Electron localization in a multichannel conductor

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The localization of an electron in a conductor consisting of N elementary metallic chains is considered. The transverse size of the conductor is assumed to be small compared with the mean free path l. It is shown that N scattering channels define N different localizing lengths. The major role in the kinetic properties at the largest distances and at low frequencies is played by a single channel with the largest localization length l_0^*. In the multichannel case $N \gg 1$ the problem reduces to one-dimensional localization and it follows hence that $l_0^* = Nl/2$. Other solutions such as $l_0^* = Nl$ or $l_0^* = 2Nl$ are obtained in the presence of an external magnetic field or of magnetic impurities, respectively. At a small number of channels, $N \sim 1$, the behavior differs significantly from that in the one-dimensional case. Two-channel localization corresponding to two chains with potential impurities or to a single chain with magnetic impurities is investigated.

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1. INTRODUCTION

The possibility of electron localization in a disordered metal was first indicated by Anderson.[1] Mott and Twose[2] postulated that all electron states are localized in the one-dimensional case, i.e., in an elementary metallic chain. Not so long ago, arguments were advanced favoring the localization of all the electrons in a sufficiently long wire[3] and in the two-dimensional case.[4]

Localization in a disordered metallic chain and, as a consequence, the vanishing of the frequency dependence of the conductance $G(\omega) \propto \omega^2 \ln^2 \omega$ as $\omega \to 0$, was proved by Berezinskiĭ.[5] A more accurate method was later obtained by his method to obtain a more accurate pre-exponential factor in the asymptotic relation for the density-fluctuation correlator[6]:

$$\langle\langle \delta\rho(x, t=\infty), \delta\rho(0,0)\rangle_T\rangle_U|_{x\gg l} = \pi^{5/2}\frac{(x/l)^{-1/2}\exp(-x/4l)}{32 v_F l}\int d\varepsilon n_\varepsilon(1-n_\varepsilon), \quad (1)$$

where v_F is the Fermi velocity, l is the mean free path, $n_\varepsilon = [1 + \exp(-\varepsilon/T)]^{-1}$ is the Fermi function, and averaging over the thermodynamic ensemble ($\langle\ \rangle_T$) and over the random potential ($\langle\ \rangle_U$) is implied in the right-hand side. A different approach[7,8] to the problem of one-dimensional localization considers in place the density-density correlator (1) the coefficient of electron transport to a disordered-chain segment of length L. The transport coefficient (L) must decrease exponentially, $T(L) \sim \exp(-L/l_{\text{loc}})$, over the characteristic localization length l_{loc}. Indeed, at $L \gg l$ the following asymptotic formulas is valid[7,8]:

$$\langle T(L)\rangle_U|_{L\gg l} = 2^{-1}\pi^{5/2}(L/l)^{-1/2}\exp(-L/4l). \quad (2)$$

In the one-dimensional case the transport coefficient determines the static conductance $G(L)$ of the system in accord with the Landauer formula[9]

$$G(L) = \frac{e^2}{2\pi\hbar}T(L)/[1-T(L)]. \quad (2')$$

Relation (1) or (2) can thus be used to determine the localization length ($4l$ in this case). There is, however, also another reasonable definition. A hypothesis has been advanced[10] that a suitable scaling parameter in localization is the quantity ε^*:

$$\langle\varepsilon^*\rangle_U = v_F/l^* = -v_F\frac{d}{dL}\langle\ln T(L)\rangle_U, \quad (3)$$

and self-averaging of the parameter ε^* was proved in the one-dimensional case for $L \to \infty$. We regard it is natural to call ε^* the localization energy (see Sec. 3 of the present article), and call $l^* = v_F/\langle\varepsilon^*\rangle$ the scaling localization length, in the sense of Ref. 10. For a single chain we have[10] $l^* = l$.

We consider here several models of a wire in the form of a certain number N of elementary chains packed in some manner or another. We assume that a certain anisotropy exists in the wire and is due to the predominant motion of the electron along the chains. In other words the binding energy t between neighboring chains is assumed to be small compared with the Fermi level ε_F ($\varepsilon_F \gg t$). In addition, the disorder in the wire is assumed to be quite small: $l \gg Na$ (i.e., $\varepsilon_F/N \gg v_F/l$), where a is the lattice constant. Under these conditions, the relation between the reciprocal free-path time $1/\tau = v_F/l$ and the characteristic spacing $\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n$ between the transverse quantization levels ε_n becomes particularly important (generally speaking, $\Delta\varepsilon \sim t/N$). The point is that under the conditions $\Delta\varepsilon \ll 1/\tau$ and $N \gg 1$ classical diffusion takes place over short distances in the wire, and the localization is the result of interaction of the diffusion modes.[11–13] In this situation the principal role is played by the so-called fan diagrams.[11,12] In the opposite pure quantum limit $\Delta\varepsilon \gg 1/\tau$ there is no classical diffusion and to investigate the localization it becomes necessary to take into account all possible scattering diagrams. My preceding papers[14–16] are devoted mainly to this limit $\Delta\varepsilon \gg 1/\tau$. An investigation of the very same problem was started in Ref. 17 in which, however, the answer for the localization length without a magnetic field was obtained in the form $l^*(N) \approx Nl$, which contradicts for two chains my result[14] $l^*(2) = l/(1 - 1/\pi)$ as well as Eq. (38′) of the present article (see also

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Ref. 16). Localization in wires without anisotropy ($t\sim\varepsilon_F$) is investigated in the limit $\Delta\varepsilon\ll 1/\tau$ in Ref. 18. We shall not confine ourselves here to the case $\Delta\varepsilon\gg 1/\tau$, but use a much weaker condition of the type $t\gg\tau^{-1}$.

2. FORMULATION OF PROBLEM

We consider several different manners of packing the chains into a wire and the approximation of band electrons in the field of the impurities along the chains in the approximation of tight binding of the electron on the chain in the perpendicular direction. The Hamiltonian is

$$H_{nn'}=\left[-\varepsilon-\frac{1}{2m}\frac{d^2}{dx^2}+U_n(x)\right]\delta_{nn'}+t(\delta_{n-1,n'}+\delta_{n+1,n'}), \quad (4)$$

where $U_n(x)$ is a random impurity potential in the n-th chain, and t is the binding energy between the chain. The boundary conditions on the wave functions $\psi_n(x)$ can be chosen in one of two forms:

$$\psi_{N+1}(x)=\psi_0(x)=0, \quad (4a)$$

$$\psi_N(x)=\psi_0(x)\exp(i\chi), \quad (4b)$$

the first of which corresponds to packing of N chains on a plane, and the second to packing on a cylinder. The phase shift χ in the second case occurs in the presence of a magnetic flux Φ inside the cylinder ($\chi=e\Phi/c$). If the magnetic flux is not a multiple of the flux quantum $\Phi_0=\pi c/e$, i.e., $\Phi\neq n_0\Phi_0$ and $\chi=\pi n_0$, the condition (4b) violates time-reversal symmetry.

Besides the usual potential impurities we shall introduce at times into the system magnetic impurities which are regarded as a random magnetic field $\mathbf{H}_n(x)$. In other words, we neglect the mutual correlations of the magnetic impurities and the quantum character of the impurity spin. This approximation becomes valid at not too low temperatures exceeding, in particular the Kondo temperature. Thus, under these conditions the potentials $U_n(x)$ must be regarded as matrices in spin space

$$U_n(x)\to(U_n^0(x)\hat{1}+U_n^x(x)\hat{\sigma}^x+U_n^y(x)\hat{\sigma}^y+U_n^z(x)\hat{\sigma}^z)_{ss'}, \quad (5)$$

where $U_n^i(x)=-\mu_B H_n^i(x)$ and μ_B is the Bohr magneton.

Corresponding to a rectangular wire consisting of $N=N_1N_2$ chains packed into a simple quadratic lattice is the Hamiltonian

$$H_{n_1n_2n_1'n_2'}=\left[-\varepsilon-\frac{1}{2m}\frac{d^2}{dx^2}+U_{n_1n_2}(x)\right]\delta_{n_1n_1'}\delta_{n_2n_2'}$$
$$+t[(\delta_{n_1-1,n_1'}+\delta_{n_1+1,n_1'})\delta_{n_2n_2'}+\delta_{n_1n_1'}(\delta_{n_2-1,n_2'}+\delta_{n_2+1,n_2'})] \quad (6)$$

with zero boundary conditions

$$\psi_{N_1+1,\,n_2}(x)=\psi_{0,\,n_2}(x)=0, \qquad \psi_{n_1,\,N_2+1}(x)=\psi_{n_1,\,0}(x)=0. \quad (6c)$$

Other boundary conditions

$$\psi_{N_1+1,\,n_2}(x)=\psi_{0,\,n_2}(x)=0, \qquad \psi_{n_1,\,N_2}(x)=\psi_{n_1,\,0}(x)\exp(i\chi) \quad (6d)$$

on the Hamiltonian lead to packing of $N=N_1N_2$ chains in the form of a hollow cylinder with a magnetic flux Φ inside it.

