Modulation echo in dispersive media

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It is shown that twofold modulation of a stationary noise wave in a linear dispersive medium can lead to the onset of a modulation echo, i.e., to reconstruction of the coherent response and to change of the wave envelope at the difference frequency of the modulation. A general interpretation of the considered effect is presented. An expression is obtained for the dispersion of a modulated one-dimensional noise wave with account taken of the irreversible phase and amplitude relaxations. Multiple modulation of stationary noise can lead to the onset of a modulation echo at combination frequencies of the modulation.

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INTRODUCTION

A characteristic feature of the phenomena known jointly as the "echo effect" can be taken to be the restoration of the coherent response of the medium to a sequence of signals under conditions when the response to each of them separately vanishes as a result of one collective process or another, even in the absence of losses. Echo manifestations were investigated, in particular, in optics[1,2] and in solid-state physics.[3] In these cases the signal interaction that leads to the onset of the echo is the result of the nonlinearity of the medium. Similar effects, however, are possible also in linear media. The echo manifests itself then relative to such signal characteristics (say, its intensity) that can be subject to nonlinear transformation. In essence, one such type is the model proposed by Vedenov and Dykhne[4] to illustrate the following echo effects: after the light passes through a periodic lattice (raster), characteristic bands of light and shadow are produced, which become smeared out with increasing distance from the raster; in the presence of a second raster, however, bands with a period $L = (L_2^{-1} - L_1^{-1})^{-1}$, where $L_{1,2}$ are the periods of the rasters, are again produced at a definite distance behind the second raster. In this case the cause of the vanishing of the coherent response is the fanning out of the waves, and nonlinear transformation of the light intensity (modulation by the rasters) results in a "three-dimensional" echo. Plasma echo,[5] a nonlinear phenomenon of kinematic origin, can be described in similar fashion.

Of interest in its own right, in our opinion is the "modulation echo" effect produced when modulated waves propagate in a linear medium with dispersion. When such a wave propagates, the modulation is "washed out" by dispersion, while a second modulation of the wave at a different frequency produces an effect at the modulation difference frequency (Fig. 1). This "temporal echo" can combine with the spatial one to form rather complicated modulation structures. The present paper is devoted to this question.

§1. MODULATION ECHO IN TWOFOLD MODULATION

We consider the problem in the following two-dimensional formulation. A wave $U(t, y, x)$ propagating in a linear isotropic uniform dispersive medium is subjected in the plane $x = a_1$ to harmonic modulation both in space and in time:

$$U(t,\ y,\ a_1+0)=U(t,\ y,\ a_1-0)\,m_1\cos(\Omega_1 t-K_1 y+\varphi_1), \tag{1.1}$$

where $U(t, y, a_1 - 0)$ is a rapidly oscillating function which we shall hereafter call the carrier wave. If the spectrum of the carrier wave is specified in the plane $x_0 < a_1$

$$C(\omega,k_y,x_0)=\iint_{-\infty}^{\infty} U(t,y,x_0)\exp(-i\omega t+ik_y y)\,dy\,dt, \tag{1.2}$$

the modulated wave at $x_1 > a_1$ can be represented in the form (see the Appendix)

$$U(t,y,x_1)=\frac{m_1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,x_0)\exp\{i\omega t-ik_y y -i(x_1-x_0)[k^2(\omega)-k_y^2]^{1/2}\} \times\cos[\Omega_1 t-K_1 y-(x_1-a_1)H_1(\omega,k_y)+\varphi_1]\,dk_y\,d\omega; \tag{1.3}$$

here

$$H_i(\omega,k_y)=\left\{1-\left[\frac{k_y}{k(\omega)}\right]^2\right\}^{-1/2}\left[\frac{\Omega_i}{v_{gr}(\omega)}-K_i\frac{k_y}{k(\omega)}\right], \tag{1.4}$$

where $k(\omega)$ is the dispersion law of the medium, $v_{gr}(\omega) = [k'(\omega)]^{-1}$ is the group velocity, and the prime denotes dif-

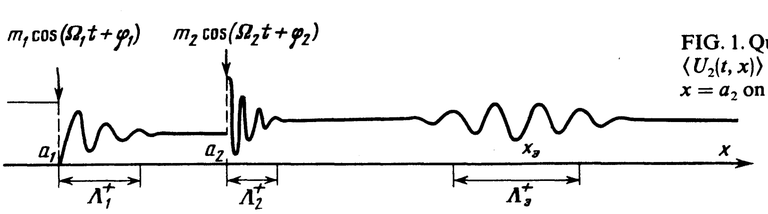

FIG. 1. Qualitative dependence of the dispersion of one-dimensional noise $\langle U_2(t, x)\rangle$ harmonically modulated in amplitude at the points $x = a_1$ and $x = a_2$ on the spatial coordinate x.

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ferentiation with respect to ω. Relation (1.3) means that the modulated wave can be approximately represented in the form of a superposition of amplitude-modulated plane waves whose envelopes propagate at the group velocity.

