_0)^2}{N_0} \sim \delta^{1/2}(T=0); \quad \frac{T}{\mu}(T>0);$$

$$\langle(\Delta\varphi)^2\rangle \sim \frac{1}{N\delta^{1/2}}(T=0), \quad \frac{T}{\mu N\delta}(T>0),$$

whereas the removal of the infrared anomaly requires an external field of finite amplitude[8] $h > h_c > p_c^2/m$.

4. ON THE PHASE-TRANSITION ORDER PARAMETER

1. A natural generalization $H^{(\mu)} \to \Omega$ of the BA (29) is easily obtained for $T>0$ by replacing $|\psi|^2$ in the potential $V(|\psi|^2)$ contained in

$$H^{(\mu)}=H(\psi,\psi^*)-\mu\int\psi^*\psi\,d\mathbf{r}\equiv\int\frac{|\nabla\psi|^2}{2m}d\mathbf{r}+V(|\psi|^2),$$

by $|\psi|^2+n'_T$ and also by taking account the thermodynamic potential $\Omega_1(T,\mu,\psi)$ of the excitation gas (including the zero-point oscillation energy

$$-\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}(\varepsilon_p{}^0+n_0V_p-\varepsilon_p{}^{\rm B})$$

and the dependence of n'_T and Ω_1 on $n_0=|\psi|^2$):

$$\Omega(T,\mu,\psi)=H^{(\mu)}(\psi,\psi^*)\,|_{|\psi|^2\to|\psi|^2+n'_T}+\Omega_1$$

$$\to \tilde{H}_T^{(\mu)}(\psi,\psi^*)$$

$$=\int d^3r\left[\frac{|\nabla\psi|^2}{2m}+A(T,\mu)|\psi|^2+\frac{1}{2}B(T,\mu)|\psi|^4\right],$$

$$n_T'=\int\frac{d^3p}{(2\pi)^3}\left[\frac{\varepsilon_p{}^0+n_0V_p-\varepsilon_p{}^{\rm B}}{2\varepsilon_p{}^{\rm B}}+\frac{\varepsilon_p{}^0+n_0V_p+\varepsilon_p{}^{\rm B}}{2\varepsilon_p{}^{\rm B}}n_G(\varepsilon_p{}^{\rm B})\right],$$

$$n_G(\varepsilon_p{}^{\rm B})=[\exp(\varepsilon_p{}^{\rm B}/T)-1]^{-1}. \tag{47}$$

At $T\sim T_c$ allowance for the thermal "smearing" ("depletion") of the condensate, i.e., the substitution $|\psi|^2\to|\psi|^2+n'_T$, is the main thing: here $n'_T\gg n_0$; T_c in the present approximation coincides with the degeneracy temperature T_0, of an ideal gas of $N=\mu V/V_0$ bosons (in (47) $n_0\to 0$, $n'_T\to n$):

$$n_0(T_c,\mu)=0,\quad T_c\sim n^{2/3}/m\sim\alpha^{-2/3}\varepsilon_{p_0}{}^0. \tag{48}$$

Near T_c (cf. (31) at $T=0$)

$$\tau\equiv(T_c-T)/T\ll 1,\quad A\approx a\tau; \tag{49}$$

$$\chi_\parallel=-\frac{1}{2[\varepsilon_p{}^0+2A(T,\mu)]}=-\frac{m}{p^2+\varkappa^2(T)},$$

$$\varkappa^2(T)=4mA(T,\mu); \tag{50}$$

$$\chi_\perp=m/p^2;\quad \varkappa\sim p_0\tau^{1/2},$$

as at $T=0$, $\varkappa(T)\sim p_0(T)\sim p_h(T)=mc_{\rm B}(T)$

$$(\varepsilon^{\rm B}_{p\to0}=[\varepsilon_p{}^0(\varepsilon_p{}^0+2A(T,\mu))]^{1/2}\approx p[A(T,\mu)/m]^{1/2}=c_{\rm B}(T)p).$$

The ratio (determining Gi) of the fluctuations averaged in a volume radius equal to the correlation length $r_c\sim\varkappa^{-1}$) to the square of the order parameter coincides here with the ratio of the number of supercondensate particles with momentum $p\lesssim\varkappa$ to the number of particles in the condensate

$$\Delta n'_{p\lesssim\varkappa(T)}/n_0(T)\sim\alpha^{1/3}/\sqrt{\tau}\sim({\rm Gi}/\tau)^{1/2},\quad {\rm Gi}\sim\alpha^{2/3}. \tag{51}$$

Notice that $\chi_\parallel$ and Gi correspond to the same combinations of the parameters m, a, and B as in the nondegenerate case.

It is just inside the fluctuation region $\tau<{\rm Gi}\sim\alpha^{2/3}$ that the model (27) loses its small parameter: the parameter $\alpha(T)$ characterizing the anharmonicity corrections increases as $T\to T_c$ as a result of the "softening" of the mode $c_{\rm B}(T)\sim c_{\rm B}\tau^{1/2}$; see, for example, the ratio of the contributions of the diagrams in Fig. 2 at $\tau\sim\alpha^{2/3}$:

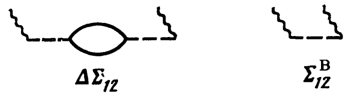

FIG. 2.

$$\alpha(T)\sim\frac{\Delta\Sigma_{12}}{\Sigma_{12}{}^{\rm B}}\sim\frac{n_0(T)V_0{}^2T_c\varkappa^3/(\varkappa^2/m)^2}{n_0(T)V_0}\sim\frac{\alpha^{1/3}}{\tau^{1/2}}\sim 1. \tag{52}$$

The same "fluctuation-region boundary" $\tau\sim{\rm Gi}\sim\alpha^{2/3}$ corresponds to the temperature at which the momentum $\varkappa(T)\sim p_0\tau^{1/2}$, decreasing as $T\to T_c$, coincides in order of magnitude with the infrared-anomaly momentum p_c $(V_0\Pi_0(p_c)\sim 1, p_c(T\sim T_c)\sim\alpha^{1/3}p_0$; the loop $\Pi_0\sim m^2T_c/p_c$ is constructed from the Bogolyulov Green functions $G_T^{\rm B}(p)$ corresponding to the Hamiltonian (47)). Notice that the relation

$$p_c\lessdot\varkappa(T), \tag{53}$$

which is valid at the "boundaries of the fluctuation region," is maintained also inside the region (although the behavior of $\varkappa(\tau)\sim\tau^{1/2}$ should change, the formula for the estimate of p_c

$$V_0\Pi_0(p_c)\sim V_0m^2T_c(1/p_c-1/\varkappa(T))\sim 1$$

remains valid).

2. The analysis carried out with $\psi=\langle\tilde\psi\rangle$ as the order parameter, just as at $T=0$, the exact formula $\chi_\parallel^{-1}(0)=0$ (see (50), where $\chi_\parallel^{-1}(0)=-\varkappa^2(T)/m$). But in the $T>0$ case we cannot correct the situation by going over to $\tilde\psi=\langle\tilde\psi\rangle$ (42) $(|\tilde\psi|=(\langle|\tilde\psi|^2\rangle)^{1/2}$); here the divergence of the phase fluctuations is not the only source of the infrared anomalies in the anharmonicity; there arises, besides the "noninvariant" anomaly, an "invariant" anharmonicity anomaly connected with the excitation gas; here the $\tilde S$-expansion coefficients (the exact vertices) have anomalies—pole singularities characterizing sound in the excitation gas—in the variables n and φ as well. In order to obtain $\tilde H_T(\tilde\psi,\tilde\psi^*)$ without singularities, we must separate the entropyless (coherent) component of the superfluid from the excitation gas. If we consider the steady-state motion of the condensate relative to the stationary excitation gas, we can easily obtain an expression for the kinetic energy of the field $\tilde\psi$ in the form $\rho_s v_s^2/2$, $\rho_s=\rho-\rho_n$ (see Sec. 5), which corresponds to $|\nabla\tilde\psi|^2/2m$, where

$$|\tilde\Psi|^2=\rho_s/m,\quad \mathbf{v}_s=\hbar m^{-1}\nabla\varphi,\quad \tilde\Psi=|\tilde\Psi|e^{i\varphi}.$$

Hence it is natural to propose as a generalization for the "correct harmonic approximation"

$$H^{(\mu)}(\tilde\Psi,\tilde\Psi^*)=H(\tilde\Psi,\tilde\Psi^*)-\mu\int\tilde\Psi^*\tilde\Psi\,d\mathbf{r}$$

the expression

$$\Omega(T,\mu,\tilde\psi)=H^{(\mu)}(\tilde\psi,\tilde\psi^*)\,|_{|\tilde\psi|^2\to|\tilde\psi|^2+\rho_n/m}+\Omega_1\to\tilde H_T^{(\mu)}(\tilde\psi,\tilde\psi^*)$$

$$=\int d\mathbf{r}\left[\frac{|\nabla\tilde\psi|^2}{2m}+\tilde A(T,\mu)|\tilde\psi|^2+\frac{1}{2}\tilde B(T,\mu)|\tilde\psi|^4\right], \tag{54}$$

$$\rho_n=\frac{1}{3}\int p^2\left(-\frac{dn_G(\varepsilon_p{}^{\rm B})}{d\varepsilon}\right)\frac{d^3p}{(2\pi)^3}. \tag{55}$$

Then, as is easy to see, the condition $\pi_s(T_c, \mu) = 0$ will give for T_c the same value as $n_0(T_c, \mu) = 0$ (at the point $n_0 = 0$, ρ_n from (55) coincides with ρ); in the region $\tau \ll 1$ the coefficients $\tilde{A}$ and $\tilde{B}$ coincide with A and B to within α, so that the relations (50)–(53) are preserved, and the above-considered picture requires only the correction $\psi \to \tilde{\psi}$.