We diagonalize the Hamiltonians (4) and (6) with respect to the transverse motion of the electrons:

$$\hat{H}_{nn'}=\left(-\varepsilon+\varepsilon_n^{a(b)}-\frac{1}{2m}\frac{d^2}{dx^2}\right)\delta_{nn'}+\hat{U}_{nn'}^{a(b)}(x), \quad (4')$$

$$\hat{H}_{n_1n_2n_1'n_2'}=\left(-\varepsilon+\varepsilon_{n_1n_2}^{c(d)}-\frac{1}{2m}\frac{d^2}{dx^2}\right)\delta_{n_1n_1'}\delta_{n_2n_2'}+\hat{U}_{n_1n_2n_1'n_2'}^{c(d)}, \quad (6')$$

where the transverse-quantization energy levels

$$\varepsilon_n^a=2t\cos[\pi n/(N+1)], \quad (7a)$$

$$\varepsilon_n^b=2t\cos[(2\pi n+\chi)/N], \quad (7b)$$

$$\varepsilon_{n_1n_2}^c=2t\cos[\pi n_1/(N_1+1)]+2t\cos[\pi n_2/(N_2+1)], \quad (7c)$$

$$\varepsilon_{n_1n_2}^d=2t\cos[\pi n_1/(N_1+1)]+2t\cos[(2\pi n_2+\chi)/N_2] \quad (7d)$$

and the matrices of the random potentials

$$U_{nn'}^a(x)=\sum_{n_0}U_{n_0}(x)\,2(N+1)^{-1}$$
$$\times\sin[\pi n_0 n/(N+1)]\sin[\pi n_0 n'/(N+1)], \quad (8a)$$

$$U_{nn'}^b(x)=\sum_{n_0}U_{n_0}(x)N^{-1}\exp[i2\pi n_0(n-n')/N], \quad (8b)$$

$$U_{n_1n_2n_1'n_2'}^c(x)=\sum_{n_{10}n_{20}}U_{n_{10}n_{20}}(x)\,2(N_1+1)^{-1}\sin[\pi n_{10}n_1/(N_1+1)]$$
$$\times\sin[\pi n_{10}n_1'/(N_1+1)]\,2(N_2+1)^{-1}$$
$$\times\sin[\pi n_{20}n_1/(N_2+1)]\sin[\pi n_{20}n_2'/(N_2+1)], \quad (8c)$$

$$U_{n_1n_2n_1'n_2'}^d(x)=\sum_{n_{10}n_{20}}U_{n_{10}n_{20}}(x)\,2(N_1+1)^{-1}\sin[\pi n_{10}n_1/(N_1+1)]$$
$$\times\sin[\pi n_{10}n_1'/(N_1+1)]N_2^{-1}\exp[i2\pi n_{20}(n_2-n_2')/N_2] \quad (8d)$$

correspond to the packings (4a), (4b), (6c), and (6d), respectively.

We regard as the simplest approach to the localization problem a method based on the investigation of the transport coefficient $T(L)$. To the left and to the right of a disordered section of length L the solution of the Schrödinger equation with the Hamiltonian (4') or (6') takes the plane-wave form

$$\tilde{\psi}_n(x)=A_n^{L(R)}\exp(ik_nx)+B_n^{L(R)}\exp(-ik_nx)$$

with wave vectors

$$k_n=(2m\varepsilon)^{1/2}-\varepsilon_n/v_F. \quad (7')$$

The amplitudes A_n^L, B_n^L and A_n^R, B_n^R to the left and to the right of the disordered section are connected by the scattering matrix $\hat{m}$:

$$\begin{bmatrix}A_n\\B_n\end{bmatrix}^R=\hat{m}\begin{bmatrix}A_{n'}\\B_{n'}\end{bmatrix}^L, \quad (9)$$

which must satisfy the constant-flux condition

$$j=[A^+B^+]^L\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}^L$$
$$=[A^+B^+]^L\hat{m}^+\begin{bmatrix}1&0\\0&-1\end{bmatrix}\hat{m}\begin{bmatrix}A\\B\end{bmatrix}^L \quad (10)$$

at all values of the amplitudes A_n^L and B_n^L. The matrix $\hat{m}$ that satisfies the restrictions (10) is of the form (for details see Ref. 14)

$$\hat{m}=\begin{bmatrix}\hat{X}\,\mathrm{ch}(\hat{\Gamma}/2)\hat{u} & \hat{X}\,\mathrm{sh}(\hat{\Gamma}/2)\hat{v}\\ \hat{Y}\,\mathrm{sh}(\hat{\Gamma}/2)\hat{u} & \hat{Y}\,\mathrm{ch}(\hat{\Gamma}/2)\hat{v}\end{bmatrix}, \quad (11)$$

where $\hat{X}$, $\hat{Y}$, $\hat{u}$, and $\hat{v}$ are unitary matrices of dimensionality N (or $2N$ in the case of magnetic impurities), and $\hat{\Gamma}$ is a diagonal

real matrix with the same dimensionality. All the scattering information we need is contained in the matrix

$$\hat{M}=\hat{m}^{+}\hat{m}=\begin{bmatrix}\hat{u}^{+}(\operatorname{ch}\hat{\Gamma})\hat{u} & \hat{u}^{+}(\operatorname{sh}\hat{\Gamma})\hat{v}\\ \hat{v}^{+}(\operatorname{sh}\hat{\Gamma})\hat{u} & \hat{v}^{+}(\operatorname{ch}\hat{\Gamma})\hat{v}\end{bmatrix}. \tag{12}$$

In the multichannel case the role of the transport coefficient is played by the transport-coefficient matrix $T(L)$. In fact, assume that to the left of the disordered section we have an incident flux $j_A^L=(A^{+}A)^L$ and a reflected flux $j_B^L=(B^{+}B)^L$, while to the right we have only a transmitted flux $j_A^R=(A^{+}A)^R$, i.e., $B_n^R=0$. We express j_A^R in terms of the amplitudes of the incident flux A_n^L:

$$j_A{}^R=(A^{+}A)^R=(A^{+})^L\hat{u}^{+}[2(1+\operatorname{ch}\hat{\Gamma})^{-1}]\hat{u}(A)^L. \tag{13}$$

The expression in the square brackets in (13) is none other than the transmission coefficient matrix $\hat{T}\ 2/(\cosh\hat{\Gamma}+1)$, which was diagonalized with a unitary transformation $\hat{u}$ in the space of the amplitudes $A_n{}^L$. Thus, the transformation $\hat{u}$ defines N (or $2N$) independent scattering channels for the flux from left to right. In other words, if a flux is produced in one, say the nth, channel the scattering process will not affect the remaining channels and will be characterized by its own transmission coefficient $T_n=2/\cosh\Gamma_n+1)$. For a wave traveling in the opposite direction, from right to left, a similar relation can be obtained, in which the transmission coefficients in the different channels remain the same, T_n, and the channels themselves are defined by a unitary transformation $\hat{v}$ in the space of the amplitudes $B_n{}^R$.

If time-reversal symmetry exists in the system, i.e., there are no magnetic impurities or a magnetic field, the unitary matrices $\hat{u}$ and $\hat{v}$ are somehow related. Thus, in cases (4a) and (6c) we have

$$\hat{u}=\hat{v}^{*}. \tag{14ac}$$

For the cylindrical packings (4b) and (6d), when the magnetic flux in the cylinder is a multiple of the flux quantum, $\Phi=n_0\Phi_0$ and $\chi=\pi n_0$, the following relations hold:

$$u_{nn'}=v^{*}_{n,N-n'}, \tag{14b}$$

$$u_{n_1n_2n_1'n_2'}=v^{*}_{n_1,n_2,n_1',N_2-n_2'}. \tag{14d}$$

We define now the elementary act of scattering by an impurity whose position in the lattice $x_n=an$ of each of N chains is assumed random. Thus, we define the random potentials $U_n(x)$. Let the amplitude for passage of an electron through a solitary impurity be $d=\cos\gamma e^{-i\beta}$, and let the probability of finding an impurity at a given lattice site of one of the chains be q. Since we are considering weak disorder $l\gg Na$, we can assume these to be Born impurities with γ, $\beta\ll 1$ and widely spaced, $q\ll 1/N$. For the forward and backward scattering mean free paths we have then respectively $l=a/q\,\gamma^2$ and $l^f=a/q\,\beta^2$. The generalization to the case of magnetic impurities (5) is obvious:

$$\gamma\to\gamma_{ss'}=(\gamma_0\hat{1}+\gamma_x\hat{\sigma}^x+\gamma_y\hat{\sigma}^y+\gamma_z\hat{\sigma}^z)_{ss'}. \tag{5'}$$

A similar relation holds for β. The mean free path corresponding to backscattering is given by

$$l^{-1}=qa^{-1}(\gamma_0{}^2+\gamma_x{}^2+\gamma_y{}^2+\gamma_z{}^2)=l_0^{-1}+l_x^{-1}+l_y^{-1}+l_z^{-1}. \tag{15}$$

The fact that we have set the densities of the simple and magnetic impurities equal is of no fundamental significance, since combinations corresponding to the mean free paths l_i enter everywhere. The matrix $\hat{m}(x_0n_0)$ for backscattering by one impurity located at site x_0 of chain n_0 is of the form

$$\hat{m}(x_0n_0)=\hat{1}+\begin{bmatrix}-i\beta_{nn'}(x_0n_0) & \gamma_{nn'}(x_0n_0)\\ \gamma^{+}_{nn'}(x_0n_0) & i\beta_{nn'}(-x_0n_0)\end{bmatrix}, \tag{16}$$

where we have retained only the terms linear in γ and β. For the packings (4a) and (4b) we have respectively

$$\gamma^{a}_{nn'}(x_0n_0)$$

$$=-i\gamma\exp[-i(k_n+k_{n'})x_0]2(N+1)^{-1}$$

$$\times\sin[\pi n_0n/(N+1)]\sin[\pi n_0n'/(N+1)], \tag{17a}$$

$$\gamma^{b}_{nn'}(x_0n_0)=-i\gamma N^{-1}\exp[-i(k_n+k_{n'})x_0+i2\pi n_0(n-n')/N], \tag{17b}$$

$$\beta^{a}_{nn'}(x_0n_0)$$

$$=\beta\exp[-i(k_n-k_{n'})x_0]2(N+1)^{-1}$$

$$\times\sin[\pi n_0n/(N+1)]\sin[\pi n_0n'/(N+1)], \tag{18a}$$

$$\beta^{b}_{nn'}(x_0n_0)=\beta N^{-1}\exp[-i(k_n-k_{n'})x_0+i2\pi n_0(n-n')/N]. \tag{18b}$$

Similar relations for the packings (6c) and (6d) can be easily obtained by comparing (17), (18), and (8).