Assume that the carrier signal is a stationary (in t and y) noise, so that the dispersion of such a noise wave behind the modulator satisfies the equation (see II.19)

$$\langle U^2(t,y,x_1)\rangle=\frac{m_1^2}{(2\pi)^4}\iint\limits_{-\infty}^{\infty} G(\omega,k_y,x_0)\cos^2[\Omega_1 t -K_1 y-(x_1-a_1)H_1(\omega,k_y)+\varphi_1]\,dk_y\,d\omega, \tag{1.5}$$

where $G(\omega, k_y, x_0)$ is the spectral density of the power of the carrier wave, and the angle brackets denote averaging over the ensemble. Expression (1.5) can be easily transformed into

$$\langle U^2(t,y,x_1)\rangle = {}^1/_2 m_1^2\{W_0+{}^1/_2[\chi_1(x_1-a_1)\exp(2i(\Omega_1 t-K_1 y+\varphi_1))+\text{c.c.}]\}. \tag{1.6}$$

Here

$$\chi_i(\zeta>0)=\frac{1}{(2\pi)^4}\iint\limits_{-\infty}^{\infty} G(\omega,k_y,x_0)\exp(-2iH_i(\omega,k_y)\zeta)\,dk_y\,d\omega,$$

$$W_0=\langle U^2(t,y,x_0)\rangle=\text{const.}$$

At large (x_1-a_1) the integrand in the expression for $\chi_1(x_1-a_1)$ contains a function that oscillates rapidly in k_y and ω, so that the expression in the square brackets in (1.6) becomes small compared with W_0, and the modulation vanishes. The characteristic distance from the modulator, Λ_1^+, at which this takes place is equal in order of magnitude to the interval in which the plane wave envelopes become dephased because of the difference between their group velocities, and is consequently determined from the condition

$$(H_{1\,max}-H_{1\,min})\Lambda_1^+\approx\pi.$$

This interval depends on the width of the frequency and angular spectra of the wave and is equal in the general case to

$$\Lambda_1^+\approx\pi\{\Omega_1[k'_{max}/\cos\theta_{max}-k'_{min}]-K_1\operatorname{tg}\theta_{max}\}^{-1},$$
$$\theta=\arcsin(k_y/k(\omega)), \tag{1.7}$$

where we have introduced an angle variable θ that characterizes the direction of the wave vector relative to the x axis. For a one-dimensional noise wave ($\theta\equiv 0$) the interval over which the modulation vanishes is equal to $\pi/(\Omega_1\Delta k')$, where $\Delta k'=k'_{max}-k'_{min}$.

We assume now that in a certain plane the noise field is again modulated

$$U(t,y,a_2+0)=U(t,y,a_2-0)\,m_2\cos(\Omega_2 t-K_2 y+\varphi_2). \tag{1.8}$$

In the same approximation as above, the envelope of the doubly modulated noise wave is described by the expression [see (II.19)]

$$\langle U^2(t,y,x_2)\rangle=\frac{(m_1m_2)^2}{(2\pi)^4}\iint\limits_{-\infty}^{\infty} G(\omega,k_y,x_0) \times\cos^2[\Omega_1 t-K_1 y-(x_2-a_1)H_1(\omega,k_y)+\varphi_1]\cos^2[\Omega_2 t-K_2 y -(x_2-a_2)H_2(\omega,k_y)+\varphi_2]\,dk_y\,d\omega=\left(\frac{m_1m_2}{2}\right)^2\left(W_0-W_1-W_2 +\frac{1}{2}W_{1+2}+\frac{1}{2}W_{2-1}\right), \tag{1.9}$$

where

$$W_{i\pm j}=\frac{1}{(2\pi)^4}\iint\limits_{-\infty}^{\infty} G(\omega,k_y,x_0)\cos 2\{(\Omega_i\pm\Omega_j)t-(K_i\pm K_j)y -[H_i(\omega,k_y)(x_2-a_i)\pm H_j(\omega,k_y)(x_2-a_j)]+(\varphi_i\pm\varphi_j)\}\,dk_y\,d\omega; \tag{1.10}$$

here $i,j=0,1,2$; $a_0=\Omega_0=K_0=\varphi_0=0$, $m_0=1$. With increasing distance (x_2-a_2), the second modulation also vanishes, since at large (x_2-a_2) the integrands in all $W_{i\pm j}$ (except W_0) again becomes a rapidly oscillating function, and the amplitude-modulated noise is transformed into a stationary one.

Under the condition $\Omega_2/\Omega_1=K_2/K_1=\mu>1$, however, there exists an "echo plane" with coordinate $x_e=(\mu a_2-a_1)/(\mu-1)$, in which the explicit dependence on x vanishes in the integrand of the expression for W_{2-1}. In the vicinity of this point the function W_{2-1} is comparable in magnitude with the dc component of the noise dispersion, and this leads to the appearance of amplitude modulation at a difference frequency $\Omega_2-\Omega_1$ and a wave number K_2-K_1. In the $x=x_e$ plane the deviation of the noise from the stationary one is a maximum and is equal to

$$\langle U^2(t,y,x_e)\rangle=\left(\frac{m_1m_2}{2}\right)^2\left\{1+\frac{1}{2}\cos 2[(\Omega_2-\Omega_1)t -(K_2-K_1)y+(\varphi_2-\varphi_1)]\right\}W_0. \tag{1.11}$$