The assumption made above amounts to the assertion that the field $\tilde{\psi} = \tilde{\psi}_T$ ($|\psi_T| = \sqrt{\rho_s/m}$) plays at $T > 0$ the same role as played by $\tilde{\psi}(|\tilde{\psi}| = \sqrt{\rho/m})$ at $T = 0$: the linear combinations of the long-wave Fourier components of $\tilde{\psi}_T$ ($\tilde{\psi}_T = \langle \tilde{\psi}_T \rangle$) correspond to the "true" normal modes (the quasiparticle operators in the case when $T \to 0$); consequently, the use of $\tilde{\psi}_T$ eliminates the infrared anomaly in $\tilde{H}$. Actually, the above definition of $\tilde{\psi}_T$ requires some refinement in the case of temperatures far from T_c: the field ("Goldstone") mode here is not the only "slow" mode; the oscillatory modes of the field interact with the analogous modes of the excitation gas, so that the degrees of freedom that are separated in the steady-state motion ($\varepsilon/p = 0$) intermix. At low T the field mode corresponds to first sound, and the "normal-component" mode to second sound (the structures of both modes are determined by the asymptotic expression for the Green functions (see Sec. 5)). But near T_c (i.e., for $\tau \ll 1$) the coupling between the two subsystems (the c-number field and the excitation gas) is weak because of the large difference between the frequencies of the sound velocities, second sound corresponding here to the Goldstone mode (the "unshifted" Goldstone mode—fourth sound—is close in frequency to first sound when $T \to 0$ and to second sound when $T \to T_c$). The weakening of the coupling between the subsystems and the smallness of ε/p for the "soft" Goldstone mode (the closeness to the steady-state $\varepsilon/p = 0$ process) allow us to assume that, for $\tau \ll 1$, the adequate c-number field (the phase-transition order parameter) is $\tilde{\psi}_T = \langle \tilde{\psi}_T \rangle$, with $|\psi_T| = \sqrt{\rho_s/m}$.

This result is confirmed by the fact that the definition $|\tilde{\psi}_T| = \sqrt{\rho_s/m}$ (like $|\tilde{\psi}| = \sqrt{\rho/m}$ at $T = 0$) corresponds to the elimination of the long-wave phase fluctuations in the entire momentum region $p \lesssim p_c$, where the infrared anharmonicity anomaly is formed. Indeed, in the momentum (p) range characterizing the difference

$$\rho_s - n_0 m = n_T' m - \rho_n \tag{56}$$

(see (47) and (55)), when $p \gg \varkappa(T)$($\varkappa(T) \sim \sqrt{mA}$), the contributions to $n_T'(m)$ and ρ_n are nearly equal and cancel each other (their difference in the integrand is $\sim -m^2 A^2/p^4$), and when $p \ll \varkappa(T)$ the contribution on ρ_n ($\sim p/\sqrt{mA}$) is much smaller than the contribution to $n_T' m(\sim \sqrt{mA}/p)$; consequently, if we represent n_T' and ρ_n in the form of a sum of long-wave ($p \leqslant Q$) and short-wave ($p > Q$) contributions, i.e., if we set

$$n_T' = (n_T')_L + (n_T')_{sh}, \quad \rho_n = (\rho_n)_L + (\rho_n)_{sh},$$

then, as is easy to verify, there exists a momentum $Q \sim \varkappa(T)$ such that

$$\{-(\rho_n)_L + [(\dot{n}_T')_{sh} m - (\rho_n)_{sh}]\} = 0,$$

i.e., such that

$$\rho_s = [n_0 + (n_T')_L] m \tag{57}$$

(for $p \sim Q$ the contribution to $n_T' m - \rho_n$ is positive and not small; in comparing $n_T' m$ and ρ_n it is more convenient to use in place of (55) the formula

$$\rho_n = \int \left[\tilde{m} + \frac{1}{3} p \frac{d\tilde{m}}{dp} \right] n_G(\varepsilon_p{}^B) \frac{d^3p}{(2\pi)^3}, \tag{58}$$

where

$$\tilde{m} = p(d\varepsilon_p/dp)^{-1}, \quad \varepsilon_p = \varepsilon_p{}^B = [\varepsilon_p{}^0(\varepsilon_p{}^0 + 2A)]^{1/2};$$

concerning the causes and manifestations of the "nonadditivity" of the quasiparticle masses in (58) see Appendix 2).

Determining with the aid of the quantity Q contained in (57) the "long-wave field operator" $\hat{\psi}_L = \langle \hat{\psi} \rangle_{sh}$ (obtained by averaging the original operator $\hat{\psi}$ over the short-wave $p > Q$ fluctuations), we find

$$|\tilde{\psi}_T|^2 = \rho_s/m = n_0 + (n_T')_L = \langle |\hat{\psi}|_L{}^2 \rangle, \tag{59}$$

so that the phase fluctuations in the region $p < Q \sim \varkappa(T)$, which includes $p < p_c$ (see (53)), are indeed eliminated from the definition of $\tilde{\psi}_T$ by $|\tilde{\psi}_T| = \sqrt{\rho_s/m}$. The explicit form of $\tilde{\psi}_T$—the "adequate field operator"—is similar to the form (42), (43) in the case $T = 0$:

$$\hat{\tilde{\psi}}_T = \sqrt{\langle \hat{\tilde{n}}_L \rangle} \exp(i \langle \hat{\tilde{\varphi}}_L \rangle) \left[1 + \frac{\hat{\tilde{n}}_L - \langle \hat{\tilde{n}}_L \rangle}{2 \langle \hat{\tilde{n}}_L \rangle} + i(\hat{\tilde{\varphi}}_L - \langle \hat{\tilde{\varphi}}_L \rangle) \right],$$

$$\tilde{\psi}_T = \langle \hat{\tilde{\psi}}_T \rangle; \tag{60}$$

$$\hat{\psi}_L = \sqrt{\hat{\tilde{n}}_L} \exp(i \hat{\tilde{\varphi}}_L) \tag{61}$$

(the relation connecting $\hat{\tilde{n}}_L$ and $\hat{\mathbf{v}}_s = \hbar m^{-1} \nabla \hat{\tilde{\varphi}}_L$ with $\hat{\psi}_L$ and $\hat{\psi}_L^+$ is similar to (2)).

The above method of eliminating the infrared anomalies from the effective Hamiltonian $\tilde{H}_T(\psi, \psi^*)$ by choosing an adequate field variable $\hat{\psi} - \tilde{\psi}_T$ does not depend on the interaction strength. A stongly interacting Bose system (real He^4) differs from the model (27) only in that $|\tilde{\psi}_T|$ differs essentially from $|\psi|$ in the region of low T (where $n_0/n \lesssim 1$) and the fluctuation region is broad (it encompasses the entire region $\tau \ll 1$:Gi ~ 1). The foregoing analysis supports the choice of a field with modulus $\sqrt{\rho_s/m}$ as the order parameter of the λ transition in the GP theory[17] (including its modification that takes into account phenomenologically the contribution of the fluctuations to the coefficients $A, B, \ldots$, i.e., the ψ theory[23]).

3. The described situation is in many respects common to all phases transitions with spontaneously broken continuous symmetry: the infinite degeneracy leads to an infrared anomaly. In particular, it is easy to follow the analogy between the λ transition and the ferromagnetic transition in the isotropic Heisenberg model; the analogy between ψ and $\mathbf{M}$ (especially in the "planar model") is not affected in this scheme by the fact that $\int \mathbf{M} \cdot d\mathbf{r}$ is conserved while $\int \psi d\mathbf{r}$ is not, nor is it affected by the arbitrariness of the superfluid analog of the external magnetic field and by the special role played by $\mathbf{v}_s = \hbar m^{-1} \nabla \varphi$ as a result of the connection with the broken Galilean symmetry (see Ref. 43a).

In the general case we should distinguish between the original order parameter x, which is suitable for a renormalization-group analysis of the "microscopic Hamiltonian,"[44] but which corresponds to "exotic" characteristics of the system because of the infrared anharmonicity anomaly (the divergence of the longitudinal susceptibility, etc.), on the one hand, and the "effective" order parameter $\tilde{x}$, which is the only one that should figure in the effective Hamiltonian of the theory of phase transitions, on the other. The relation between x and $\tilde{x}$ is similar to that in the case of the λ transition:

$$x=\langle\hat{x}\rangle,\quad \tilde{x}=(\langle|\hat{x}_L|^2\rangle)^{1/2}\exp\{i\langle\hat{\varphi}_L\rangle\},\quad \tilde{x}=\langle\hat{\tilde{x}}\rangle, \tag{62}$$

$$\hat{x}_L=\langle\hat{x}\rangle_{sh},\quad \hat{\tilde{x}}=\tilde{x}\left[1+\frac{|\hat{x}_L|^2-\langle|\hat{x}_L|^2\rangle}{2\langle|\hat{x}_L|^2\rangle}+i(\hat{\varphi}_L-\langle\hat{\varphi}_L\rangle)\right]; \tag{63}$$

the region L corresponds to $p\ll Q\sim\varkappa(T)$, where $\varkappa(T)$ is the reciprocal correlation length.[9)]

Notice that the superfluid (like the superconducting) state occupies a special place among states with broken continuous symmetry. Here the order parameter is directly connected with the particles' translational degrees of freedom (which constitute all the thermodynamically important ones for a superfluid), and characterizes the wave nature of the particle motion at the macroscopic level, forming, similarly to the condensate of noninteracting bosons, a special macroscopic field. Even for an ideal gas of bosons with $m\neq 0$ the c-number field equation contains $\hbar$, in contrast to the normal macroscopic fields (sound, radio waves, etc.). This is the basis for the macroscopic manifestation of quantum relations (e.g., the macroscopic quantization of the angular momentum of a thin cylindrical layer). The case in which interaction occurs is different in that a field is formed as a result of an ordinary type of second-order phase transition with broken continuous symmetry, which gives rise to the following additional specific properties: a linear spectrum, superfluidity, the possibility of undamped motion (this particular manifestation of the quantum nature of matter has for decades provided one of the profound motivations for the study of superfluidity). As follows from the above analyses, in the case of interaction the coherent component does not coincide with the condensate; at $T=0$ it is the entire fluid (its long-wave macroscopic degrees of freedom), and at $T>0$ it is the "field component" (whose definition at temperatures far from T_c depends somewhat on ε/p). Because of the presence of the infrared anomaly, only the use of the effective c-number field characteristic $\tilde{\psi}$ (the MWF),[10] and not ψ (the CWF), allows us to reveal the physically important analogy between the superfluid state and a classical nonlinear field, as well as the analogy between the λ transition and a second-order phase transition with no degeneracy.