3. BASIC RELATIONS

Expressions as, e.g., (2) and (3) are averaged over the realizations of the random potential with the aid of the function $W(L,\hat{T}\hat{u}\hat{v})$, which determines the probability density of the random quantities $\hat{T}$, $\hat{u}$, and $\hat{v}$ given the length L. Since we are dealing with weak impurity potentials ($\beta,\gamma\ll 1$), the function $W(L,\hat{T}\hat{u}\hat{v})$ is described by a Fokker–Planck partial differential equation. To obtain this equation we must define a Markov process for $\hat{T}$, $\hat{u}$, $\hat{v}$. In other words, starting from the distribution density $W(L,\hat{T}\hat{u}\hat{v})$ we must find $W(L+a,\hat{T}\hat{u}\hat{v})$. Expanding the relation obtained in terms of the small increments of a (up to the linear term), $\Delta\hat{T}$, $\Delta\hat{u}$, and $\Delta\hat{v}$ (up to the quadratic terms) and averaging over the rapidly oscillating functions, we obtain the sought Fokker–Planck equation. We thus add one lattice constant a to the disordered section of length L. A new lattice site of one of the N chains (say, the chain n_0) will contain, with probability $Nq\ll 1$, an impurity. The matrix (12) then acquires an increment

$$\Delta M(L)=m^{+}(L+a,n_0)M(L)m(L+a,n_0)-M(L).$$

From this we get the corresponding increments of the random quantities $\hat{T}=2/\cosh\hat{\Gamma}+1)$, $\hat{u}$, and $\hat{v}$. We present expressions for the increments $\Delta\hat{\Gamma}$, $\Delta\hat{u}$, and $\Delta\hat{v}$, accurate to terms linear in $\hat{\gamma}$ and $\hat{\beta}$:

$$\Delta\Gamma_n=R_{nn}+R_{nn}^{*}, \tag{19}$$

$$\Delta u_{nn'}=\sum_{n_1}\{-iu_{nn_1}\beta_{n_1n'}+[\delta_{nn_1}(R_{nn}^{*}-R_{nn})\operatorname{cth}\Gamma_n/2$$

$$+(1-\delta_{nn_1})(R_{nn_1}\operatorname{sh}\Gamma_{n_1}+R_{n_1n}^{*}\operatorname{sh}\Gamma_n)/(\operatorname{ch}\Gamma_n-\operatorname{ch}\Gamma_{n_1})]u_{n_1n'}\}, \tag{20}$$

$$\Delta v_{nn'}=\sum_{n_1}\{iv_{nn_1}\beta_{n_1n'}+[\delta_{nn_1}(R_{nn}-R_{nn}^{*})\operatorname{cth}\Gamma_n/2$$

$$+(1-\delta_{nn_1})(\operatorname{sh}\Gamma_nR_{nn_1}+\operatorname{sh}\Gamma_{n_1}R_{n_1n}^{*})/(\operatorname{ch}\Gamma_n-\operatorname{ch}\Gamma_{n_1})]v_{n_1n'}\}, \tag{20'}$$

where we have introduced the notation

$$\hat{R}=\hat{u}\hat{\gamma}(L+a, n_0)\hat{v}^+, \quad \hat{\beta}=\hat{\beta}(L+a, n_0), \quad \hat{\tilde{\beta}}=\hat{\beta}(-L-a, n_0). \tag{21}$$

It is important to note that in relations (19)–(21), obtained on the basis of Eqs. (12) and (16), we did not use so far the actual form of the matrices $\hat{\gamma}(x_0 n_0)$ and $\hat{\beta}(x_0 n_0)$, which depends on the packing of the chains and on the coupling energy t between them. This remark applies to all the results of the present section, which are therefore quite general.

The Fokker–Planck equation is obtained with the aid of (19)–(21) by the method described above. It is more convenient for us to deal with an equation for the distribution density of the quantities $F = \cosh \hat{\Gamma}$, $\hat{u}$, and $\hat{v}$, i.e., for the function $W(L, \hat{F}\hat{u}\hat{v})$. We present the first few terms of this equation:

$$\frac{a}{Nq}\frac{\partial}{\partial L}W(L, \hat{F}\hat{u}\hat{v}) = \sum_n \left\{ \frac{\partial}{\partial F_n}(F_n^2-1)\left[\frac{\partial}{\partial F_n}\overline{|R_{nn}|^2} - \sum_{n'\neq n}(\overline{|R_{nn'}|^2}+\overline{|R_{n'n}|^2})/(F_n-F_{n'}) \right] \right.$$
$$\left. + \sum_{n'\neq n}\frac{\partial}{\partial F_n}(F_n^2-1)^{1/2}\frac{\partial}{\partial F_{n'}}(F_{n'}^2-1)^{1/2}(\overline{R_{nn}R^*_{n'n'}}+\overline{R_{n'n'}R^*_{nn}}) \right\} W + \dots, \tag{22}$$

where the bar over the terms quadratic in $\hat{\gamma}(x_0 n_0)$ denotes averaging over the rapidly oscillating dependence on x_0 and n_0. Following next in (22) are terms with the derivatives $\partial^2/\partial\hat{F}\partial\hat{u}, \partial^2/\partial\hat{F}\partial\hat{v}$ and all the remaining derivatives with respect to $\hat{u}$ and $\hat{v}$. Since the variables F_n determine the transmission coefficients, $T_n = 2/(F_n + 1)$, even the terms retained in (22) contain important information on the localization in various channels. In fact, if we insert $\ln T_n = \ln[2/(F_n + 1)]$ in both halves of (22 and integrate with respect to $\hat{F}$, $\hat{u}$, and $\hat{v}$, only the terms written out in (22) make a nonzero contribution. The result can be represented in the form

$$\langle \varepsilon_n^* \rangle = v_F/l_n^* = -v_F\frac{d}{dL}\langle \ln T_n \rangle$$
$$= v_F a^{-1} Nq\left\langle \overline{|R_{nn}|^2} + \sum_{n'\neq n}(\overline{|R_{nn'}|^2}+\overline{|R_{n'n}|^2})(F_n+1)/(F_n-F_{n'}) \right\rangle, \tag{23}$$

where we have introduced, in accordance with (3), the localization energies ε_n^* and the localization lengths l_n^* corresponding to the different channels. We now average (23) over all N (or $2N$) channels. We rewrite first the factor that contains F_n and $F_{n'}$ in the form of two terms:

$$\frac{F_n+1}{F_n-F_{n'}} = \frac{F_n+F_{n'}+2}{2(F_n-F_{n'})} + \frac{1}{2},$$

the first of which reverses sign upon the substitution $n \leftrightarrow n'$ and therefore makes no contribution to the answer averaged over the channels:

$$N^{-1}\sum_n \langle \varepsilon_n^* \rangle = v_F N^{-1}\sum_n 1/l_n^* = v_F a^{-1} q\left\langle \sum_{n,n'} \overline{|R_{nn'}|^2} \right\rangle = 1/\tau. \tag{24}$$