Near this plane, the shape of the echo is described by the expression

$$\langle U^2(t,y,x_2)\rangle=\left(\frac{m_1m_2}{2}\right)^2\left[W_0+\frac{1}{2}W_{2-1}\right]=\frac{(m_1m_2)^2}{4}\left\{W_0 +\frac{1}{4}[\Gamma((\mu-1)(x_2-x_e))\exp(2i(\mu-1)(\Omega_1 t-K_1 y) +2i(\varphi_2-\varphi_1))+\text{c.c.}]\right\},$$

where

$$\Gamma((\mu-1)(x_2-x_e))=\chi^*((\mu-1)(x_e-x_2))\quad\text{at}\quad x_2<x_e,$$
$$\Gamma((\mu-1)(x_2-x_e))=\chi((\mu-1)(x_2-x_e))\quad\text{at}\quad x_2\geqslant x_e. \tag{1.12}$$

From a comparison of (1.12) and (1.6) it can be seen that when the plane x_e is approached from the left the stationary noise becomes modulated, and its dispersion passes through the same stages as when it moves away from the modulator, but in reversed sequence. To the right of the plane x_e the modulated beam again becomes stationary, and the noise dispersion behaves in the same way as with increasing distance from the modulator.

§2. GENERAL INTERPRETATION OF THE MODULATION ECHO

It is possible to present for echo effects of various types a simple but quite general interpretation, based on the following circumstance. If two functions with conjugate temporal spectra

$$U(t,\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}C(\omega,\mathbf{r})\exp(i\omega t)\,d\omega,$$

$$U_{\text{c.c.}}(t,\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}C^{*}(\omega,\mathbf{r})\exp(i\omega t)\,d\omega, \tag{2.1}$$

are specified, then obviously $U_{\text{c.c.}}(t,\mathbf{r}) = U(-t,\mathbf{r})$. It is thus possible to reconstruct with time reversal the response $U(t,\mathbf{r})$ of the medium to a certain external action, if a signal of the form $U_{\text{c.c.}}(t,\mathbf{r})$ is again applied to the medium.

In the experiments described in Refs. 6 and 7 they observed a photon echo pulse that reproduced, with time reversal, the shapes of two successive laser pulses incident on a ruby crystal. In the general case, however, the coherent response may also not reproduce the shape of the external signal, as was indeed observed in most studies of photon echo as well as of other types of echo. In such cases the conditions that ensure an exact transformation of the spectrum into its complex conjugate were not satisfied.

At the same time in optics there is a well known spatial transformation of a spectrum in the form

$$C(\omega,\mathbf{r})\to C^{*}(\omega,\mathbf{r}). \tag{2.2}$$

It occurs in processes named wave-front reversal.[8] In this case the spectrum of a monochromatic wave

$$\mathbf{R}(\mathbf{r})\delta(\omega-\omega_0)+\mathbf{R}^{*}(\mathbf{r})\delta(\omega+\omega_0) \tag{2.3}$$

is transformed into its complex conjugate, which is again equivalent to time reversal.

Thus, echo and wave-front inversion effects, at least in the case of restoration of the signal structure, constitute one and the same transformation that effects a complex conjugation of the spectrum. Recent investigations dealt also with mixed processes,[9] when application of two laser pulses with complicated spatial structure produced in a nonlinear medium a photon echo that duplicated the spatial structure of the laser pulse.

The modulation echo considered here has that important feature that the complex conjugate spectrum was not of the signal itself but of its dispersion:

$$S(\Omega,y,x)=\int_{-\infty}^{\infty}\langle U^{2}(t,y,x)\rangle\exp(-i\Omega t)\,dt. \tag{2.4}$$

As follows from (1.6), after the first modulation the spectrum of the alternating component of the noise dispersion takes the form

$$S_1^{\sim}(\Omega,y,x_1-a_1)$$
$$=\left(\frac{m_1}{2}\right)^2\{[\chi(x_1-a_1)\exp(-2i(K_1y-\varphi_1))]\delta(\Omega-2\Omega_1)$$
$$+[\chi(x_1-a_1)\exp(-2i(K_1y-\varphi_1))]^{*}\delta(\Omega+2\Omega_1)\}. \tag{2.5}$$

After the second modulation in accord with the law $\cos 2(\Omega_1 t - K_1 y)$ we have at $x_2 < x_e$ and $x_2 - a_2 \gg \Lambda_2^{+}$ [see (1.12)]

$$S_2^{\sim}(\Omega,y,x_e-a_2)$$
$$=\left(\frac{m_1m_2}{4}\right)^2\{[\chi(x_e-x_2)\exp(-2i(K_1y-\varphi_1))]^{*}$$
$$\times\delta(\Omega-2\Omega_1)+[\chi(x_e-x_2)\exp(-2i(K_1y-\varphi_1))]\delta(\Omega+2\Omega_1)\}. \tag{2.6}$$

The second modulation transforms the spectrum of the noise dispersion

$$S_1^{\sim}(\Omega,y,x_1-a_1)\to\left(\frac{m_2}{2}\right)^2[S_1^{\sim}(\Omega,-y,x_e-x_2)]^{*}$$
$$=S_2^{\sim}(\Omega,y,x_e-x_2). \tag{2.7}$$

The interchange of the spatial variables $y\to -y$, $x_1 - a_1 \to x_e - x_2$ is of no significance and is explained by the fact that modulation causes time reversal only of the envelopes of the space-time harmonics, whereas the direction of propagation of the plane waves themselves remains unchanged. This means that the plane where the echo is produced does not coincide with the plane of the first modulation, as was the case when the sign of the time was reversed. However, as follows from (2.7), the time-reversed behavior of the noise dispersion with increasing distance from the modulator is identical with the behavior of the dispersion as the plane x_0 is approached.