5. THE EFFECTIVE c-NUMBER FIELD AT LOW $T>0$

Using the method[14] of combined variables (21), let us compute the temperature corrections to the Green functions $g_{ab}(p)$, where $a,b=(\pi,\varphi)$ [Eq. (23)]. The interaction is, generally speaking, not assumed to be weak; the only limitation on the model is the requirement $p_c\ll p_h$, owing to which we can choose q_0 in accordance with (25). The elements of the inverse matrix $(\hat{g}^{-1})_{ab}$ are the coefficients in the quadratic part of the action $\tilde{S}_2(\pi,\varphi)$, the "two-ray vertices." The condition $q_0\ll p_h$ from (25) allows us to write simple expressions for the vertices that take account of the contribution of the "short-wave" $(p>q_0)$ anharmonicity, i.e., for the $\tilde{S}_2^{(q_0)}$-expansion coefficients at $T=0$ (see Ref. 14):

$$S^{(q_0)}-\beta P(n_0,\mu,0)\approx\tilde{S}_2^{(q_0)}=\int\left[-\frac{P_\mu}{2m}(\nabla\varphi)^2-\frac{P_{\mu\mu}}{2}\dot{\varphi}^2+iP_{\mu n_0}\pi\dot{\varphi}+\frac{1}{2}P_{n_0n_0}\pi^2\right]d\mathbf{r}\,d\tau$$

$$=\frac{1}{2}\sum_{p=(\varepsilon',\mathbf{p})}\Big\{\Big(-\frac{1}{m}P_\mu p^2-P_{\mu\mu}\varepsilon'^2\Big)\varphi(p)\varphi(-p)-2P_{\mu n_0}\varepsilon'\varphi(p)\pi(-p)+P_{n_0n_0}\pi(p)\pi(-p)\Big\}$$

$$=\frac{1}{2}\sum_p(\varphi^*(p)\pi^*(p))\hat{g}^{-1}(p)\begin{pmatrix}\varphi(p)\\ \pi(p)\end{pmatrix}, \tag{64}$$

$$\hat{g}^{-1}(p)=\begin{pmatrix}-\frac{1}{m}P_\mu p^2-P_{\mu\mu}\varepsilon'^2 & P_{\mu n_0}\varepsilon'\\ -P_{\mu n_0}\varepsilon' & P_{n_0n_0}\end{pmatrix}, \tag{65}$$

where $P(n_0(\mathbf{r},\tau),\ \mu(\mathbf{r},\tau),\ \mathbf{v}(\mathbf{r},\tau)$ is the local pressure, $\mu(\mathbf{r},\tau)\equiv\mu+i\dot{\varphi}-(\nabla\varphi)^2/2m$, and ε' is the frequency corresponding to the imaginary time τ.

The condition $q_0\gg p_c$ excludes from the derivatives of P in $\tilde{S}_2^{(q_0)}$ the infrared anomalies connected with the phase degeneracy, thus facilitating subsequently the computation of the excitation-gas-related anharmonicity in $\tilde{S}_2$ for $T>0$. The matrix $\hat{g}(p)$ from (65) coincides then with (23). On account of the conditions (25), the contribution $\Sigma_{ab}^{T=0}$ of the long-wave $(p\ll q_0)$ anharmonicity to $\hat{g}_{ab}^{-1}$ at $T=0$ can be neglected, i.e., $\tilde{S}_2\approx\tilde{S}_2^{(q_0)}$. On the other hand, it is precisely this contribution that is important at $T>0$; the condition $T\ll cq_0$ allows us to neglect the effect of temperature on the contribution of the anharmonicity of the modes with $p>q_0$: $\tilde{S}_2^{(q_0)}(T)\approx\tilde{S}_2^{(q_0)}(T=0)$, (64). Because of the inequality $(T/cp_n)\ll 1$, in computing the contribution of the modes with $p\ll q_0$ at $T>0$ we need consider only the diagrams with the smallest number of integrations, i.e., the situation in the arbitrary case is the same as for the $\alpha\ll 1$ model (27). Therefore, let us, for definiteness, return to the model (27), in which

$$P_\mu=n,\quad P_{\mu\mu}=0,\quad P_{\mu n_0}=1,\quad P_{n_0n_0}=-V_0 \tag{66}$$

(the derivatives of P correspond to the BA); $p_c\sim p_he^{-1/\alpha}\ll p_h\sim p_0$. Notice that the elimination of the "noninvariant" anharmonicity anomaly in $\tilde{S}(T)\,(q_0\gg p_c)$ does not prevent its being taken into account together with the "invariant" anomaly in the subsequent computation of $\Sigma_{ik}^{T>0}(p)$.

Using (66), we obtain, similarly to (64) and (65), the expression

$$\tilde{S}_2(T)=\frac{1}{2}\sum_p(\varphi^*(p)\pi^*(p))\hat{g}^{-1}(p)\begin{pmatrix}\varphi(p)\\ \pi(p)\end{pmatrix}$$

$$\hat{g}^{-1}=\hat{g}_B^{-1}-\hat{\Sigma}=\begin{pmatrix}-\frac{n}{m}p^2-\Sigma_{\varphi\varphi} & \varepsilon'-\Sigma_{\varphi\pi}\\ -\varepsilon'-\Sigma_{\pi\varphi} & -V_0-\Sigma_{\pi\pi}\end{pmatrix} \tag{67}$$

The corrections Σ_{ab}, which makes $\tilde{S}_2(T)$ different from

$\tilde{S}_2^{(q_0)}(T) \approx S_2 0T = 0)$, are the result of the interaction between the modes with $p < q_0$, an interaction whose vertices correspond to the nonquadratic term in $\tilde{S}^{(q_0)} \approx S$; the most important term

$$\int d\mathbf{r}\, d\tau P_{\mu n_0}\pi(\nabla\varphi)^2/2m \approx -\int d\mathbf{r}\, d\tau\pi(\nabla\varphi)^2/2m$$

$$= \frac{1}{\sqrt{\beta}} \sum_{p_1 p_2 p_3} \frac{\mathbf{p}_1\mathbf{p}_2}{2m} \varphi(p_1)\varphi(p_2)\pi(p_3) \qquad (68)$$

(the vertex $d(p_1, p_2, -p_3) = \mathbf{p}_1 \cdot \mathbf{p}_2/2m$ (see Fig. 3a)).

In the single-loop approximation (Fig. 3b), after the summation over the frequencies we can set $\Sigma_{ab} = \Sigma^{\rm I}_{ab} + \Sigma^{\rm II}_{ab}$, where in the integrand $\Sigma^{\rm I}_{ab}$ contains the factor $[1 + n_G(\varepsilon^{\rm B}_{p_1}) + n_G(\varepsilon^{\rm B}_{\mathbf{p}+\mathbf{p}1})$, while Σ^{II}_{ab} contains $[n_G(\varepsilon^{\rm B}_{\mathbf{p}+\mathbf{p}_1} - n_g(\varepsilon^{\rm B}_{p_1})]$ $0 n_G(x) = (e^{x/T} - 1)^{-1}$). In the region $cp \ll T$ we find

$$\Sigma_{\varphi\varphi}{}^{\rm I} = \Sigma_{\varphi\pi}{}^{\rm I} = 0, \quad \Sigma_{\pi\pi}{}^{\rm I} = -\frac{3}{16}\frac{c^2}{n^2}\rho_n; \qquad (69)$$

$$\Sigma^{\rm II}_{\varphi\varphi}(p) = -\frac{\rho_n}{m^2}p^2 - \int \frac{d^3p_1}{(2\pi)^3}\left(\frac{\mathbf{p}\mathbf{p}_1}{m}\right)^2 \frac{dn_G}{d\varepsilon_{p_1}{}^{\rm B}} \frac{i\varepsilon'}{\varepsilon^{\rm B}_{\mathbf{p}+\mathbf{p}_1} - \varepsilon_{p_1}{}^{\rm B} - i\varepsilon'},$$

$$\Sigma^{\rm II}_{\varphi\pi}(p) = \int \frac{d^3p_1}{(2\pi)^3} \frac{\mathbf{p}\mathbf{p}_1}{m} \frac{\varepsilon_{p_1}{}^{\rm B}}{2n} \frac{dn_G}{d\varepsilon_{p_1}{}^{\rm B}} \frac{\varepsilon'}{\varepsilon^{\rm B}_{\mathbf{p}+\mathbf{p}_1} - \varepsilon_{p_1}{}^{\rm B} - i\varepsilon'}, \qquad (70)$$

$$\Sigma^{\rm II}_{\pi\pi}(p) = -\frac{3}{4}\frac{c^2}{n^2}\rho_n - \int \frac{d^3p_1}{(2\pi)^3} \frac{(\varepsilon_{p_1}{}^{\rm B})^2}{4n^2} \frac{dn_G}{d\varepsilon_{p_1}{}^{\rm B}} \frac{i\varepsilon'}{\varepsilon^{\rm B}_{\mathbf{p}+\mathbf{p}_1} - \varepsilon_{p_1}{}^{\rm B} - i\varepsilon'}.$$

The dominant contribution to $\Sigma_{ab}(p)$ is made by the integrals (70); their computation with logarithmic accuracy yields a nonanalytic—in T—correction to $\varepsilon^{\rm B}_p \approx c_{\rm B}\, p$, i.e., to the Bogolyubov sound velocity $c_{\rm B} = (nV_0/m)^{1/2}$:

$$\frac{c - c_{\rm B}}{c_{\rm B}} \approx \frac{27}{16}\frac{\rho_n}{\rho} \ln \frac{c_{\rm B}{}^2}{\gamma T^2} \qquad (71)$$

(γ is given by the relation $\varepsilon^{\rm B}_p \approx c_{\rm B}\, p(1 + \gamma p^2)$). The formula (71) coincides with the result obtained from the Andreev-Khalatnikov kinetic equation[45]:

$$\Delta c = \frac{3c\rho_n}{4\rho} A \ln \frac{c^2}{\gamma T^2}, \quad A = \left(1 + \frac{\rho}{c}\frac{dc}{d\rho}\right)^2$$

($A = 9/4$ for the $\alpha \ll 1$ model).