The important and unexpected result (24) shows that the localization energy averaged over the realizations of the random potential and over the channels is equal to the reciprocal free path time and *is independent* of the number of channels, of the method of packing, the chains, and of their binding energy t. At the same time we expect that some localization should decrease, and a localization length should increase, when N or t is increased. The answer is that spontaneous symmetry breaking between the channels takes place, and as a result the channels have different localization lengths $\langle \varepsilon_n^* \rangle$ and localization lengths l_n^*.[14–16] In other words, the localization-energy degeneracy existing at $t = 0$ and in the absence of magnetic impurities is lifted with respect to the transverse quantization index because $t \neq 0$, and with respect to the spin index on account of the magnetic impurities. One of the N (or $2N$) channels receives in this case the smallest localization energy $\langle \varepsilon_{n_0}^* \rangle$ and the largest localization length $l_{n_0}^*$. It is just this channel alone which plays the principal role in the kinetic phenomena at the very longest wavelengths and lowest frequencies. Thus, for sufficiently long samples the channel in question makes the main contribution to the transmission coefficient $T = N^{-1}\Sigma_n T_n = N^{-1} T_{n_0}$ averaged over the channels. All this obviates the intensively discussed[10,19,20] question of how to use the Landauer formula[9] for the conductivity (2′) in the multichannel case. The total conductivity of long samples is determined by the very same channel with maximum localization length:

$$G(L\to\infty) = \frac{e^2}{2\pi\hbar} T_{n_0}(L)/[1-T_{n_0}(L)]. \tag{25}$$

We now renumber the channels in decreasing order of their transmission coefficients, i.e., we have $F_0 < F_1 < \dots < F_{N-1}$. As a result, as can be seen from (23), the localization energy $\langle \varepsilon_n^* \rangle$ is arranged in increasing order. This means that all the transmission coefficients decrease with increasing L in accord with different exponential relations $T_n(L) \sim \exp(-\varepsilon_n^* L/v_F)$. Thus, for sufficiently large lengths $L \gg v_F/(\varepsilon_{n+1}^* - \varepsilon_n^*)$ we get already the strong inequality

$$F_0 \ll F_1 \ll \dots \ll F_{N-1}. \tag{26}$$

Using this inequality, we can simplify the relation (23):

$$\langle \varepsilon_0^* \rangle = v_F/l_0^* = v_F a^{-1} Nq \langle \overline{|R_{00}|^2} \rangle, \tag{23′}$$

$$\langle \varepsilon_n^* \rangle = v_F/l_n^* = v_F a^{-1} Nq\left\langle \overline{|R_{nn}|^2} + \sum_{n'=0}^{n-1}(\overline{|R_{nn'}|^2}+\overline{|R_{n'n}|^2}) \right\rangle. \tag{23″}$$

Similarly, without essentially finding the total probability density $W(L, \hat{T}\hat{u}\hat{v})$, we can obtain an expression for the average logarithm of the conductivity (25). Inserting $\ln G(L\to\infty)$ in (22), integrating with respect to $\hat{F}$, $\hat{u}$, and $\hat{v}$, and using the inequality (26), we arrive at the expression

$$\langle \ln G(L\to\infty)\rangle = -La^{-1}Nq\langle \overline{|R_{00}|^2}\rangle = -L/l_0^*. \quad (27)$$

To determine the localization energies (23) as $L\to\infty$ it suffices thus to calculate the correlators, indicated in (23′) and (23″), of the random matrices u and v. At large distances $L \gg v_F/(\varepsilon_{n+1}^* - \varepsilon_n^*)$, when inequality (26) holds, the distribution of $\hat{u}$ and $\hat{v}$ ceases to depend on L and T, i.e., it is determined by the stationary distribution density $W(\hat{u},\hat{v})$. This can be seen from expressions (20) and (20′) for the increments $\Delta\hat{u}$ and $\Delta\hat{v}$, which cease to depend on $\hat{\Gamma}$ under the condition (26):

$$\Delta\hat{u} = -i\hat{u}\hat{\beta} + \hat{P}\hat{u}, \quad (28)$$

$$\Delta\hat{v} = i\hat{v}\hat{\tilde{\beta}} - \hat{\tilde{P}}\hat{v}, \quad (28')$$

where we have used the matrices ($\theta(x)$ is the step function)

$$P_{nn'} = \theta(n-n')R_{nn'} - \theta(n'-n)R_{nn'}, \quad (29)$$

$$\tilde{P}_{nn'} = \theta(n'-n)R_{nn'} - \theta(n-n')R_{nn'}. \quad (29')$$

Starting from relations (28) and (29) we can find a closed Fokker-Planck equation for the function $W(\hat{u}\hat{v})$ used to average in (23′), (23″), and (27). Generally speaking, however, the stationary density $W(\hat{u}\hat{v})$ is not sufficient for the averaging of the transmission coefficient $T_0(L)$ or of the conductivity (25). The inequality (26) permits simplification of this question, too. The point is that under condition (26) Eq. (22) for the function $W(L, \hat{F}\hat{u}\hat{v})$ splits into N (or $2N$) equations for the function $W_n(L, F_n\hat{u}\hat{v})$. Using the inequality (26) and integrating with respect to all the variables $F_{n'}$ except for one F_n, we obtain

$$\frac{a}{Nq}\frac{\partial}{\partial L}W_0(L, F_0\hat{u}\hat{v}) = \frac{\partial}{\partial F_0}(F_0^2-1)\frac{\partial}{\partial F_0}\overline{|R_{00}|^2}\,W_0 + \dots, \quad (22')$$

$$\frac{a}{Nq}\frac{\partial}{\partial L}W_n(L, F_n\hat{\tilde{u}}\hat{\tilde{v}}) = \frac{\partial}{\partial F_n}F_n^2\left[\frac{\partial}{\partial F_n}\overline{|R_{nn}|^2} - \frac{1}{F_n}\sum_{n'=0}^{n-1}(\overline{|R_{nn'}|^2} + \overline{|R_{n'n}|^2})\right]W + \dots. \quad (22'')$$

Equation (22′) for the channel with maximum localization length is already very similar to the equation for the function $W(L,F)$ in the case of one chain[7,8]:

$$l\frac{\partial}{\partial L}W(L,F) = \frac{\partial}{\partial F}(F^2-1)\frac{\partial}{\partial F}W, \quad (30)$$

from which follows, in particular, the asymptotic form (2).[7,8] Equation (22′) reduces exactly to (30) only if $\overline{|R_{00}|^2}$ is in fact a number independent of $\hat{u}$ and $\hat{v}$, so that (22′) can be integrated with respect to $\hat{u}$ and $\hat{v}$. Two cases of this kind (without and with magnetic impurities) are presented below. As an example of a correlated distribution of the parameters F_0 and $\hat{u}$ or $\hat{v}$, when it is impossible to obtain (30) from (22′), we cite the two-channel case considered in Sec. 6. More general, however, is the situation wherein it is possible to reduce Eq. (22′) approximately (relative to some parameter) to the one-dimensional case (30). Thus, if we are dealing with smooth impurity potentials forward scattering is much more effective than backward, $l_0^* \ll l$. The characteristic distance l^f over which the matrices $\hat{u}$ and $\hat{v}$ change substantially turns out then to be much less than the localization length l_0^*, since $l_0^* > l$. This circumstance permits averaging in (22′) over the fast dependences on $\hat{u}$ and $\hat{v}$, and obtain Eq. (30) accurate to the substitution

$$l \to a\langle Nq\overline{|R_{00}|^2}\rangle^{-1}, \quad (30')$$

which agrees with Eq. (23′).

Of greater importance is the situation $N \gg 1$, when the localization length $l_0^* \sim Nl$ is again much larger than the distance l over which the matrices $\hat{u}$ and $\hat{v}$ vary. Consequently[16] the localization in sufficiently thick wires with $N \gg 1$ is described by the same equations (30), (30′), (2) (1), etc. as in a single chain, whereas in very thin wires with $N \sim 1$ this is not the case.

4. LOCALIZATION IN THE MULTICHANNEL CASE WITHOUT MAGNETIC IMPURITIES

Having at our disposal Eqs. (23′), (23″), and (30′) we can proceed to calculate the localization length. The question is, over which of the characteristic distances is it necessary to average over the rapidly oscillating dependence on x_0 and n_0 in expressions of the type

$$\overline{R_{nn'}R_{\bar{n}\bar{n}'}^*} = u_{nn_1}v_{n'n_1'}^* u_{\bar{n}\bar{n}_1}^* v_{\bar{n}'\bar{n}_1'} \overline{\gamma_{n_1n_1'}(x_0n_0)\,\gamma_{\bar{n}_1\bar{n}_1'}^*(x_0n_0)}. \quad (31)$$

The answer is that, since we are dealing with the nth channel, the averaging in (31) should be over characteristic lengths x_0 of the order of the corresponding localization length l_n^*. Since we shall be interested hereafter mainly in the contribution from a short localization length $l_0^* \sim Nl$, we average in (31) over distances $x_0 \sim Nl$.