§3. MODULATION ECHO IN MULTIPLE MODULATION

The analysis carried out in §1 for the case of two successive modulations of a noise wave can be easily generalized to the case of multiple modulation of noise [see (A.19)]. Both the wave number and the frequency of the modulation echo are then combinations of wave numbers and modulation frequencies. Thus, in triple modulation of a noise wave at points a_i with frequencies Ω_i and wave numbers K_i ($i = 1, 2, 3$) the behavior of the envelope is described by the expression

$$\langle U^2(t,y,x_3)\rangle={}^1/_8(m_1m_2m_3)^2\{W_0-(W_1+W_2+W_3)$$
$$+{}^1/_2(W_{2+1}+W_{2-1}+W_{3+1}+W_{3-1}+W_{3-2}+W_{3+2})$$
$$+{}^1/_4(W_{3+2+1}+W_{3-2-1}+W_{3+2-1}+W_{3-2+1})\}, \tag{3.1}$$

where

$$W_{i\pm j\pm l}=\frac{1}{(2\pi)^4}\iint_{-\infty}^{\infty}G(\omega,k_y,x_0)\cos 2\{(\Omega_i\pm\Omega_j\pm\Omega_l)t$$
$$-(K_i\pm K_j\pm K_l)y+[H_i(\omega,k_y)(x_3-a_i)\pm H_j(\omega,k_y)(x_3-a_j)$$
$$\pm H_l(\omega,k_y)(x_3-a_l)]+(\varphi_i\pm\varphi_j\pm\varphi_l)\}\,dk_y\,d\omega;$$

here $i, j, l = 1, 2, 3$, with $\Omega_0 = K_0 = \varphi_0 = a_0 = 0$, $m_0 = 1$.

We present below different types of echo responses that can arise in triple action on noise, depending on the temporal and spatial modulation periods, and on the positions of the modulators (we designate here $\mu = \Omega_2/\Omega_1$, $\eta = \Omega_3/\Omega_1$, $\beta = \Omega_3/\Omega_2$). Each numbered case designates first the echo coordinate, next the expression for the noise dispersion in the plane x_e, and on a separate line the conditions for the onset of the echo:

1) Coordinate $(\mu a_2-a_1)/(\mu-1)$:

$$\tfrac{1}{4}(m_1m_2)^2\{1+\tfrac{1}{2}\cos 2[(\Omega_2-\Omega_1)t-(K_2-K_1)y+(\varphi_2-\varphi_1)]\}W_0,$$

$$\Omega_1/K_1=\Omega_2/K_2,\quad \mu>1,\quad a_3>x_e.$$

2) Coordinate $(\mu a_2-a_1)/(\mu-1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{2}\cos 2[(\Omega_2-\Omega_1)t-(K_2-K_1)y+(\varphi_2-\varphi_1)]\}W_0,$$

$$\Omega_1/K_1=\Omega_2/K_2,\quad \mu>1,\quad a_3<x_e.$$

3) Coordinate $(\eta a_3-a_1)/(\eta-1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{2}\cos 2[(\Omega_3-\Omega_1)t-(K_3-K_1)y+(\varphi_3-\varphi_1)]\}W_0,$$

$$\Omega_1/K_1=\Omega_3/K_3,\quad \eta>1.$$

4) Coordinate $(\beta a_3-a_2)/(\beta-1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{2}\cos 2[(\Omega_3-\Omega_2)t-(K_3-K_2)y+(\varphi_3-\varphi_2)]\}W_0,$$

$$\Omega_2/K_2=\Omega_3/K_3,\quad \beta>1.$$

5) Coordinate $(\eta a_3-\mu a_2-a_1)/(\eta-\mu-1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{4}\cos 2[(\Omega_3-\Omega_2-\Omega_1)t-(K_3-K_2-K_1)y+(\varphi_3-\varphi_2-\varphi_1)]\}W_0,$$

$$\Omega_1/K_1=\Omega_2/K_2=\Omega_3/K_3,\quad \eta-\mu-1>0.$$

6) Coordinate $(\eta a_3+\mu a_2-a_1)/(\eta+\mu-1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{4}\cos 2[(\Omega_3-\Omega_2-\Omega_1)t-(K_3-K_2-K_1)y+(\varphi_3-\varphi_2-\varphi_1)]\}W_0,$$

$$\Omega_1/K_1=\Omega_2/K_2=\Omega_3/K_3,\quad \eta+\mu-1>0,\quad (a_3-a_1)/(a_3-a_2)>\mu.$$

7) Coordinate $(\eta a_3-\mu a_2+a_1)/(\eta-\mu+1)$:

$$\tfrac{1}{8}(m_1m_2m_3)^2\{1+\tfrac{1}{4}\cos 2[(\Omega_3-\Omega_2+\Omega_1)t-(K_3-K_2+K_1)y+(\varphi_3-\varphi_2+\varphi_1)]\}W_0,$$

a) $\Omega_1/K_1=\Omega_2/K_2=\Omega_3/K_3,\ \eta-\mu+1>0,\ (a_3-a_1)/(a_3-a_2)<\mu$;

b) $\Omega_1/K_1=\Omega_2/K_2=\Omega_3/K_3,\ \eta-\mu+1<0,\ (a_3-a_1)/(a_3-a_2)>\mu$.