Although for the $\alpha \ll 1$ model each integration over momentum contributes an additional small factor, with no divergences occurring in terms of the variables φ and π, the single-loop approximation for $\Sigma_{ab}(p)$ ($T > 0$) is not suitable for all p. Indeed, in the region $cp \ll \Gamma_T$ (Γ_T is the damping constant for the characteristic excitations at the given T) we should take into account in the $\Sigma^{\rm II}$ integrals the excitation

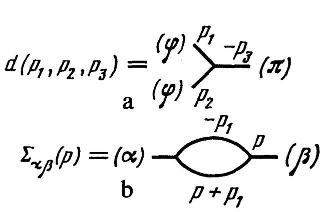

FIG. 3.

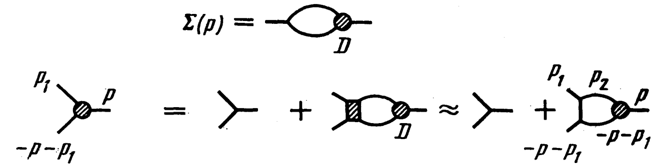

FIG. 4.

damping constant that is greater than the Bogolyubov contribution in the denominator. At the same time, we should also not neglect the changes in the vertex (Fig. 4). Following Ref. 14 (see, however, footnote 11)), let us, using the explicit form of the Green functions associated with the vertex D in $\Sigma_{ab}(p)$ (the Bogolyubov $g = g_{\rm B}$, but with allowance for the damping), separate out the two linear combinations (h_3 and g_3) of the vertices (with different ray compositions) in terms of which we can express all the Σ_{ab}; in the combinations h_3 and g_3 the vertices are taken with factors obtained from the residues at the poles of the Green functions corresponding to the rays. The equation for D assumes the form of the kinetic equations for h_3 and g_3, whose solutions yield ($\tau \to it$, $\varepsilon' \to -i\varepsilon$)

$$h_3(\mathbf{p}_1, \mathbf{p}_2, \varepsilon) \approx \frac{(p^2\varepsilon_{p_1}{}^{\rm B}/3m) + (\mathbf{p}\mathbf{p}_1\varepsilon/m)}{\varepsilon^2 - {}^1/_3 c_{\rm B}{}^2 p^2},$$

$$g_3(\mathbf{p}_1, \mathbf{p}_2, \varepsilon) \approx \frac{c^2\mathbf{p}\mathbf{p}_1 + \varepsilon_{p_1}{}^{\rm B}\varepsilon}{2(\varepsilon^2 - {}^1/_3 c_{\rm B}{}^2 p^2)}; \qquad (72)$$

$$\Sigma_{\varphi\varphi} = \Sigma^{\rm II}_{\varphi\varphi} = -\frac{\rho_n}{m^2}\left[p^2 - \frac{\varepsilon^2 p^2}{\varepsilon^2 - {}^1/_3 c_{\rm B}{}^2 p^2}\right],$$

$$\Sigma_{\varphi\pi} = \Sigma^{\rm II}_{\varphi\pi} = -i\frac{\rho_n c_{\rm B}{}^2}{2mn} \frac{\varepsilon p^2}{\varepsilon^2 - {}^1/_3 c_{\rm B}{}^2 p^2}, \qquad (73)$$

$$\Sigma_{\pi\pi} = \Sigma_{\pi\pi}{}^{\rm I} + \Sigma^{\rm II}_{\pi\pi} = \frac{3\rho_n c_{\rm B}{}^2}{4n^2}\left[-\frac{5}{4} + \frac{\varepsilon^2}{\varepsilon^2 - {}^1/_3 c_{\rm B}{}^2 p^2}\right].$$

Notice that the result

$$\Sigma_{\varphi\varphi}(p, \varepsilon = 0) = -\rho_n p^2/m^2, \qquad (74)$$

obtained from (73) in the lowest approximation in α is in fact exact: it corresponds to the exact formula for the system's kinetic energy (entering into the expression for $\tilde{S}$) at $T > 0$—for constant condensate velocity $\mathbf{v}_s = \hbar m^{-1}\nabla\langle\varphi\rangle$:

$$H_{kin} = \int \frac{\rho_s v_s{}^2}{2} d\mathbf{r}, \quad \rho_s = \rho - \rho_n. \qquad (75)$$

It follows from (74) that (75) is valid in the general case of steady-state motion ($\varepsilon = 0$, $\mathbf{p} \neq 0$) as well; accordingly, the role of the longitudinal component of the field $\tilde{\psi} = |\tilde{\psi}| e^{i\langle\varphi\rangle}$ is played here by

$$|\tilde{\psi}| = \left(\frac{\rho_s}{m}\right)^{1/2} \left(H_{kin} = \int \frac{\hbar^2 |\nabla\tilde{\psi}|^2}{2m} d\mathbf{r}\right).$$

With the aid of (67) and (73) we find

$$g_{\varphi\varphi}=\frac{mc_B^2}{\rho}\left[\left(1-\frac{15}{8}\frac{\rho_n}{\rho}\right)\Big/z_1+\frac{27}{16}\frac{\rho_n}{\rho}\Big/z_2\right],$$

$$g_{\varphi\pi}=-g_{\pi\varphi}=-i\varepsilon\left[\left(1-\frac{45}{16}\frac{\rho_n}{\rho}\right)\Big/z_1+\frac{45}{16}\frac{\rho_n}{\rho}\Big/z_2\right], \quad (76)$$

$$g_{\pi\pi}=\frac{\rho p^2}{m}\left[\left(1-\frac{25}{16}\frac{\rho_n}{\rho}\right)\Big/z_1+\frac{25}{16}\frac{\rho_n}{\rho}\Big/z_2\right];$$

$$z_1=\varepsilon^2-c_1^2p^2, \quad z_2=\varepsilon^2-c_2^2p^2,$$

$$c_1=c_B\left(1+\frac{35}{32}\frac{\rho_n}{\rho}\right), \quad c_2=\frac{c_B}{\sqrt{3}}\left(1-\frac{33}{16}\frac{\rho_n}{\rho}\right);$$

c_1 and c_2, the first- and second-sound velocities with allowance for their temperature dependence, are computed from the equation for $\varepsilon=\varepsilon_{1,2}\equiv c_{1,2}\,p$:

$$\det\hat{g}^{-1}\sim\left\{\left[\varepsilon^2-c_B^2p^2\left(1-\frac{31}{16}\frac{\rho_n}{\rho}\right)\right]\right.$$

$$\left.\times\left(\varepsilon^2-\frac{1}{3}c_B^2p^2\right)-\frac{11}{4}\frac{\rho_n}{\rho}\varepsilon^2c_B^2p^2\right\}=0. \quad (77)$$

The results (76) satisfies the Bogolyubov identity

$$G_{11}(p,\ \varepsilon=0)\approx ng_{\varphi\varphi}(p,\ \varepsilon=0)\approx m\rho/\rho_s p^2, \quad (78)$$

and agrees exactly with the predictions of two-velocity hydrodynamics: if we substitute into the equation[25]

$$c^4-c^2\left[\left(\frac{\partial P}{\partial\rho}\right)_\sigma-\frac{\rho_s}{\rho_n}\sigma^2\left(\frac{\partial T}{\partial\sigma}\right)_\rho\right]+\frac{\rho_s}{\rho_n}\sigma^2\left(\frac{\partial T}{\partial\sigma}\right)_\rho\left(\frac{\partial P}{\partial\rho}\right)_T=0 \quad (79)$$

the thermodynamic characteristics of a gas of phonons with energy $\varepsilon_p=c_B\,p$, we obtain for $c_{1,2}$ an equation that is equivalent to (77).[11)]

It follows from the formulas (76) that the variables φ and π are linear combinations of the canonical variables $P_{1,2}$ and $Q_{1,2}$ of two oscillators—with frequencies equal to those of first and second sounds $\varepsilon=c_{1,2}\,p$:

$$\varphi=\left(1-\frac{65}{32}\frac{\rho_n}{\rho}\right)P_1+\frac{9}{4}\left(\frac{\rho_n}{\rho}\right)^{1/2}P_2,$$

$$\pi=\left(1-\frac{25}{32}\frac{\rho_n}{\rho}\right)Q_1+\frac{5}{4}\left(\frac{\rho_n}{\rho}\right)^{1/2}Q_2. \quad (80)$$

Indeed, for an oscillator, with

$$H=\frac{1}{2}\left(\alpha p^2+\frac{\varepsilon_0^2}{\alpha}q^2\right)=\varepsilon_0\left(a^+a+\frac{1}{2}\right),$$

$$p=(\varepsilon_0/2\alpha)^{1/2}(a+a^+), \quad q=-i(\alpha/2\varepsilon_0)^{1/2}(a^+-a),$$

$$G=(\varepsilon-\varepsilon_0+i\delta)^{-1},$$

we have

$$G_{pp}(\varepsilon)=\frac{\varepsilon_0}{2\alpha}[G(\varepsilon)+G(-\varepsilon)]=\frac{\varepsilon_0^2}{\alpha}(\varepsilon^2-\varepsilon_0^2+i\delta)^{-1},$$

$$G_{qq}(\varepsilon)=\frac{\alpha}{2\varepsilon_0}[G(\varepsilon)+G(-\varepsilon)]=\alpha(\varepsilon^2-\varepsilon_0^2+i\delta)^{-1}, \quad (81)$$

$$G_{pq}(\varepsilon)=-G_{qp}(\varepsilon)=\frac{1}{2i}[G(\varepsilon)-G(-\varepsilon)]=-i\varepsilon(\varepsilon^2-\varepsilon_0^2+i\delta)^{-1}.$$