Using (17a) and (17b) we obtain the following relations for the packings (4a), (4b), (6c) and (6d):

$$(\overline{\gamma_{nn'}\gamma_{\bar{n}\bar{n}'}^*})^a = \gamma^2 f^a(nn', \bar{n}\bar{n}')/[N(N+1)], \quad (32a)$$

$$(\overline{\gamma_{nn'}\gamma_{\bar{n}\bar{n}'}^*})^b = \gamma^2 f^b(nn', \bar{n}\bar{n}')/N^2, \quad (32b)$$

$$\overline{(\gamma_{n_1n_2n_1'n_2'}\gamma_{\bar{n}_1\bar{n}_2\bar{n}_1'\bar{n}_2'}^*)^c} = \gamma^2 f^a(n_1n_1', \bar{n}_1\bar{n}_1')\, f^a(n_2n_2', \bar{n}_2\bar{n}_2')/[N_1(N_1+1)\,N_2(N_2+1)], \quad (32c)$$

$$\overline{(\gamma_{n_1n_2n_1'n_2'}\gamma_{\bar{n}_1\bar{n}_2\bar{n}_1'\bar{n}_2'}^*)^d} = \gamma^2 f^a(n_1n_1', \bar{n}_1\bar{n}_1')\, f^b(n_2n_2', \bar{n}_2\bar{n}_2')/[N_1+1)\,N_2^2], \quad (32d)$$

where we have introduced the notation

$$f^a(nn', \bar{n}\bar{n}') = \overline{\exp}\{-ix_0(\varepsilon_n^a + \varepsilon_{n'}^a - \varepsilon_{\bar{n}}^a - \varepsilon_{\bar{n}'}^a)/v_F\}[\delta_{n-n',\bar{n}-\bar{n}'} + \delta_{n-n',\bar{n}'-\bar{n}} + \delta_{n+n',\bar{n}+\bar{n}'} + \delta_{n+n',2(N+1)-\bar{n}-\bar{n}'}], \quad (33a)$$

$$f^b(nn', \bar{n}\bar{n}') = \overline{\exp}\{-ix_0(\varepsilon_n^b + \varepsilon_{n'}^b - \varepsilon_{\bar{n}}^b - \varepsilon_{\bar{n}'}^b)/v_F\}\tilde{\delta}_{n-n',\bar{n}-\bar{n}'}, \quad (33b)$$

$$\tilde{\delta}_{n,n'} = N^{-1}\sum_{n_0}\exp[i2\pi n_0(n-n')/N] = \sum_{n_0}\delta_{n,n'+n_0N}.$$

We can similarly obtain expressions for the terms quadratic in $\hat{\beta}$. The essential difference is that there appear not only the mean value $\overline{\beta\beta}^*$, but also $\overline{\beta\beta}$ and $\overline{\beta^*\beta^*}$.

We shall illustrate the averaging over the fast dependence on x_0 using as an example expression (33b), in which the exponential can be rewritten in the form

$$\overline{\exp}\{-8ix_0tv_F^{-1}\sin[\pi(n-\bar{n})/N]\sin[\pi(n+\bar{n}'+\chi/\pi)/N]\sin[\pi(n-n'-N/2)/N]\}. \quad (34)$$

If there is no time-reversal symmetry ($\chi \neq \pi n_0$) and N is odd, the second and third sine function in the exponential of (34) cannot vanish exactly. The minimum values that they assume at $n+\bar{n}'=N$ and $n-n'=(N\pm 1)/2$ are of the order of $1/N$. Therefore, if we want a fast dependence on x_0 in the exponential (34) at arbitrary parameters $n\neq\bar{n}$, n', $\bar{n}'$ we must to stipulate satisfaction of the condition

$$x_0t/(v_FN^3)\gg 1. \quad (35)$$

This restriction, which takes at $x_0\sim Nl$ the form

$$\min \Delta\varepsilon\sim t/N^2\gg 1/\tau \quad (35')$$

and means total absence of classical diffusion, is precisely the one assumed by us in the earlier papers.[15,16] Here we obtain first the answers under the condition (35), and then show that for the results (38), (38') and (38") to be valid we need the much weaker restriction (42). Thus, for odd N and $\chi\neq\pi n_0$ we have under the conditions of (35)

$$f^b(nn',\ \bar{n}\bar{n}')=\delta_{n\bar{n}}\delta_{n'\bar{n}'}. \quad (36)$$

In the presence of time-reversal symmetry ($\chi=\pi n_0$) at odd N, expression (33b) takes a different form

$$f^b(nn',\ \bar{n}\bar{n}')=\delta_{n\bar{n}}\delta_{n'\bar{n}'}+\delta_{n+\bar{n}',N}\delta_{n'+\bar{n},N}-\delta_{n\bar{n}}\delta_{n'\bar{n}'}\delta_{n+\bar{n}',N}. \quad (36')$$

Similar formulas are easily obtained also for even N, when the last sine function in the exponential (34) can also vanish.

Substituting (32b), (36), and (36') in (31) and using at $\chi=\pi n_0$ the relations (14b), we obtain

$$\overline{(R_{nn'}R^{*}_{\bar{n}\bar{n}'})^b_{\chi\neq\pi n_0}}=\gamma^2\delta_{n\bar{n}}\delta_{n'\bar{n}'}/N^2, \quad (37)$$

$$(R_{nn'}R^{*}_{\bar{n}\bar{n}'})^b_{\chi=\pi n_0}=\gamma^2(\delta_{n\bar{n}}\delta_{n'\bar{n}'}+\delta_{n\bar{n}'}\delta_{n'\bar{n}}-\sum_{n_1}u_{nn_1}u_{n'n_1}u^{*}_{\bar{n}n_1}u^{*}_{\bar{n}'n_1})/N^2. \quad (37')$$

Since the mean value (37) does not contain the matrices $\hat{u}$ and $\hat{v}$, the answers for $\chi=\pi n_0$ are directly obtained[15] from (23') and (23"):

$$l_0^{*}=Nl,\quad l_n^{*}=Nl/(1+2n), \quad (38)$$

where the last relation requires, generally speaking, satisfaction of the inequality (35) with $x_0\sim l_n^{*}$. We have here an exact reduction to the problem of one-dimensional localization and, in particular, we get an asymptotic form with l replaced by Nl.

If $\chi=\pi n_0$, the matrix $\hat{u}$, strictly speaking, enters in the expression for the localization length (packing (4b), N odd):

$$l_0^{*}=Nl\Big/\Big(2-\Big\langle\sum_n|u_{0n}|^4\Big\rangle\Big). \quad (39)$$

Since u is a random unitary matrix, we have $|u_{nn'}|^2\sim 1/N$ (for details see Appendix I) and we get the estimate

$$\Big\langle\sum_n|u_{0n}|^4\Big\rangle\sim N\langle|u_{nn'}|^4\rangle\sim 1/N. \quad (40)$$

At $N\gg 1$ we get thus the approximate reduction, accurate to the parameter $1/N$ and described in the preceding section, to the one-dimensional problem. In the presence of time-reversal symmetry the localization length is then[16]

$$l_0^{*}=Nl/2, \quad (38')$$

or half the value in the absence of this symmetry (cf. Ref. 18).

In Appendix I, starting from Eqs. (37') and (39) and calculating the necessary mean values of the matrix $\hat{u}$, we obtained the corrections to (38'):

$$l_0^{*}=Nl/(2-2/N+\ldots). \quad (41)$$

In the limit of strong forward scattering ($\beta^2\gg\gamma^2$) it is possible to obtain in the case of (39) for the localization length an expression that is valid for all N:

$$l_0^{*}=(N+1)l/2. \quad (41')$$

We now stop to examine whether the rather strong restrictions (35) and (35') are really so necessary for the validity of the answers (38) and (38') at $N\gg 1$. The cube of N in the inequality (35) is necessary for the dependence on x_0 to remain fast in the rather untypical case when all three sine functions in the exponential (34) are small. It is understood that the phase space corresponding to the situation in which two or three sine functions in the exponential (34) are close to zero is definitely small. We confine ourselves therefore to a condition that is much weaker than the inequality (35), namely

$$[\ln(v_FN^3/x_0t)]^{-2}x_0t/(v_FN)\gg 1, \quad (42)$$

As a result, a smooth term corresponding to classical diffusion appears in relations (36) and (36'). This function, however, differs from zero in a small region of the space of the variables n and n', in analogy with the third term of (36'), which gives a small correction, $\sim 1/N$, to the localization length. Thus, the use of the weak restriction (42) leads to corrections for the localization length in (38) and (28'), but these corrections are small in terms of the parameter (42). Since $x_0\sim Nl$, the inequality (42) is equivalent in the case of the two-dimensional packings (4a) and (4b) to the condition

$$[\ln(N^2/t\tau)]^{-2}t\tau\gg 1, \quad (42ab)$$

For the three-dimensional packings (6a) and (6d), similar reasoning leads to the restriction

$$t\tau\gg 1. \quad (42cd)$$

It is convenient here to estimate the characteristic values of the magnetic flux in a cylinder (more accurately, its differences $\Delta\Phi=\Phi-n_0\Phi$) from the total number of flux quanta at which the interchange of the regimes (38) and (38') takes place. This estimate can be obtained by considering the second sine function in the exponential (34):

$$\Delta\Phi_c\sim\Phi_0/t\tau, \quad (43b)$$

$$\Delta\Phi_c\sim\Phi_0/t\tau N_1. \quad (43d)$$

Strictly speaking, Eqs. (38) and (38') were obtained so far only for the cylindrical packing (4b). At $N\gg 1$, however, these results are universal in the sense that they do not depend on the packing method. On the other hand, the corrections to the answers, such as, e.g., in Eqs. (41) and (41'), are by far not universal, i.e., they depend on whether the number of chains N is odd and on the shape of the wire cross section.

We present expression (33a) averaged over x_0 under the condition (33):

$$f^a(nn', \bar{n}\bar{n}')$$
$$=\delta_{n\bar{n}}\delta_{n'\bar{n}'}+\delta_{n\bar{n}'}\delta_{n'\bar{n}}+2^{-1}\{(2-\delta_{n\bar{n}}-\delta_{\bar{n}\bar{n}'})\,\delta_{n+n',\,N+1}\delta_{\bar{n}+\bar{n}',\,N+1}$$
$$-(1-\delta_{n,\,(N+1)/2})\,\delta_{n\bar{n}}\delta_{n'\bar{n}'}\delta_{nn'}\}. \qquad (36a)$$

Relation (31) now takes the form

$$(\overline{R_{nn'}R^{*}_{\bar{n}\bar{n}'}})^a=\gamma^2(\delta_{n\bar{n}}\delta_{n'\bar{n}'}+\delta_{n\bar{n}'}\delta_{n'\bar{n}}+\ldots)/[N(N+1)], \qquad (37a)$$

where we have used the relation (14a) and the three dots denote the terms $\sim 1/N$ that stem from the expression in the curly brackets of (36a). These terms are estimated in analogy with (40). Thus, for chains packed on a plane we get the answer (38′). The same results (38) and (38′) are obtained (in principal order in $1/N$) when (32c) and (32d) are used for three-dimensional packings. Just as before, by foregoing the condition (35) and changing over to the restrictions (42cd) we arrive at universal corrections that are small in the parameter (42cd).