§4. INFLUENCE OF IRREVERSIBLE EFFECTS ON THE BEHAVIOR OF A MODULATED SOUND WAVE

Among the physical questions connected with the modulation-echo effect, those of importance are the influence of the dissipation on the behavior of the wave and on the shape of the echo, and the possibility of the appearance of an echo on going outside the framework of the group approximation (of space-time geometric optics), when dispersion spreading comes into play for each modulated spectral components. We consider these questions as applied to a one-dimensional wave of type $U(t,x)$, which is modulated twice harmonically in amplitude at the points a_1 and a_2 with frequencies Ω_1 and Ω_2. In a homogeneous isotropic medium with losses, the frequency spectrum of the wave behind the second modulator is of the form [see (II.4)]

$$\begin{aligned}C(\omega,x_2)&=\tfrac{1}{4}m_1m_2\exp[-i(x_1-a_1)k(\omega)]\\&\times\{[C(\omega+\Omega_2+\Omega_1,x_0)\exp(-i(a_1-x_0)k(\omega+\Omega_2+\Omega_1)-i\varphi_1)\\&+C(\omega+\Omega_2-\Omega_1,x_0)\exp(-i(a_1-x_0)k(\omega+\Omega_2-\Omega_1)+i\varphi_1)]\\&\times\exp(-i(a_2-a_1)k(\omega+\Omega_2)-i\varphi_2)\\&+[C(\omega-\Omega_2+\Omega_1,x_0)\exp(-i(a_1-x_0)k(\omega-\Omega_2+\Omega_1)-i\varphi_1)\\&+C(\omega-\Omega_2-\Omega_1)\exp(-i(a_1-x_0)k(\omega-\Omega_2-\Omega_1)\\&+i\varphi_1)]\exp(-i(a_2-a_1)k(\omega-\Omega_2)+i\varphi_2)\}.\end{aligned}\tag{4.1}$$

Here $k(\omega)=\varkappa(\omega)+i\gamma(\omega)$ is the dispersion law of the medium, including also the losses. The field of the wave behind the second modulator is described by the expression

$$U(t,x_2)=U_{1+2}+U_{-1+2}+U_{1-2}+U_{-1-2},\tag{4.2}$$

where

$$\begin{aligned}U_{\pm1\pm2}=\frac{m_1m_2}{4\cdot 2\pi}\int_{-\infty}^{\infty}C(\omega\pm\Omega_1\pm\Omega_2)\exp[\mp i\varphi_1\mp i\varphi_2+i\omega t\\-i(a_2-x_0)k(\omega\pm\Omega_1\pm\Omega_2)-i(a_2-a_1)k(\omega\pm\Omega_2)\\-i(x_2-a_2)k(\omega)]d\omega.\end{aligned}$$

We make in U_{1+2} the change of variable $\omega^{\mathrm{ir}}=\omega+\Omega_1+\Omega_2$, and expand the functions $k(\omega+\Omega_2)$ and $k(\omega)$ in a Taylor series in the vicinity of the point $\omega^{\mathrm{ir}}(\Omega_{1,2}/\omega\ll1)$. We assume the losses to be small. Then, to assess the effects of the losses and of the violation of the group approximation on the behavior of the modulated noise, it suffices to take into account four terms of the expansion for $\varkappa(\omega)$ and two terms for $\gamma(\omega)$. Transforming similarly the other terms of (4.2), we have

$$\begin{aligned}U(t,x_2)=\frac{m_1m_2}{2\pi}\int_{-\infty}^{\infty}C(\omega,x_0)\exp\Big[i\omega t-i(x_2-x_0)k(\omega)\\-\frac{i}{2}\varkappa''(\omega)(\Omega_1^2(x_2-a_1)+\Omega_2^2(x_2-a_2))\Big][\cos(A_1+iB_1)\\\times\cos(A_2+iB_2)\cos E+i\sin(A_1+iB_1)\sin(A_2+iB_2)\sin E]d\omega,\end{aligned}\tag{4.3}$$

where

$$A_{1,2}=\Omega_{1,2}t+\varphi_{1,2}-\Omega_{1,2}(x_2-a_{1,2})\Big[\varkappa'(\omega)+\frac{1}{2}\varkappa'''(\omega)\times\Big(\Omega_{2,1}^2+\frac{1}{3}\Omega_{1,2}^2\Big)\Big],$$

$$B_{1,2}=-\Omega_{1,2}\gamma(\omega)(x_2-a_{1,2}),\quad E=\Omega_1\Omega_2\varkappa''(\omega)(x_2-a_2).$$