The formulas (80) follow directly from a comparison of (76) and (81). The variables P_2 and Q_2 characterize collisional-hydrodynamic sound in a system of quasiparticles, and can consequently be expressed in terms of the number density n_n and velocity $\mathbf{v}_n$ of the quasiparticles (the small correction characterizing the influence of the superfluid component can be neglected in the present case). In their turn n_n and $\mathbf{v}_n$ can be regarded as the mean values of operators connected with the quasiparticle creation and annihilation operators $B_\mathbf{p}^+$ and $B_\mathbf{p}$ (this connection, which is similar to the formulas (2) for the variables $\hat{n}$ and $\hat{\mathbf{v}}$ for a particle system, is free of the mathematical difficulties associated with the δ function, since the moment admissible here are clearly finite). In the approximation (76) both the oscillator modes turn out to be undamped (this corresponds to the local-equilibrium approximation in the phenomenological approach). The formulas (80) allow us to express the canonical variables $\hat{P}_1$ and $\hat{Q}_1$ of the quantum-hydrodynamic-sound oscillators in terms of $\hat{\varphi},\hat{\pi},\hat{n}_n$, and $\mathbf{v}_n$ (or $\hat{B}_\mathbf{p}^+$ and $\hat{B}_\mathbf{p}$). On the other hand, the Fourier components of the effective field $\tilde{\psi}$ are complex canonical variables of the same oscillators, and can thus be expressed in terms of $\hat{\varphi}$, $\hat{\pi}$, $\hat{n}_n$, and $\hat{\mathbf{v}}_n$ (or $\hat{B}_\mathbf{p}^+$ and $\vec{B}_\mathbf{p}$). Notice that, in contrast to $\tilde{\psi}=\langle\tilde{\psi}\rangle$, the operator $\tilde{\psi}$ for $T>0$ itself has in the collisional-hydrodynamic region only a formal meaning (just like $\hat{n}_n$ and $\mathbf{v}_n$)—sound in an excitation gas is always classical: $\varepsilon\ll k_BT$. The effective action (or the effective Hamiltonian), expressed in terms of the variables $\tilde{\psi}$, n_n, and $\mathbf{v}_n$, does not contain singularities.

6. ANOMALOUS SELF-ENERGY PART $\Sigma_{ik}(p)$ AT $T>0$

Using the relation between Σ_{ik} and g_{ab} (Refs. 14 and 28), we find (see (23) and (24))

$$\Sigma_{12}^T(p\to0)\approx n_0[(V_0+\Sigma_{\pi\pi})^{-1}-\Pi_0^T(p)]^{-1} \quad (82)$$

(here we have taken account of the fact that $g_{\pi\pi}=g_{\varphi\pi}^2g_{\varphi\varphi}^{-1}=(g^{-1})_{\pi\pi}^{-1}=-(V_0+\Sigma_{\pi\pi})^{-1}$);

$$\Sigma_{11}^T(p)-\Sigma_{12}^T(p)\approx G_0^{-1}(p)+\frac{ig_{\varphi\pi}(-p)}{g_{\varphi\varphi}(p)\Pi_0^T(p)}-\frac{1}{2n_0g_{\varphi\varphi}(p)}$$

$$=\mu+\varepsilon\left[1-\frac{n(1-3\rho_nA/4\rho)}{m\bar{c}^2\Pi_0^T(p)}\right]+\frac{n_s'p^2}{2n_0m}\left[1-\frac{27}{8}\frac{\rho_s\rho_n}{\rho_s'\rho}\frac{\varepsilon^2}{c_B^2p^2}A\right]$$

$$-\frac{n\varepsilon^2}{2n_0m\bar{c}^2}\left[1+\frac{9\rho_n}{8\rho}A\right], \quad (83)$$

$$A=c_B^2p^2\left[\varepsilon^2-\frac{1}{3}\left(1-\frac{3\rho_n}{4\rho}\right)c_B^2p^2\right]^{-1}.$$

The expression for $\Sigma_{12}^T(p)$ reflects both anharmonicity anomalies, the one connected with the phase degeneracy (cf. the $T=0$ case[1]):

$$\Sigma_{12}(p)=\Sigma_{11}(p)-\mu-\varepsilon-\frac{n'p^2}{2n_0m}-\frac{\Pi(0)}{2n_0}\varepsilon^2$$

$$=\Delta\Sigma(p)+O(\varepsilon^2,p^2)\approx n_0V_0[1-V_0\Pi_0(p)]^{-1}, \quad \Pi(0)=-n/mc^2; \quad (84)$$

($\Pi_0^T(p)$ exhibits the main divergence

$$\left(\frac{n_0}{n}\right)^2\frac{m^2cT}{\bar{p}}\left(\bar{p}=\max\left(\frac{\varepsilon}{c},p\right)\right),$$

at $p\to0$ and nonanalyticity when $\varepsilon\pm cp\to0$), and the other the "invariant" anomaly caused by second sound (see $\Sigma_{\pi\pi}(p)$ in (73)); just as at $T=0$, we have $\Sigma_{12}^T(0)=0$. The formulas (82) and (83) go over into (84) as $T\to0$, supplementing (84) (a result of Gavoret and Nozières's analysis[8] with allowance made for the equation[1] $\Sigma_{12}(0)=0$) by a small nonanalytic correction in the term $\sim\varepsilon$:

$$\frac{ig_{\varphi\pi}(-p)}{g_{\varphi\varphi}(p)\Pi_0(p)}\approx-\frac{dn_0}{d\mu}\varepsilon/\Pi_0(p)\approx-\frac{n\varepsilon}{mc_B^2\Pi_0(p)}. \quad (85)$$

The occurrence in $[\Sigma_{11}^T(p)-\Sigma_{12}^T(p)]$, at $T>0$, of nonanalytic terms that are not small at $p\to 0$ indicates that a direct generalization of Gavoret and Nozières's analysis[8] to the $T>0$ case is ineffective: the difference $(\Sigma_{11}^T-\Sigma_{12}^T)$ for $T>0$ can no longer be determined from its values for $p\to 0$, $\varepsilon=0$ and $p=0$, $\varepsilon\to 0$. The only cause of the complications is the appearance of second sound. Indeed, if allowance is made in $g_{ab}(p)$ for only the nonsingular contributions of the $\Sigma_{ab}(p)$, i.e., for the $\Sigma_{ab}(\mathbf{p},\varepsilon=0)$ (see (73)), the expression for $(\Sigma_{11}^T-\Sigma_{12}^T)$ coincides with the result obtained in a simple generalization of Gavoret and Nozières's analysis[8]—the formula (84) in which we have made the substitutions

$$n\to n_s,\quad c\to c_T\quad \left(c_T^2=\left(\frac{\partial P}{\partial\rho}\right)_T=\left(\frac{\partial\mu}{\partial n}\right)_T\approx\left(1-\frac{15}{16}\frac{\rho_n}{\rho}\right)c_B^2\right)$$

and the correction in $\varepsilon/mc_T^2\Pi_0^T(p)$ (cf. (85)). The validity of this result at $\varepsilon=0$ is apparent: at $p\to 0$, $\varepsilon=0$ second sound does not affect Gavoret and Nozières's analysis,[8] which employs only Galilean invariance. Notice, finally, that if $\Pi(0)$ (a factor in the ε^2 term; see (84)) is equal to $-n/mc^2$ at $T=0$, then the limit $\Pi^T(p\to 0)$ is not defined at $T>0$ (since $\Pi^T(p)$ contains an unbounded term (with a second-sound pole); see footnote 2 in Ref. 1).

The singular terms in $(\Sigma_{11}-\Sigma_{12})$ complicate the computation of the field Green functions $G_{ik}^T(p\to 0)$, which of course also possess first- and second-sound poles, in comparison with the corresponding computation in the $T=0$ case (see (42) in Ref. 1); in computing the $G_{ik}^T(p\to 0)$, it is convenient to use their connection, indicated in Refs. 14 and 28, with the $g_{ab}(p)$.

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APPENDIX

1. The infrared anharmonicity anomaly in the "combined" variables

The infrared anharmonicity anomaly limits the possibility of an approximate identification (see Ref. 14) of the effective Hamiltonian $\tilde{H}(\tilde{n}_L,\tilde{\mathbf{v}}_L)$ expressed in terms of the "combined" variables (see (21)) with the hydrodynamic Landau Hamiltonian $\tilde{H}(n_L,\mathbf{v}_L)$: in the region $q_0\lesssim p_c$ both Hamiltonians describe physically quite different long-wave subsystems—the field subsystem $\tilde{\psi}_L$ ($\tilde{\psi}_L=\sqrt{\tilde{n}_L}\exp(i\tilde{\varphi}_L)$, $\tilde{\mathbf{v}}_L=\hbar m^{-1}\nabla\tilde{\varphi}_L$) and the hydrodynamic subsystem n_L, $\mathbf{v}_L$ ($\psi_L=\sqrt{n_L}\exp(i\varphi_L)$, $v_L=\hbar m^{-1}\nabla\varphi_L$), which respectively reflect at $T=0$ the properties of the condensate and the superfluid component (see Sec. 2); if the static susceptibilities ($\chi_{\parallel}(\mathbf{p}\to 0,0)$, $F_{44}^{(q_0\to 0)}(p)$ (see (18) and (35) above)) to perturbations of the condensate $|\tilde{\psi}_L|$, $\tilde{n}_L$ diverge, then the analogous characteristics of the superfluid component $|\psi_L|$, n_L are finite:

$$\tilde{\chi}_{\parallel}(p,0)=(1/4n)F_{44}(p,0)=-1/4mc^2,$$

and are given in the $\alpha\ll 1$ case (27) by the BA:

$$\chi_{\parallel}^B=(1/4n)F_{44}=-1/4mc_B^2.$$

Let us compute explicity the quadratic part of $\tilde{H}_2(\tilde{n}_L,\tilde{\mathbf{v}}_L)=\tilde{H}_2(\psi,\psi^*)$. It is easy to relate the derivatives of P in the expression for $\tilde{S}^{(q_0)}(\pi,\varphi)\equiv\tilde{S}^{(q_0)}(\tilde{n}_L,\tilde{\varphi}_L)$—the result of the path integration over ψ_{sh} and ψ_{sh}^* (21) (see (64) and (65))—with the derivatives of $E'^{(q_0)}(n_0,\mu)$, a quantity that differs from the one investigated in Ref. 1 only by the limitation placed on the momentum range over which the integration is performed (i.e., by the requirement that $p_1>q_0$); $E'^{(q_0)}(n_0,\mu)$ characterizes directly the contribution made by the anharmonicity of the short-wave modes to the effective action:

$$\Delta\tilde{S}^{(q_0)}=\Delta S-\Delta E'^{(q_0)}\Delta\tau. \quad (A.1)$$

With the aid of Ref. 1 we find

$$P_\mu=n,\quad P_{\mu\mu}=\frac{dn}{d\mu}-\frac{1}{n_0}\Sigma_{12}^{(q_0)}(0)\left(\frac{dn_0}{d\mu}\right)^2,$$
$$P_{\mu n_0}=\frac{1}{n_0}\Sigma_{12}^{(q_0)}(0)\frac{dn_0}{d\mu},\quad P_{n_0n_0}=-\frac{1}{n_0}\Sigma_{12}^{(q_0)}(0). \quad (A.2)$$