5. MAGNETIC IMPURITIES AND MULTICHANNEL LOCALIZATION

It can be stated right away that the influence of magnetic impurities of general form reduces to two effects. First, they eliminate the time-reversal symmetry. Second, they lift the degeneracy in the spin index of the electron, leading thereby to a doubling of the number of channels. Even these general considerations allow us to write down a correction equation for the localization length[16] (cf. Ref. 18):

$$l_0^{*}=2Nl. \qquad (38'')$$

Deviations from this result can occur on account of the strongly anisotropic character of the magnetic impurities. A difference is obvious, e.g., for Ising magnetic impurities ($\gamma_x=\gamma_y=\beta_x=\beta_y=0$). In this case all the matrices are diagonal in the spin indices and the results for nonmagnetic impurities remain in force if account is taken of the change of the mean free path (15). A different formula is obtained also for magnetic impurities in the form of plane rotators ($\gamma_y=\beta_y=0$). A feature of these impurities is that they leave the Hamiltonian (4) or (6) real (if $\chi=\pi n_0$), i.e., they do no violate the symmetry with respect to complex conjugation of the wave functions.

If the system contains magnetic impurities, the amplitudes γ and β must be regarded as spin matrices (5′). The averaging in Eqs. (32) is not only over the impurity position, but also over the direction of the random magnetic field. It is necessary to make in (32) the substitution

$$\gamma^2\to\overline{\gamma_{ss'}\gamma^{*}_{\bar{s}\bar{s}'}}=2^{-1}(\gamma_0^2+\gamma_x^2+\gamma_y^2+\gamma_z^2)\,\delta_{s\bar{s}}\delta_{s'\bar{s}'}$$
$$+2^{-1}\{(\gamma_0^2+\gamma_x^2-\gamma_y^2-\gamma_z^2)$$
$$\times\sigma^x_{s\bar{s}}\sigma^x_{s'\bar{s}'}-(\gamma_0^2-\gamma_x^2+\gamma_y^2-\gamma_z^2)\,\sigma^y_{s\bar{s}}\sigma^y_{s'\bar{s}'}$$
$$+(\gamma_0^2-\gamma_x^2-\gamma_y^2+\gamma_z^2)\,\sigma^z_{s\bar{s}}\sigma^z_{s'\bar{s}'}\}. \qquad (44)$$

It follows therefore that the simplest case of Heisenberg impurities is the model (4b) under the conditions $l_0=l_x=l_y=l_z$, $\chi\neq\pi n_0$ and N is odd, when we obtain, using (37) and (44), the result (38″) even at $N\sim 1$. In the general case of magnetic impurities, when $l_0\neq l_x\neq l_y\neq l_z$, the terms in the curly brackets of (44) add a small correction, in terms of $1/N$, to Eq. (38″), in analogy with the reasoning of the preceding section. For Heisenberg impurities $l_0\neq l_x=l_y=l_z$ we have obtained in Appendix II the following expansion, which makes more accurate the result (38″) at $\chi\neq\pi n_0$, at odd N, and in the absence of forward scattering ($l^f=\infty$):

$$l_0^{*}=2Nl/\{1+3(\alpha/2)^2/[(2N)^2-1]+\ldots\}, \qquad (45)$$

where

$$\alpha=(l_0^{-1}-l_x^{-1})/[l_0^{-1}+l_x^{-1}+4(l_x^{f})^{-1}]. \qquad (46)$$

The expansion of (45) is in powers of α and is valid even for $N\sim 1$. Eq. (45) shows directly how the universal answer (38″) is arrived at when the number of chains is increased.

We consider now the multichannel case without a magnetic field, but with magnetic impurities in the form of plane rotators ($\gamma_x^2=\gamma_z^2$, $\beta_x^2=\beta_z^2$, $\gamma_y=\beta_y=0$), when it is convenient to rewrite (44) in a somewhat different form:

$$\gamma^2\to\overline{\gamma_{ss'}\gamma_{\bar{s}\bar{s}'}}=4^{-1}(\gamma_0^2+3\gamma_x^2)\,(\delta_{s\bar{s}}\delta_{s'\bar{s}'}+\delta_{s\bar{s}'}\delta_{s'\bar{s}})$$
$$+4^{-1}(\gamma_0^2-\gamma_x^2)\,\{(\delta_{s\bar{s}}\delta_{s'\bar{s}'}+\delta_{s\bar{s}'}\delta_{s'\bar{s}})\,(-1)^{s+s'}+4\delta_{ss'}\delta_{\bar{s}\bar{s}'}(1-\delta_{s\bar{s}})\}. \qquad (47)$$

Substituting this expression in (37) or (37a) and retaining only terms of principal order in $1/N$, we obtain

$$l_0^{*}=2N/(l_0^{-1}+3l_x^{-1}). \qquad (48)$$

Since the mean free path (15) is equal here to $l=1/(l_0^{-1}+2l_x^{-1})$, the result (38) does not agree with (38″).

Relations (38″) and (48) should go over into (38′) if the effectiveness of the magnetic impurities is greatly decreased, $\alpha\to 1$. Under these conditions a tendency appears toward degeneracy of the localization energies in channel pairs that differ only in spin. We were unfortunately unable to calculate to correction proportional to $N(1-\alpha)^{1/2}$, to expression (38), i.e., to track the instant when the spin degeneracy is lifted. The difficulty here is that one must deal with two channels that can in principle not be uncoupled. To estimate the characteristic value $(1-\alpha)_c$ at which the interchange of the regimes (38′) and (38″) takes place, it is necessary to use the following approximate considerations. Change of the symmetry of the matrices u and v from the case (38′) to (38″) requires one act of total scattering by magnetic impurities in at least one of the N channels. Therefore the distance over which this takes place is equal to l_m/N, where l_m is the scattering mean free path for the magnetic impurities. If the scale l_m/N exceeds the localization length, $l_m/N>l_0^{*}(N)$, the symmetry of the matrices $\hat{u}$ and $\hat{v}$ cannot change before $l_0^{*}(N)$ is formed, i.e., the regime (38′) takes place. We thus obtain the estimate (cf. Ref. 18)

$$(1-\alpha)_c\sim 1/N^2.$$

6. TWO-CHANNEL LOCALIZATION

It is of interest to consider in greater detail two-channel localization, which does not reduce generally speaking to the one-dimensional situation. We shall not find the behavior of the mean value $\langle T(L)\rangle$, whose asymptotic form does not coincide here with Eq. (2). It will be much simpler to calculate the scaling localization length l_0^{*}. Two-channel scattering occurs either in two chains with potential impurities, or in one chain in the presence of magnetic impurities. We consider first the former case.

For two chains, both models (4a) and (4b) are equivalent apart from a certain redefinition of the binding energy t. At $N=2$ the localization length is specified by expression (39), as can be easily verified by using relations (23'), (32a), and (32b). For a sufficiently long disordered segment $L \gg l$ we can obtain with the aid of (28) and (29), by the method described above, a Fokker-Planck equation for the stationary distribution function $W(\xi)$ of the random parameter $\xi = |u_{00}|^2 - |u_{01}|^2$. This equation is of the form

$$\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}\left(1+\frac{2\beta^2}{\gamma^2}+\xi^2\right)W(\xi)=0. \tag{49}$$

The solution of (49), satisfying the conditions

$$W(\xi)>0, \quad \int_{-1}^{1} d\xi W(\xi)=1,$$

is obviously

$$W(\xi)=(1+2\beta^2/\gamma^2)^{1/2}/\{2\,\mathrm{arctg}\,[(1+2\beta^2/\gamma^2)^{-1/2}](1+2\beta^2/\gamma^2+\xi^2)\}. \tag{50}$$

Averaging in (39) with the aid of the distribution function (50), we obtain the localization length:

$$l_0^*(N=2)=l\Big/\left\{1+\frac{\beta^2}{2\gamma^2}-\frac{(1+2\beta^2/\gamma^2)^{1/2}}{4\,\mathrm{arctg}[(1+2\beta^2/\gamma^2)^{-1/2}]}\right\}. \tag{51}$$

We present two limiting values of (51) for weak and strong forward scattering:

$$l_0^*(N=2)=\begin{cases} l/(1-1/\pi), & \beta^2 \ll \gamma^2 \\ l/(1-1/3), & \beta^2 \gg \gamma^2. \end{cases} \tag{51'}$$

The first of these values was obtained by us earlier,[14] and the second is obtained in accord with the result (41'). As can be seen from (51) and (51'), the localization length in the two coupled chains depends on the forward scattering amplitude β, this being a new feature compared with the one-dimensional situation and with the case $N \gg 1$. For numerical reasons, however, this dependence is quite weak. As already noted in Sec. 3, in the case of strong forward scattering ($\beta^2 \gg \gamma^2$) the distribution function $W(L,\xi)$ the stationary distribution (50) at distances $L \sim l^f = a/q\beta^2$, much shorter than the localization length $l_0^* = 2l/3$. This reduces the problem to that of one-dimensional localization (after substituting $l \to 2l/3$) and yields, in particular, the asymptotic form (2). For an impurity potential of general form ($\beta^2 \sim \gamma^2$) localization in two chains is described by equations that cannot be reduced to the one-dimensional case (for details see Ref. 14).