If stationary noise is used as the carrier wave, the dispersion of the noise far from the second modulator $(x_2-a_2\gg\Lambda_2^+)$ takes the form

$$\begin{aligned}\langle U^2(t,x_2)\rangle&=\frac{1}{8}\Big(\frac{m_1m_2}{2\pi}\Big)^2\int_{-\infty}^{\infty}G(\omega,x_0)\\&\times\exp[2\gamma(\omega)(x_2-x_0)][\cos 2(A_1-A_2)+\mathrm{ch}\,2(B_1-B_2)\\&+\mathrm{ch}\,2(B_1+B_2)]d\omega=\frac{1}{8}\Big(\frac{m_1m_2}{2\pi}\Big)^2\int_{-\infty}^{\infty}G(\omega,x_0)\\&\times\exp[2\gamma(\omega)(x_2-x_0)]\Big\{\mathrm{ch}\,2\gamma'(\omega)[\Omega_1(x_2-a_1)-\Omega_2(x_2-a_2)]\\&+\mathrm{ch}\,2\gamma'(\omega)[\Omega_1(x_2-a_1)+\Omega_2(x_2-a_2)]\\&+\cos 2\Big[(\Omega_2-\Omega_1)t+(\varphi_2-\varphi_1)\\&+\varkappa'(\omega)(\Omega_2(x_2-a_2)-\Omega_1(x_2-a_1))-\frac{1}{2}\varkappa'''(\omega)\Big(\Omega_2(x_2-a_2)\\&\times\Big(\Omega_1^2+\frac{1}{3}\Omega_2^2\Big)-\Omega_1(x_2-a_1)\Big(\Omega_2^2+\frac{1}{3}\Omega_1^2\Big)\Big)\Big]\Big\}d\omega.\end{aligned}\tag{4.4}$$

Relation (4.4) makes it possible to find the spatial scales that characterize the behavior of a modulated noise wave. First, the characteristic width Λ_e^+ of the region in which the echo is localized can be defined in analogy with the definition, in

(1.7), of the interval at which modulation vanishes behind each modulator:

$$\Lambda_e^+ \approx \pi/\Delta\varkappa'(\Omega_2-\Omega_1), \tag{4.5}$$

where

$$\Delta\varkappa' = \varkappa'_{max} - \varkappa'_{min}.$$

Second, if the condition

$$|{}^1/_2\Delta\varkappa'''\{\Omega_2(x_2-a_2)(\Omega_1^2+{}^1/_3\Omega_2^2) - \Omega_1(x_2-a_1)(\Omega_2^2+{}^1/_3\Omega_1^2)\}| \geqslant \pi,$$

is satisfied, where $\Delta\varkappa''' = \varkappa'''_{\max} - \varkappa'''_{\min}$, the dephasing of the envelopes is no longer reversible. In this case, as can be seen from (4.4), the argument of the cosine depends substantially on the frequency even in the vicinity of the point $x_e = (\Omega_2 a_2 - \Omega_1 a_1)/(\Omega_2 - \Omega_1)$. This means that a modulation echo can appear only under the condition

$$|(x_e-a_2)\eta(1+{}^1/_3\eta^2)-(x_e-a_1)(\eta^2+{}^1/_3)| \ll \Lambda,$$

where $\Lambda = 2\pi(\Delta\varkappa'''\Omega_1^3)^{-1}$ is the characteristic interval of the irreversible phase relaxation.

Third, the point in the vicinity of which an echo is produced can be far enough from the modulators ($x_e - a_{1,2} > \Lambda_{1,2}^+$) to permit vanishing of the modulation with frequencies $2\Omega_1$ and $2\Omega_2$, and then the noise dispersion will vary harmonically with the echo frequency $2(\Omega_2 - \Omega_1)$.

Finally, the losses in the medium lead in this approximation only to damping of the wave at the characteristic distance $Z \sim (2\gamma)^{-1}$.

The effect considered can be observed in a great variety of physical situations. We note, for example, the case of propagation of a one-dimensional electromagnetic noise wave in the ionosphere[10] (electron density 8×10^{15} cm^{-3}), with a spectrum in the interval 10–100 MHz. If the noise is modulated at frequencies $\Omega_1/2\pi = 100$ and $\Omega_2/2\pi = 200$ kHz at two points separated by a distance $a_2 - a_1 = 20$ km, it is easy to see that the characteristic scales $x_e - a_1 = 40$ *km*, $\Lambda_1^+ = 2$ km, $\Lambda = 1300$ km, $Z = 200$ km satisfy the requirements indicated above.

Another example pertains to a noise microwave system in a laboratory facility for plasma heating. If the plasma and electron frequencies are assumed to be 10^{11} Hz and 10^{-3} sec^{-1}, respectively (see, e.g., Ref. 10), then for a spectrum concentrated in the interval $\omega_{\min}/2\pi = 1.1\cdot10^{11}$ Hz, $\omega_{\max}/2\pi = 2\cdot10^{11}$ Hz, at modulation frequencies $\Omega_1/2\pi = 10^9$ and $\Omega_2/2\pi = 2\cdot10^9$ Hz, the scales indicate are $\Lambda_1^+ \approx 0.1$ m, $\Lambda \approx 60$ m, $Z \approx 5\cdot10^5$ m.

APPENDIX

We shall show that a two-dimensional wave amplitude modulated in the planes $x = a_i$ $(1 \leqslant i \leqslant N)$ in accordance with the laws

$$m_i\cos(\Omega_i t - K_i y + \varphi_i) \tag{A.1}$$

and propagating in a homogeneous isotropic lossless medium is described approximately in the region $x_N > a_N$ by the expression

$$U(t,y,x_N) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,x_{M-1}) \times\exp\{i\omega t - ik_y y - i(x_N - x_{M-1})[k^2(\omega)-k_y^2]^{1/2}\}$$
$$\times\prod_{i=M}^{N} m_i\cos\left\{[\Omega_i t - K_i y+\varphi_i] - (x_N-a_i)\left[1-\frac{k_y^2}{k^2(\omega)}\right]^{-1/2} \times\left[\Omega_i k'(\omega) - K_i\frac{k_y}{k(\omega)}\right]\right\} dk_y\, d\omega, \tag{A.2}$$

where $k(\omega)$ is the dispersion law of the medium, the primes denote derivatives with respect to ω, and $C(\omega, k_y, x_{M-1})$ is the space-time spectrum of the wave at the point $x_{M-1}(a_{M-1} < x_{M-1} < a_M)$, where M is an integer, i.e.,

$$C(\omega,k_y,x_{M-1}) = \iint_{-\infty}^{\infty} U(t,y,x_{M-1})\exp(-i\omega t + ik_y y)\,dy\,dt. \tag{A.3}$$