The form of $\tilde{S}_2^{(q_0)}$, as given in (64), (65) ($p<q_0$, $\varepsilon'\lesssim cq_0$), allows us to find the hydrodynamic Green functions:

$$\hat{g}(p)=\begin{pmatrix}g_{\varphi\varphi} & g_{\varphi\pi}\\ g_{\pi\varphi} & g_{\pi\pi}\end{pmatrix}=\begin{pmatrix}\dfrac{d\mu}{dn} & -i\varepsilon\dfrac{dn_0}{dn}\\ i\varepsilon\dfrac{dn_0}{dn} & c^2p^2\dfrac{dn_0}{dn}\dfrac{dn_0}{d\mu}\end{pmatrix}(\varepsilon^2-c^2p^2)^{-1}$$
$$+\left[-\frac{n_0}{\Sigma_{12}^{(q_0)}(0)}+\frac{dn_0}{dn}\frac{dn_0}{d\mu}\right]\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$
$$\Sigma_{12}^{(q_0)}(0)\sim-n_0[\Pi_0(q_0)]^{-1}, \quad (A.3)$$
$$c^2=\frac{n}{m}\left(P_{\mu\mu}-\frac{P_{\mu n_0}}{P_{n_0n_0}}\right)^{-1}=\frac{n}{m}\frac{d\mu}{dn};$$

the time here is real: $\tau\to it$, $\varepsilon'\to-i\varepsilon$.

The nonpole term $g_{\pi\pi}$, which increases as the limiting momentum q_0 of the long-wave subsystem decreases, is quite noticeable against the background of the pole term, whose residue vanishes as $p\to 0$. The off-diagonal terms of $\hat{g}$ determine the Poisson bracket of $\pi_{p\alpha}$ and $-\varphi_{p\alpha}$:

$$f_p=\sum_{\varepsilon'}e^{-i\varepsilon'\tau}f(p),\quad f_{\pm p}=f_{p1}\pm if_{p2}:$$
$$[\pi_{p\alpha},-\varphi_{p'\beta}]=-i\frac{dn_0}{dn}\delta_{pp'}\delta_{\alpha\beta},\quad \alpha,\beta=(1,2). \quad (A.4)$$

The pole term in $\hat{g}$, (A.3), corresponds to an oscillator with canonical variables $R_{p\alpha}\equiv\pi_{p\alpha}dn/dn_0$, $-\varphi_{p\alpha}$ (see (A.4)); the nonpole term shows that $\pi_{p\alpha}dn/dn_0$ does not exactly coincide with the canonical momentum $R_{p\alpha}$, and we should set

$$R_{p\alpha}=\pi_{p\alpha}dn/dn_0+r_{p\alpha}, \quad (A.5)$$

where the quantity $r_{p\alpha}$ commutes with $-\varphi_{p\alpha}$; the appearance of $r_{p\alpha}$ in the Hamiltonian cannot change the result of the commutation of $H(R_{p\alpha},-\varphi_{p\alpha})$ with $\varphi_{p\alpha}$ (expressed in terms of $R_{p\alpha}$), so that, as before,

$$-\dot{\varphi}_{p\alpha}=\frac{d\mu}{dn}R_{p\alpha}. \quad (A.6)$$

Writing the Lagrangian in accordance with (64), (65), (A.2), (A.5), and (A.6):

$$\tilde{L}_2=\sum_{p\alpha}{}' \left\{ \frac{1}{2}\left[\frac{dn}{d\mu}\dot{\varphi}_{p\alpha}{}^2-\frac{np^2}{m}\varphi_{p\alpha}{}^2\right]-\frac{\Sigma_{12}^{(q_0)}(0)}{2n_0}\left(\frac{dn_0}{dn}\right)^2 r_{p\alpha}{}^2\right\}, \quad \text{(A.7)}$$

and assuming in the computation of

$$\tilde{H}_2=\sum_{p\alpha}\dot{\varphi}_{p\alpha}\frac{\partial \tilde{L}_2}{\partial\dot{\varphi}_{p\alpha}}-\tilde{L}_2,$$

that $r_{p\alpha}$ does not depend on $\dot{\varphi}_{p\alpha}$, we find

$$\tilde{H}_2=\sum_{p\alpha}{}' \left\{\frac{1}{2}\left[\frac{d\mu}{dn}R_{p\alpha}{}^2+\frac{np^2}{m}\varphi_{p\alpha}{}^2\right] +\frac{\Sigma_{12}^{(q_0)}(0)}{2n_0}\left(\frac{dn_0}{dn}\right)^2 r_{p\alpha}{}^2\right\} \quad \text{(A.8)}$$

(the prime on Σ denotes summation over half-space). At the same time, in the BA

$$\tilde{L}_2^{\mathrm{B}}=L_2=\sum_{p\alpha}{}' \left\{\pi_{p\alpha}(-\dot{\varphi}_{p\alpha})-\frac{1}{2}\left[\frac{np^2}{m}\varphi_{p\alpha}{}^2+V_0\pi_{p\alpha}{}^2\right]\right\},$$

$$\tilde{H}_2^{\mathrm{B}}=H^{\mathrm{B}}=\frac{1}{2}\sum_{p\alpha}{}'\left[V_0\pi_{p\alpha}{}^2+\frac{np^2}{m}\varphi_{p\alpha}{}^2\right].$$

If we neglect the singularities connected with the appearance of second sound, the formulas (A.7) and (A.8) can be applied at $T>0$ as well; the treatment of $r_{p\alpha}$ as an independently fluctuating quantity yields then the correct result (which diverges as $q_0\to 0$ like $[\Sigma_{12}^{(q_0)}(0)]^{-1}$) for the fluctuations of this quantity.

In conclusion, let us discuss the contribution made to $\tilde{S}$ by the long-wave fluctuations with $p<q_0$ (which were neglected in $\tilde{S}^{(q_0)}$). The coefficients of the expansion of $\tilde{S}(\pi,\varphi)$ in powers of $\pi=\tilde{n}_L-\langle\tilde{n}_L\rangle$, $\varphi=\tilde{\varphi}_L$ differ from the analogous coefficients in the expansion of $\tilde{S}^{(q_0)}(\pi,\varphi)$, i.e., from the field vertices, by corrections that are "hydrodynamic" diagrams with integrations over $p_1\leqslant q_0$, in which the field vertices are connected by the "hydrodynamic" Green functions $g_{ab}(p_1)$. It can be shown that in the general case these corrections possess at $q_0\to 0$ the same degree of divergence as the field vertices themselves, but in the case of the "two-ray" vertices the corrections are negligibly small (this latter fact can be proved by taking into consideration the field-theory formula[1,8] for the sound velocity $c=(dP/d\rho)^{1/2}$). Thus, although $\tilde{S}(\pi,\varphi)\neq\tilde{S}^{(q_0)}(\pi,\varphi)$, in the quadratic approximation at $T=0$ we have $\tilde{S}_2(\pi,\varphi)\approx\tilde{S}_2^{(q_0)}(\pi,\varphi)$, so that $\hat{g}(p)$ (A.3), $\tilde{L}$, (A.7), and $\tilde{H}$ (A.8) are exact expressions (their dependence on q_0 arises only because of the presence of q_0 in the definitions of π and φ).

2. Peculiarities of the inertial properties of the quasiparticles

Equation (58) raises a number of questions concerning the specific nature of the concepts "normal" and "superfluid" component. The quasiparticle mass $\tilde{m}=p(d\varepsilon_p/dp)^{-1}$, as the coefficient of proportionality between the quasiparticle momentum and (group) velocity, turns out to be nonadditive in the analysis of a gas of quasiparticles in thermal equilibrium:

$$\frac{|\mathbf{P}|}{|\mathbf{V}|}\neq\int\frac{d^3p}{(2\pi)^3}\tilde{m}n_G(\varepsilon(p)-\mathbf{pV})\approx\int\frac{d^3p}{(2\pi)^3}\tilde{m}n_G(\varepsilon(p))\equiv\tilde{M},$$

although it is, of course, additive for "condensation" of quasiparticles into a single $\mathbf{p}$ state. Thus, during the nondissipative "thermalization" of a quasiparticle condensate as a result of internal interactions, the mass and velocity of the quasiparticle gas change while the total momentum is kept unchanged. What is the source of this peculiarity of the inertial properties of the quasiparticles?