We consider now magnetic impurities in one chain. Here, as in the preceding section, the localization depends on the assumption that the magnetic impurities are anisotropic. We consider first the case of isotropic Heisenberg impurities ($\gamma_x^2=\gamma_y^2=\gamma_z^2$, $\beta_x^2=\beta_y^2=\beta_z^2$), when it follows from (31) and (44) that

$$\overline{R_{ss'}R^*_{\bar{s}\bar{s}'}} = 2\gamma_x^2\delta_{s\bar{s}}\delta_{s'\bar{s}'} + (\gamma_0^2-\gamma_x^2)S_{ss'}S^*_{\bar{s}\bar{s}'}, \tag{52}$$

where we have introduced the unitary matrix $\hat{S}=\hat{u}\hat{v}^+$. Thus, to calculate the localization length in accord with (23') we must find the mean value of $|S_{00}|^2=(1+\eta)/2$. The equation constructed for the stationary distribution function $W(\eta)$ with the aid of (28) and (28') takes the form

$$\frac{d}{d\eta}(1-\eta^2)\left\{(1-\alpha\eta)\frac{d}{d\eta}+\frac{3\alpha}{2}\right\}W(\eta)=0, \tag{53}$$

where the parameter α is defined in (46). The solution of (53), which satisfies the condition

$$W(\eta)>0, \quad \int_{-1}^{1} d\eta\, W(\eta)=1,$$

can be easily obtained

$$W(\eta)=\alpha(1-\alpha^2)^{1/2}/\{2[(1+\alpha)^{1/2}-(1-\alpha)^{1/2}](1-\alpha\eta)^{3/2}\}. \tag{54}$$

We thus obtain for the localization length the expression

$$l_0^*(N=1)=2l\Big/\left\{1+\left(1+\frac{8l}{l_x^f}\right)\frac{1-(1-\alpha^2)^{1/2}}{2-\alpha}\right\}, \tag{55}$$

which agrees at $\alpha \ll 1$ with the expansion (45).

To complete the picture we consider also the case of magnetic impurities in the form of planar rotators ($\gamma_x^2=\gamma_z^2$, $\beta_x^2=\beta_z^2$, $\gamma_y=\beta_y=0$). Since the Hamiltonian is real, we have $\hat{u}=\hat{v}^*$. Using this circumstance we obtain

$$\overline{|R_{00}|^2}=(\gamma_0^2+\gamma_x^2)-(\gamma_0^2-\gamma_x^2)\zeta^2, \quad \zeta=|u_{00}u_{01}^*-u_{00}^*u_{01}|.$$

We seek the equation for $W(\zeta)$, as before, with the aid of (28) and (29):

$$\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta}[1-\alpha(1-2\zeta^2)]W(\zeta)=0, \tag{56}$$

where α is again defined by (46). A solution of this equation, satisfying the positiveness and normalization conditions

$$\int_0^1 d\zeta\, W(\zeta)=1,$$

is

$$W(\zeta)=\left(\frac{|1-\alpha|}{2\alpha}\right)^{1/2}\Big/\left\{\mathrm{arctg}\left[\left(\frac{2\alpha}{1-\alpha}\right)^{1/2}\right](1-\alpha+2\alpha\zeta^2)\right\}. \tag{57}$$

Using the distribution function $W(\zeta)$ we obtain the following expression for the localization length:

$$l_0^*=\left\{l^{-1}+\frac{2}{l_x^f}-\frac{|l_0^{-1}-l_x^{-1}|^{1/2}(l_x^{-1}+2/l_x^f)^{1/2}}{\mathrm{arctg}[(l_0^{-1}-l_x^{-1})^{1/2}/(l_x^{-1}+2/l_x^f)^{1/2}]}\right\}^{-1}. \tag{58}$$

If $l_0 > l_x$, the arctangent in (55) and (58) is replaced in natural fashion by the hyperbolic arctangent.

We note that the results (55) and (58) for the localization length in one chain with magnetic impurities contain the mean free path l_0^* for forward scattering by magnetic impurities. Just as before, at $l_x^f \ll l$ the problem becomes equivalent to the one-dimensional one.

7. CONCLUSION

We have considered multichannel localization of an electron in a conductor. It is assumed that the transverse dimensions of the conductor are smaller than the mean free path. When the number N of channels is large, localization at the largest distances (lowest frequencies) is described by the same equations as in the purely one-dimensional case. The expressions (38), (38') and (38") obtained for the localization

length do not depend at $N \gg 1$ on the shape of the conductor cross section.

The central point is the statement (34), that the localization energy (reciprocal localization scaling length) averaged over the channels is a constant quantity independent of the model and of the number N of channels. We advance the hypothesis that this statement holds also for three-dimensional samples. Then Anderson's transition from localized to delocalized states would have a somewhat different character than previously assumed in the literature. In fact, if we go with respect to a certain parameter from localized to delocalized states, condition (24) remains in force at all times but an increase takes place in the splitting of the localization energies corresponding to different channels. At a certain value of the parameter of one of the channels, the localization length becomes larger than the sample size. At this point, which would correspond to the Anderson transition, the localization length in the remaining channels remains finite. With further increase of the parameter, the number of channels in which the electron is delocalized will increase. These channels, however, do not contribute to (24). Therefore channels with localized electrons must exist on both sides of the transition point. The situation is analogous to the percolation transition, on both sides of which there are clusters of finite dimensions. These considerations help understand the experimentally observed [21–23] absence of a minimal metallic conductivity.

Our hypothesis contradicts thus the statement (see [24] that localized and delocalized states cannot coexist at a given energy, and predicts the presence of localized wave functions in the metallic phase. A possible experimental test could be a search for localized paramagnetic centers near a metal–insulator transition on the metallic side. These singly occupied centers usually arise[24,25] near the Fermi level on account of the weak Hubbard repulsion of electrons in one localized state.

APPENDIX I

For the packing 4(b) at $\chi = \pi n_0$ and odd N the localization length is given by Eq. (39), which contains the mean value $\langle |u_{nn'}|^4 \rangle$. The Fokker-Planck equation for the distribution function $W(L, |u_{nn'}|^2)$ is constructed under conditions (26) with the aid of relations (28) and (39). In the principal order in $1/N \ll 1$ there is no correlation between $|u_{nn'}|^2$ and the other matrix elements, and the equation for $W(L, |u_{nn'}|^2)$ becomes closed:

$$\frac{N}{l^{-1}+1/l^f}\frac{\partial}{\partial L} W(L, |u_{nn'}|^2) = \left\{ \frac{\partial^2}{\partial (|u_{nn'}|^2)^2} |u_{nn'}|^2 + \frac{\partial}{\partial |u_{nn'}|^2} (N|u_{nn'}|^2 - 1) \right\} W. \quad \text{(AI.1)}$$

Inserting $|u_{nn'}|^2$ in both sides of this equation and integrating we obtain

$$\left(l^{-1} + \frac{1}{l^f} \right)^{-1} \frac{d}{dL} \langle |u_{nn'}|^2 \rangle = N^{-1} - \langle |u_{nn'}|^2 \rangle. \quad \text{(AI.2)}$$

We obtain similarly the relation

$$2^{-1} \left(l^{-1} + \frac{1}{l^f} \right)^{-1} \frac{d}{dL} \langle |u_{nn}|^4 \rangle = 2N^{-1} \langle |u_{nn}|^2 \rangle - \langle |u_{nn'}|^4 \rangle. \quad \text{(AI.3)}$$

The stationary values of the averages takes in this approximation the form

$$\langle |u_{nn'}|^2 \rangle = N^{-1} + O(N^{-2}), \quad \langle |u_{nn'}|^4 \rangle = 2N^{-2} + O(N^{-3}). \quad \text{(AI.4)}$$

When $l^f \ll l$ we can obtain the equation

$$Nl^f \frac{\partial}{\partial L} W(L, |u_{nn'}|^2) = \left\{ \frac{\partial^2}{\partial (|u_{nn'}|^2)^2} (|u_{nn'}|^2 - |u_{nn'}|^4) + \frac{\partial}{\partial |u_{nn'}|^2} (N|u_{nn'}|^2 - 1) \right\} W, \quad \text{(AI.5)}$$

which is valid even at $N \sim 1$. The stationary mean values

$$\langle |u_{nn'}|^2 \rangle = N^{-1}, \quad \langle |u_{nn'}|^4 \rangle = 2N^{-1}(N+1)^{-1} \quad \text{(AI.6)}$$

are obtained just as before. Expressions (AI.4) and (AI.6) for the mean values $\langle |u_{nn'}|^4 \rangle$ lead to the results (41) and (41').