At $M = 1$ Eq. (A.2) connects the initial unmodulated carrier wave $U(T, y, x_0)$ (where $x_0 < a_1$) with the wave $U(t, y, x_N)$ behind modulators arranged in tandem.

Before we proceed to prove (A.3) we present a relation that connects the spectrum $C(\omega, k_y, x_{i-1})$ of the wave in an arbitrary plane x_{i-1} modulator ($a_{i-1} < x_{i-1} < a_i$, Fig. 2) and the spectrum in the plane x_i located behind this modulator ($a_i < x_i < a_{i+1}$).[11] In a homogeneous medium this relation is of the form[12]

$$C(\omega,k_y,x_i) = {}^1/_2 m_i\{C(\omega+\Omega_i, k_y+K_i, x_{i-1}) \times\exp[-i(a_i-x_{i-1})(k^2(\omega+\Omega_i)-(k_y+K_i)^2)^{1/2} - i\varphi_i]$$
$$+C(\omega-\Omega_i, k_y-K_i, x_{i-1})\exp[-i(a_i-x_{i-1}) \times(k^2(\omega-\Omega_i)-(k_y-K_i)^2)^{1/2}+i\varphi_i]\} \times\exp\{-i(x_i-a_i)[k^2(\omega)-k_y^2]^{1/2}\}. \tag{A.4}$$

In Eqs. (A.1)–(A.4) and in the main text of the article the modulators are numbered along the propagation direction of the wave, i.e., a_i increases with increasing i. Here, however, to simplify the proof it is convenient to number the modulators in opposite order (see Fig. 2). To this end we introduce a new index $j = N - i + 1$, and then the signal is amplitude modulated in the plane $x = \tilde{a}_j$ in accord with the law

$$\tilde{m}_j\cos(\tilde{\Omega}_j t - \tilde{K}_j y + \tilde{\varphi}_j),$$

where, obviously,

$$(\tilde{x}, \tilde{a}, \tilde{m}, \tilde{\Omega}, \tilde{K}, \tilde{\varphi})_j = (x, a, m, \Omega, K, \varphi)_{N-j+1}$$

and, conversely,

$$(x, a, m, \Omega, K, \varphi)_i = (\tilde{x}, \tilde{a}, \tilde{m}, \tilde{\Omega}, \tilde{K}, \tilde{\varphi})_{N-i+1}.$$

Now (A.2) takes the form

$$U(t,y,\tilde{x}_1) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_{N-M+2}) \times\exp\{i\omega t - ik_y y - i(\tilde{x}_1-\tilde{x}_{N-M+2})[k^2(\omega)-k_y^2]^{1/2}\}$$
$$\times\prod_{j=1}^{N-M+1}\tilde{m}_j\cos\left\{[\tilde{\Omega}_j t - \tilde{K}_j y + \tilde{\varphi}_j] - (\tilde{x}_1 - \tilde{a}_j) \times\left[1-\frac{k_y^2}{k^2(\omega)}\right]^{-1/2}\left[\tilde{\Omega}_j k'(\omega) - \tilde{K}_j\frac{k_y}{k(\omega)}\right]\right\} dk_y\,d\omega. \tag{A.5}$$

We prove (A.5) by induction. We first show that (A.5) is valid at $N-M+1=1$. We expand the two-dimensional wave $U(T,y,\tilde{x}_1)$ in an arbitrary plane $\tilde{x}_1>\tilde{a}_1$ in a space-time Fourier integral

$$U(t,y,\tilde{x}_1)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_1)\exp[-ik_yy+i\omega t]dk_y\,d\omega.$$

It follows from (A.4) that

$$U(t,y,\tilde{x}_1)=U_1(t,y,\tilde{x}_1)+U_2(t,y,\tilde{x}_1),$$

where

$$U_1(t,y,\tilde{x}_1)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega+\tilde{\Omega}_1,k_y+\tilde{K}_1,\tilde{x}_2)$$
$$\times\exp[i\omega t-ik_yy]\exp\{-i(\tilde{x}_2-\tilde{a}_1)[k^2(\omega)-k_y^2]^{1/2}$$
$$-i(\tilde{a}_1-\tilde{x}_1)[k^2(\omega+\tilde{\Omega}_1)-(k_y+\tilde{K}_1)^2]^{1/2}-i\tilde{\varphi}_1\}dk_y\,d\omega. \quad \text{(A.6)}$$