The nonadditivity of the mass is all the more surprising, since additivity is assumed in the definition $\rho_s=\rho-\rho_n$ itself. And the physical adequacy of the definition of ρ_s can be confirmed by a direct microscopic computation: we can, by considering at $T>0$ a Bose system with a moving condensate (using a generalization of the Belyaev technique[2] (see Refs. 46 and 40)), easily obtain the system's total momentum $\mathbf{P}=(\rho_s V)\mathbf{V}$; similarly, from the form of $\Sigma\varphi\varphi$ (74) we obtain $E=\frac{1}{2}\rho_s\mathbf{V}^2$, (75). In this case a quasiparticle gas with mass $\rho_n V$ is entirely analogous to the particle gas in, for example, the computation of second sound at low T as local-equilibrium sound for a gas of phonons $\varepsilon=sp$:

$$\left(\frac{dP_{ph}}{d\rho_n}\right)_S=\left(\frac{dP_{ph}}{d(M_{ph}/V)}\right)_{S,\,M_{ph}}=-\frac{dP_{ph}}{dT}\left(\frac{dT}{dV}\right)_S\frac{V^2}{M_{ph}} =S\frac{dP_{ph}}{dT}\Big/\rho_n\frac{dS}{dT}=\frac{s^2}{3}.$$

Other questions arise. The Lagrangian, energy, and momentum of the phonons are (neglecting the polarization) "isomorphic" to the case of free photons with c replaced by s; we can even introduce a conditional "phonon" pseudo-Euclidean metric (using "phonon" rules and clocks)—so long as the particles are free (and $\varepsilon=sp$), the isomorphism cannot be destroyed, and the conditionality of the metric cannot be established. Why then does the isomorphism in the relation between the energy and the inertia ($\rho_n=4/3E_{ph}/s^2$) disappear? Further, what "symmetry character" does the quasiparticle mass $\tilde{m}$ have, and how does this mass compare with the relativistic "excitation-energy mass" $\mu\equiv\varepsilon/c^2$ (i.e., is it very much smaller than $\tilde{m}$)? Finally, why does the quadratic quasiparticle dispersion law $\varepsilon\sim p^2$ "restore" the additivity (i.e., is the disappearance here of the difference between particles and quasiparticles accidental)?

The nonadditivity of the quasiparticle mass (in particular, the coefficient 4/3 for phonons) is due to the "unusual" momentum transformation law $\mathbf{p}'=\mathbf{p}$ used in the derivation of the formula for ρ_n; this transformation is implicitly carried out in the formula

$$\mathbf{P}=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}n_G(\varepsilon(p)-\mathbf{pV}) =\mathbf{V}\int\frac{d^3p}{(2\pi)^3}\left(\tilde{m}+\frac{1}{3}p\frac{d\tilde{m}}{dp}\right)n_G(\varepsilon(p)), \quad \text{(A.9)}$$

since the original expression is

$$\mathbf{P}=\int\frac{d^3p'}{(2\pi)^3}\mathbf{p}n_G(\varepsilon'),\quad \varepsilon'=\varepsilon-\mathbf{pV} \quad \text{(A.10)}$$

(the Gibbs distribution is given in the "moving" reference system K', but the momentum $\mathbf{P} = \rho_n \mathbf{V}$ is computed in the stationary system K).

It is precisely the "usual" momentum transformation law $\mathbf{p}' = \mathbf{p} - \tilde{m}\mathbf{V}$ for particles which guarantees, as can easily be verified, the additivity: substituting

$$d^3p' = D d^3p \equiv \left(1 - \frac{d\tilde{m}}{dp}\frac{\mathbf{pV}}{p}\right) d^3p,$$

into (A.10), we find

$$\mathbf{P} = \mathbf{V}\int \frac{d^3p}{(2\pi)^3}\tilde{m} n_G(\varepsilon(p)). \tag{A.11}$$

From where does the unusual quasiparticle-momentum transformation law $\mathbf{p}' = \mathbf{p}$ arise when we have the usual formula $\varepsilon' = \varepsilon - \mathbf{p}\cdot\mathbf{V}$ for the energy? Why should there be different transformation laws for particles and quasiparticles (and why should a special quasiparticle-mass concept differing from $\mu = \varepsilon/c^2$ arise) at all when the energy and momentum of any excitation (like the original energy and momentum) of the system form a 4-vector? Actually, the ε- and p-transformation laws for particles do not differ from those for quasiparticles:

$$\varepsilon' = (\varepsilon - p_x V) \Big/ \left(1 - \frac{V^2}{c^2}\right)^{1/2},$$

$$p_x' = \left(p_x - \frac{\varepsilon}{c^2}V\right) \Big/ \left(1 - \frac{V^2}{c^2}\right)^{1/2}, \ldots, \text{i.e.,}$$

$$\varepsilon' \approx \varepsilon - pV, \quad p' \approx p - \varepsilon V/c^2. \tag{A.12}$$

The whole difference lies in the fact that for particles $\mu \equiv \varepsilon/c^2 = \tilde{m}$ ($\tilde{m} \equiv p(d\varepsilon/dp)^{-1}$), while for quasiparticles $\mu \equiv \varepsilon/c^2 \ll \tilde{m}$ (which has been taken into account in the law $p' = p$). The two formulas (A.9) and (A.11) can be reduced to one:

$$\mathbf{P} = \int \frac{d^3p'}{(2\pi)^3}\mathbf{p} n_G(\varepsilon') = \int D \frac{d^3p}{(2\pi)^3}\mathbf{p} n_G[\varepsilon(p) - \mathbf{pV}]$$

$$= \int \left(1 - \frac{d\mu}{dp}\frac{\mathbf{pV}}{p}\right)\frac{d^3p}{(2\pi)^3}\mathbf{p} n_G[\varepsilon(\mathbf{p} - \tilde{m}\mathbf{V})]. \tag{A.13}$$

The "symmetry meaning" of $\tilde{m}$ for quasiparticles is also clear from (A.13): it is the coefficient in the conditional transformation law for the momentum, i.e., the law that preserves the form of the energy dependence

$$\varepsilon' = \varepsilon(p'_{\text{con}}), \quad \mathbf{p}'_{\text{con}} = \mathbf{p} - \tilde{m}\mathbf{V}. \tag{A.14}$$

The relation $\tilde{m} = \mu(\mathbf{p}_{\text{con}} = \mathbf{p})$ for particles corresponds to the principle of relativity: $\varepsilon(p)$ preserves its form under the transformations. Thus, the source of the peculiarity of the inertial properties of the quasiparticles is the violation of the principle of relativity (there is a preferred rest frame for the medium), or, specifically, the deviation of the form of $\varepsilon(p)$ from $(c^2p^2 + m^2c^4)^{1/2}$, leading to a situation in which the "inertial mass" $\tilde{m} = p(d\varepsilon/dp)^{-1}$ (the coefficient in the conditional momentum transformation law (A.14)) differs from the "energy mass" $\mu = \varepsilon/c^2$ (the coefficient in the true transformation law (A.12)). The other differences—in particular, in the character of the "rest mass" $m_0^2c^2 = \varepsilon^2/c^2 - p^2$ (for quasiparticles, $m_0^2 \approx -p^2/c^2 < 0$)—are unimportant here.

The violation of the principle of relativity explains the nonconservation in the general case of the velocity of the quasiparticle gas as a whole (in the case when momentum is conserved—as a result of homogeneity).

The nonadditivity does not manifest itself in the case of the excitation condensate because the expression for $\mathbf{P}$ in the K system is not obtained by means of a transformation from another reference system (the distribution is given in K).

The quadratic dispersion law $\varepsilon = ap^2$ "restores" the additivity because it imitates the relativistically invariant form $\varepsilon = (c^2p^2 + m^2c^4)^{1/2} \approx mc^2 + p^2/2m$ ($m = 1/2a$) (the constant is unimportant); a similar imitation does not occur in the case of phonons despite the profound analogy with photons: $s \neq c$, where c is a chosen quantity.

The destruction of the isomorphism between phonons and photons is due to the fact that the velocity $\mathbf{V}$ of a moving gas in thermal equilibrium is fixed by the walls, i.e., is determined, in essence, by the interaction of the quasiparticles with the surrounding matter (even if the interaction is arbitrarily weak—just strong enough for the establishment of equilibrium); the concept of mass of a gas in thermal equilibrium is beyond the scope of the free-quasiparticle model: the true metric essentially manifests itself; within the framework of the conditional "phonon" pseudo-Euclidean metric the velocity of the "center-of-mass system" K_1 differs from the wall velocity $V_1 = Ps^2/E_{ph} = 4/3V$. The mass per unit volume $\tilde{M} = E_{ph}/s^2 = 3/4\rho_n$, the distribution constructed by transforming from K not being strictly a Gibbs distribution in either K' or K_1.

That the metric dictated by free photons is the true one is manifest precisely by the fact that the metric is conserved, and is, in general, found to be universal when allowance is made for the interaction of the electromagnetic field with the matter; the symmetry (covariance) of the equations points to the true special principle of relativity; the "photon" rules and clocks, in contrast to the "phonon" ones, turn out to be true rules and clocks. Similarly, if we construct the theory of massless spin-2 particles in flat space-time, requiring that the "source" of the field be any energy (T_{ik})—such a theory coincides with the general theory of relativity in which we have artificially carried out the splitting $g_{ik} = g^0_{ik} + \varphi_{ik}$ and we interpret the Einstein pseudotensor θ_{ik} as the field-energy tensor T^g_{ik} (the result of the variation in the metric g^0_{ik})—then the preference of the original flat metric turns out to be experimentally unprovable (like the preference of the "laboratory reference system" in electrodynamics); the symmetry of the equations written without the splitting of g_{ik} (general covariance) points to the true general principle of relativity, to the curvature of a physically observable metric. But for phonons the medium has its rest frame chosen for it, which is manifested in the inertial properties of a moving thermalized gas.

In a certain (very limited) sense, the additivity of the quasiparticle mass can be "saved" by not identifying the gas velocity with the velocity of the system K'; although the energy distribution ε' in K' has the Gibbs form, this system is not the "rest frame" of the gas—the distribution, together with $\varepsilon'(\mathbf{p})$, is not isotropic: $\varepsilon' = \varepsilon(p) - \mathbf{p}\cdot\mathbf{V} = \varepsilon(p') - \mathbf{p}'\cdot\mathbf{V}$, and even in K the quasiparticle distribution differs from the "moving Gibbs distribution" of particles—we do not have the factor D attached to d^3p. The "saving of the additivity" means that we must assign the velocity $\mathbf{V}_1 = \mathbf{P}/M$ to the gas. But a profound analogy with particles cannot be achieved even at this price: K_1 is also not a rest frame everywhere within the range of the transformation law $\mathbf{p}_1 = \mathbf{p}$ (the momentum distribution is not isotropic: $n_G(\varepsilon') = n_G[\varepsilon(p) - \mathbf{p}\cdot\mathbf{V}] = n_G[\varepsilon(p_1) - \mathbf{p}\cdot\mathbf{V}]$); in the case of phonons V_1 and $\tilde{M}$ are preserved during the thermalization of the quasiparticle condensate (the manifestation of the conditional "phonon" principle of relativity), but this is generally not realized.