APPENDIX II

In the simplest model (4b) with odd N and $\chi \neq \pi n_0$ the Heisenberg magnetic impurities ($l_0 \neq l_x = l_y = l_z$) lead to the following elaboration of relation (37) in the absence of forward scattering:

$$\overline{R_{\rho\rho'} R^*_{\bar{\rho}\bar{\rho}'}} = (\gamma_0^2 + 3\gamma_x^2)(2N^2)^{-1} \times \left\{ \delta_{\rho\bar{\rho}} \delta_{\rho'\bar{\rho}'} + \alpha/(2-\alpha) \sum_i U^i_{\rho\bar{\rho}} (V^i_{\rho'\bar{\rho}'})^* \right\}, \quad \text{(AII.1)}$$

where the index ρ corresponds to the set (n,s), α is defined by (46), and we have introduced the matrices ($i = x,y,z$)

$$U^i_{\rho\rho'} = \sum_{n_1 s_1 s_2} u_{ns, n_1 s_1} \sigma^i_{s_1 s_2} u^+_{n_1 s_2, n' s'},$$

$$V^i_{\rho\rho'} = \sum_{n_1 s_1 s_2} v_{ns, n_1 s_1} \sigma^i_{s_1 s_2} v^+_{n_1 s_2, n' s'},$$

which are unitary, Hermitian, and have zero trace, $\mathrm{Sp}\, \hat{U} = \mathrm{Sp}\, \hat{V} = 0$. We note also the commutation properties:

$$\hat{U}^i \hat{U}^j + \hat{U}^j \hat{U}^i = 2\delta_{ij} \hat{1}. \quad \text{(AII.2)}$$

The localization length follows from expression (AII.1):

$$l_0^* = 2Nl \Big/ \left\{ 1 + \alpha(2-\alpha)^{-1} \left\langle \sum_i U_{00}{}^i V_{00}{}^i \right\rangle \right\}. \quad \text{(AII.3)}$$

To determine the correlator $\langle \Sigma_i U^i_{00} V^i_{00} \rangle$ it is necessary to construct an equation for the distribution function $W(\hat{U}, \hat{V})$. With an aim at inserting in this equation the quantity $\Sigma_i U^i_{00} V^i_{00}$ and integrating next over all the random variables, we present here only those terms of the equation which make a nonzero contribution to the result:

$$\sum_i \left\{ \left(\frac{\partial}{\partial U_{00}{}^i} U_{00}{}^i + \frac{\partial}{\partial V_{00}{}^i} V_{00}{}^i \right) \left[2N - \alpha/(2-\alpha) \sum_j \left(U_{00}{}^j V_{00}{}^j + \sum_{\rho,\rho'=1}^{2N-1} U^j_{\rho\rho'} V^j_{\rho'\rho} \right) \right] + 2\alpha(2-\alpha)^{-1} \frac{\partial^2}{\partial U_{00}{}^i \partial V_{00}{}^i} \times \sum_{\rho,\rho'=1}^{2N-1} U_{0\rho}{}^i V^i_{0\rho'} \sum_j U^j_{\rho' 0} V_{\rho 0}{}^j + \ldots \right\} W(\hat{U}\hat{V}) = 0. \quad \text{(AII.4)}$$

We thus obtain the relation

$$\left\langle \sum_i U_{00}{}^i V_{00}{}^i \right\rangle = \alpha [2N(2-\alpha)]^{-1} \left\langle \sum_{ij} \left\{ U_{00}{}^i V_{00}{}^i \left(U_{00}{}^j V_{00}{}^j + \sum_{\rho,\rho'=1}^{2N-1} U_{\rho\rho'}{}^j V_{\rho'\rho}{}^j \right) + \sum_{\rho,\rho'=1}^{2N-1} U_{0\rho}{}^i V_{0\rho'}{}^i U_{\rho'0}{}^j V_{\rho 0}{}^j \right\} \right\rangle . \quad \text{(AII.5)}$$

We assume now that $\alpha \ll 1$ and obtain the first correction to the localization length (38″). For this purpose it suffices to calculate the correlator in the right-hand side of (AII.5) at $\alpha = 0$. The equation for the function $W(\hat{U}, \hat{V})$ at $\alpha = 0$, while much simpler than at $\alpha \neq 0$, is still quite cumbersome:

$$\sum_{i\rho\rho'} \left\{ \left(\frac{\partial}{\partial U_{\rho\rho'}^i} U_{\rho\rho'}^i + \frac{\partial}{\partial V_{\rho\rho'}^i} V_{\rho\rho'}^i \right) \left[2N + \frac{\delta_{\rho\rho'} - 1}{2} \right] + \sum_{j\bar{\rho}\bar{\rho}'} \left[\left(1 - \frac{1}{2} \delta_{ij} \delta_{\rho\bar{\rho}} \delta_{\rho'\bar{\rho}'} \right) \times \left(\frac{\partial^2}{\partial U_{\rho\rho'}^i \partial U_{\bar{\rho}\bar{\rho}'}^j} f(\hat{U}) + \frac{\partial^2}{\partial V_{\rho\rho'}^i \partial V_{\bar{\rho}\bar{\rho}'}^j} f(\hat{V}) \right) + \frac{\partial^2}{\partial U_{\rho\rho'}^i \partial V_{\bar{\rho}\bar{\rho}'}^j} U_{\rho\rho'}^i V_{\bar{\rho}\bar{\rho}'}^j \frac{1}{2} (\delta_{\rho\bar{\rho}} + \delta_{\rho'\bar{\rho}'} - \delta_{\rho\bar{\rho}'} - \delta_{\rho'\bar{\rho}}) \right] \right\}$$

$$W(\hat{U}, \hat{V}) = 0, \quad \text{(AII.6)}$$

where we use the notation

$$f(\hat{U}) = U_{\rho\bar{\rho}}^i U_{\bar{\rho}\rho'}^j \frac{1}{2} (\delta_{\rho\bar{\rho}} + \delta_{\rho'\bar{\rho}'} - \delta_{\rho\bar{\rho}'} - \delta_{\rho'\bar{\rho}}) - U_{\bar{\rho}\rho'}^i U_{\rho\bar{\rho}'}^j - U_{\rho\bar{\rho}'}^i U_{\bar{\rho}\rho'}^j + \sum_{\rho_1} (\delta_{\rho'\bar{\rho}} U_{\rho\rho_1}^i U_{\rho_1\bar{\rho}'}^j + \delta_{\rho\bar{\rho}'} U_{\rho_1\bar{\rho}}^i U_{\bar{\rho}\rho_1}^j).$$

Inserting the terms from the curly brackets of (AII.5) in (AII.6) and allowing for the commutation relations (AII.2) we can verify that only the terms with $i = j$ and $\rho = \rho'$ remain in the sums over i and j and over ρ and ρ'. We introduce the correlators

$$\langle K_1 \rangle = \langle U_{00}{}^i U_{\rho\rho}{}^i V_{00}{}^i V_{\rho\rho}{}^i \rangle, \quad \langle K_2 \rangle = \langle U_{00}{}^i U_{\rho\rho}{}^i | V_{0\rho}{}^i |^2 \rangle,$$
$$\langle K_3 \rangle = \langle | U_{0\rho}{}^i |^2 | V_{0\rho}{}^i |^2 \rangle$$

and rewrite the mean value (AII.5) in the form

$$\left\langle \sum_i U_{00}{}^i V_{00}{}^i \right\rangle = 3\alpha (4N)^{-1} \left\{ \langle (U_{00} V_{00})^2 \rangle + \sum_{\rho=1}^{2N-1} \langle K_1 + K_3 \rangle \right\}. \quad \text{(AII.7)}$$

Inserting in succession K_1, K_2, and K_3 in (AII.6) and integrating over all the variables we obtain the system of equations

$$2N\langle K_1 \rangle = -\langle K_2 \rangle, \quad 4N\langle K_2 \rangle = \langle U_{00} U_{\rho\rho} \rangle - \langle K_1 \rangle - \langle K_3 \rangle,$$
$$2N\langle K_3 \rangle = \langle | U_{0\rho} |^2 \rangle - \langle K_2 \rangle,$$

from which it follows that

$$\langle K_1 + K_3 \rangle = (2N\langle | U_{0\rho} |^2 \rangle - \langle U_{00} U_{\rho\rho} \rangle) / [(2N)^2 - 1].$$

Summation over ρ in (AII.7) leads now to the relation

$$\left\langle \sum_i U_{00}{}^i V_{00}{}^i \right\rangle = 3\alpha (4N)^{-1} \langle U_0{}^2 V_{00}{}^2 + [2N(1 - U_{00}{}^2) + U_{00}{}^2] / [(2N)^2 - 1] \rangle. \quad \text{(AII.8)}$$

From (AII.6), just as in Appendix I, it is easy to find the simplest mean values:

$$\langle U_{00}{}^2 \rangle = (2N+1)^{-1}, \quad \langle U_{00}{}^2 V_{00}{}^2 \rangle = \langle U_{00}{}^2 \rangle^2.$$

Ultimately the expression for the sought correlation takes the form

$$\left\langle \sum_i U_{00}{}^i V_{00}{}^i \right\rangle = \frac{3\alpha}{2} \frac{1}{(2N)^2 - 1} + O(\alpha^2). \quad \text{(AII.9)}$$

The first term of the expansion of the localization length (AII.3) in powers of α is thus given at $N \sim 1$ by Eq. (45).

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