[1)]In Ref. 6 the expression for $G_{ik}(p\to 0)$ is constructed under the assumption that $\Sigma_{12}(0)\neq 0$ even though this actually contradicts the result $(dM/dh)^{-1} = 0$ obtained there in a renormalization-group analysis.

[2)]Thus, in the general case of a Bose system with a condensate $\langle\hat{\psi}\rangle \neq 0$ the Landau quantum hydrodynamics can be justified only with the aid of the field method[2] (see Gavoret and Nozières's result,[8] as corrected with allowance for $\Sigma_{12} = 0$ in Ref. 1); in the other methods we restrict ourselves in the justification to the use of the either the simplest "quasiharmonic" model (see (27) and, for example, Refs. 10 and 11), or a more general, but still additional, assumption: in the method of "combined" variables[14] (see (21)) the condition $p_c \ll p_h$ (see (25)) is in fact required for the derivation of the asymptotic formulas for $G_{ik}(p\to 0)$; here the derivation does not take into consideration the long-wave anharmonicity (of the oscillations with $p \ll q_0$), a contribution which is small only when $p_c \ll q_0 \ll p_h$: as $q_0 \lesssim p_c$ decreases, the smallness of the phase volume is compensated for by the vertex divergence due to the infrared anomaly of the field anharmonicity (see Appendix 1). In Ref. 15 the Landau quantum hydrodynamics is justified under the assumption of the existence of a "two-particle" condensate" $\langle\hat{\psi}\hat{\psi}\rangle \neq 0$.

[3)]Connected with the square-law divergence of $\chi_\perp(\mathbf{p},0)$ is the well-known power-law decrease of the fluctuation correlator for the field $\hat{\psi} = \sqrt{n_0} + \hat{\psi}'$:

$$\langle\hat{\psi}'^{+}(\mathbf{r})\hat{\psi}'(0)\rangle \sim \langle\hat{\psi}'(\mathbf{r})\hat{\psi}'(0)\rangle \sim 1/r^2 \quad (T=0); \quad \sim 1/r \quad (T \gg c\hbar/r)$$

(which corresponds to a momentum distribution $N_\mathbf{p} = \langle\hat{a}_\mathbf{p} + \hat{a}_\mathbf{p}\rangle \sim 1/p$ $(T=0)$, $\sim 1/p^2 (T>0)$ for the particles). The divergence of $\chi_\parallel(\mathbf{p},0)$ indicates that at $T>0$ the correlator for the fluctuations of the Hermitian part (or modulus) of the field also falls off according to a power law:

$$\langle (a_\mathbf{p}^+ + a_{-\mathbf{p}})(a_\mathbf{p} + a_{-\mathbf{p}}^+)\rangle \sim 1/p, \qquad \langle \mathrm{Re}\,\hat{\psi}'(\mathbf{r})\,\mathrm{Re}\,\hat{\psi}'(0)\rangle \sim 1/r^2.$$

[4)]The indicated states of a Bose system at $T\to 0$ (the superfluid state and crystalline state with localized particles) are the extreme cases of a family of possible "intermediate" states combining both types of long-range order: on the one hand, the nonlinear field can acquire a periodic structure (the anharmonicity then limits to a certain extent the rate of intersite particle tunneling; the shear elasticity here "gets accustomed" to the presence of the superfluid component[29]), and, on the other, the lattice with localized particles is not destroyed upon the appearance of defects of low concentration, even when these defects are not localized at $T=0$ (Ref. 30). More "exotic"—far removed from their classical analogs—are the states of a Bose system at $T\to 0$: a fluid without a single-particle condensate ($\langle\Psi\rangle = 0$) but with a two-particle one ($\langle\hat{\psi}\hat{\psi}\rangle \neq 0$) also obeys the laws of Landau's quantum hydrodynamics[15]; a fluid with $\rho_s(T=0) < \rho$, or it is not superfluid at all: $\rho_s(T=0) = 0$.

[5)]Let us add a number of "details" imitating in the BA the properties of real He-II: lowering of the roton minimum, crystallization, and vanishing of the anomalous dispersion with increase of P; the appearance of an anomaly in (i.e., enhancement of) the "short-range order the increase of T." A straight-forward allowance for the energy of the zero-point oscillations in the $V_0 \ll |V_p \sim p_0|$ case allows us to obtain a "self-constricted" state within the framework of the BA.

[6)]Formally, the independence of $\hat{\tilde{X}}_L$ and $\hat{x}_{sh}$ needs to be stated more precisely. For example, $\hat{\tilde{x}}_L = (\hat{\tilde{n}}_L, \hat{\tilde{\mathbf{v}}}_L)$, where $\hat{\tilde{n}}_L, \hat{\tilde{\mathbf{v}}}_L$ are connected with $\hat{\psi}_L, \hat{\psi}_L^+$, just as $\hat{n}, \hat{\mathbf{v}}$ are connected with $\hat{\psi}, \hat{\psi}^+$, (2); but the differences between $\hat{n}_L, \hat{\mathbf{v}}_L$ and $\hat{\tilde{n}}_L, \hat{\tilde{\mathbf{v}}}_L$ are unimportant in the case when $q_0 \gg p_c$.

Let us note also another way of introducing $\tilde{x}$ into the description of a condensate-containing Bose system: the use in (2) of $\hat{n} = \hat{n}_L + \hat{n}_{sh}$, $\hat{\mathbf{j}} = \hat{\mathbf{j}}_L + \hat{\mathbf{j}}_{sh}$, and not $\hat{\psi} = \hat{\psi}_L + \hat{\psi}_{sh}$, as the initial splitting; here $\hat{\tilde{x}} = (\hat{n}_L, \hat{\mathbf{v}}_L; \hat{\psi}_{sh}, \hat{\psi}_{sh}^+)$, $\hat{\psi}_{sh}$ is connected with $\hat{n}_{sh}$ and $\hat{\mathbf{j}}_{sh}$ by formulas of the type (2), and $\hat{\mathbf{v}}_L$ is given by the relation $\hat{\mathbf{j}}_L = \frac{1}{2}(\hat{n}_L\hat{\mathbf{v}}_L + \hat{\mathbf{v}}_L\hat{n}_L)$. Notice that a definition (of the type (2)) of $\hat{\tilde{\mathbf{v}}}_L$ in terms of $\hat{\psi}_L$, $\hat{\psi}_L^+$, or $\hat{\mathbf{j}}_L$ does not present any difficulties connected with the "δ-function" character of $n(\mathbf{r})$, since the $n_L(\mathbf{r})$ are smooth functions.

[7)]In view of this fact, the nonfield interpretations of the equations of an ideal fluid as applied to He-II, such as Tkachenko's interpretation[42] (which admits of a nonsingular rotation), are doubtful.

[8)]In a field $h \neq 0$ all the divergences at $\mathbf{p}\to 0$ [of $\chi_\perp(\mathbf{p},0)$, $\chi_\parallel(\mathbf{p},0)$, and of the coefficients in $\hat{\alpha}_\mathbf{p}$ (expressed in terms of $a_\mathbf{p}$, $a_{-\mathbf{p}}^{+}$)] disappear, with the $\mathbf{p} = 0$ mode corresponding to stable, and not neutral, equilibrium. There exists thus an operator $\hat{\alpha}_0$ (a Bogolyubov combination of the operators $\hat{a}_0'$ and $\hat{a}_0'^+$ $(\hat{a}_0' = \hat{a}_0 - \langle\hat{a}_0\rangle)$) such that $\hat{\alpha}_0|\Psi_0^B\rangle = 0$. But if $h < h_c$, all the system's characteristics for $p \lesssim p_c$ strongly differ, as before, from the characeristics obtained in the Ba ($\chi_\parallel$ from $\chi_\parallel^B$, $B_\mathbf{p}$ from $\alpha_\mathbf{p}$, etc.).

Notice further that as $T\to 0$ the field h need not be used in the determination of the state with the broken symmetry. Although the ground state of the system at $h=0$ is symmetric ($\langle\hat{\psi}\rangle = 0$), a "narrow packet" $\langle\hat{\psi}\rangle \neq 0$ with $\langle(\Delta\hat{\varphi})^2\rangle \equiv A^{-1} \ll 1$ possesses such a small energy uncertainty $\Delta E \sim \delta N \mu \sim A^2\mu/N$ that it spreads in the case when $N\to\infty$ for an indefinitely long time: $\tau \sim \hbar/\Delta E \sim N\hbar/\mu A^2$ (ΔE coincides in order of magnitude with the energy in a field δ that guarantees the relation $\langle(\Delta\hat{\varphi})^2\rangle = A^{-1}$).

[9)]The effective Hamiltonian can be understood in the sense that it takes into account only the short-wave anharmonicity $(p\to q_0) - \tilde{H}^{(q_0)}$. With the aid of $\tilde{H}^{(q_0)}$ we can next compute the long-wave anharmonicity (first and foremost, the contribution to the thermodynamic potential of the fluctuations (treated as independent quantities) from $\tilde{H}_2^{(q_0)}$). In the variables $\tilde{x}$ the long-wave contribution is small everywhere, except the fluctuation region. But in the original variables x it is everywhere important (the region $p \lesssim p_c$ makes a nonanalytic contribution).

[10)]Let us note that the term MWF pertains to quantum properties at the macroscopic level (the nondissipative motion, the quantization of the velocity circulation, the macroscopic quantization of angular momentum, etc.), associates with matter an equation containing, and yet is not a "genuine" wave function: interference among probability amplitudes will always remain in the microworld; from the point of view of measurements the MWF is simply a classical field.

[11)]The data presented in Ref. 14,

$$|\Sigma_{\varphi\varphi}(\mathbf{p},\ \varepsilon=0)| = 11\beta\langle p^2\rangle p^2/24m^2 = {}^{11}/_{8}\rho_n p^2/m^2$$

and, consequently,

$$c_1 = c(1 + {}^{29}/_{32}\rho_n/\rho)$$

(see the formulas (19.39) and (19.42) in Ref. 14) are at variance with the exact relation (74), with the Bogolyubov identity (78), and with the equation (79) of two-velocity hydrodynamics.

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