The function $U_2(t,y,\tilde{x}_1)$ differs from $U_1(t,y,\tilde{x}_1)$ only in the signs in front of $\tilde{\Omega}_1$, $\tilde{K}_1$, and $\tilde{\varphi}_1$. We consider therefore only U_1. We change in (A.6) the integration variables $\omega^{\text{ir}}=\omega+\tilde{\Omega}_1$, $k_y^{\text{ir}}=k_y+\tilde{K}_1$ and expand in a Taylor series the function $[k^2(\omega^{\text{ir}}-\tilde{\Omega}_1)-(k_y^{\text{ir}}-\tilde{K}_1)^2]^{1/2}$, near ω^{ir} and k_y^{ir}, confining ourselves to terms linear in $\tilde{\Omega}_1$ and $\tilde{K}_1$:

$$[k^2(\omega^{\text{ir}}-\tilde{\Omega}_1)-(k_y^{\text{ir}}-\tilde{K}_1)^2]^{1/2}$$
$$\approx[k^2(\omega^{\text{ir}})-(k_y^{\text{ir}})^2]^{1/2}-\tilde{\Omega}_1\frac{k(\omega^{\text{ir}})k'(\omega^{\text{ir}})}{[k^2(\omega^{\text{ir}})-(k_y^{\text{ir}})^2]^{1/2}}$$
$$+\tilde{K}_1\frac{k_y^{\text{ir}}}{[k^2(\omega^{\text{ir}})-(k_y^{\text{ir}})^2]^{1/2}} \quad \text{(A.7)}$$

(we omit hereafter the superscript "ir" in k_y^{ir} and ω^{ir}).

It follows from (A.6) and (A.7) that

$$U_1(t,y,\tilde{x}_1)$$
$$=\frac{1}{(2\pi)^2}\frac{\tilde{m}_1}{2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_2)\exp[i\omega t-ik_yy-i(\tilde{x}_1-\tilde{x}_2)$$
$$\times(k^2(\omega)-k_y^2)^{1/2}]\exp\Big\{-i[\tilde{\Omega}_1t-\tilde{K}_1y+\tilde{\varphi}_1]$$
$$+\frac{i(\tilde{x}_1-\tilde{a}_1)}{[1-(k_y/k(\omega))^2]^{1/2}}$$
$$\times\Big[\tilde{\Omega}_1k'(\omega)-\tilde{K}_1\frac{k_y}{k(\omega)}\Big]\Big\}dk_yd\omega. \quad \text{(A.8)}$$

Transforming in similar fashion $U_2(t,y,x_1)$, we obtain

$$U(t,y,\tilde{x}_1)$$
$$=\frac{1}{(2\pi)^2}\frac{\tilde{m}_1}{2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_2)\exp[i\omega t-ik_yy-i(\tilde{x}_1-\tilde{x}_2)$$
$$\times(k^2(\omega)-k_y^2)^{1/2}]\cos\Big\{[\tilde{\Omega}_1t-\tilde{K}_1y+\tilde{\varphi}_1]-\frac{(\tilde{x}_1-\tilde{a}_1)}{[1-(k_y/k(\omega))^2]^{1/2}}$$
$$\times\Big[\tilde{\Omega}_1k'(\omega)-\tilde{K}_1\frac{k_y}{k(\omega)}\Big]\Big\}dk_yd\omega. \quad \text{(A.9)}$$

We have thus proved the validity of (A.15) in the case $N-M+1=1$. Assume now that (A.5) is valid in the case $N-M+1=n-1$ (where $n>1$ is a positive integer), i.e.,

$$U(t,y,\tilde{x}_1)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_n)\exp[i\omega t-ik_yy-i(\tilde{x}_1-\tilde{x}_n)$$
$$\times(k^2(\omega)-k_y^2)^{1/2}]\prod_{j=1}^{n-1}\tilde{m}_j\cos\Big\{[\tilde{\Omega}_jt-\tilde{K}_jy+\tilde{\varphi}_j]$$
$$-\frac{(\tilde{x}_1-\tilde{a}_j)}{[1-(k_y/k(\omega))^2]^{1/2}}$$
$$\times\Big[\tilde{\Omega}_jk'(\omega)-\tilde{K}_j\frac{k_y}{k(\omega)}\Big]\Big\}dk_yd\omega. \quad \text{(A.10)}$$

On the basis of this assumption, we shall prove that relation (A.5) takes place at $N-M+1=n$, where n is an integer. Just as above, starting from (A.4), we represent $U(t,y,\tilde{x}_1)$ in the form

$$U(t,y,\tilde{x}_1)=U_1(t,y,\tilde{x}_1)+U_2(t,y,\tilde{x}_1),$$
$$U_1(t,y,\tilde{x}_1)$$
$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{\tilde{m}_n}{2}C(\omega+\tilde{\Omega}_n,k_y+\tilde{K}_n,\tilde{x}_{n+1})\exp\{i\omega t-ik_yy$$
$$-i(\tilde{x}_1-\tilde{a}_n)[k^2(\omega)-k_y^2]^{1/2}\}\exp\{-i(\tilde{a}_n-\tilde{x}_{n+1})[k^2(\omega+\tilde{\Omega}_n)$$
$$-(k_y+\tilde{K}_n)^2]^{1/2}-i\tilde{\varphi}_n\}\prod_{j=1}^{n-1}\tilde{m}_j\cos\Big\{[\tilde{\Omega}_jt-\tilde{K}_jy+\tilde{\varphi}_j]$$
$$-(\tilde{x}_1-\tilde{a}_j)\Big[1-\frac{k_y^2}{k(\omega)^2}\Big]^{-1/2}$$
$$\times\Big[\tilde{\Omega}_jk'(\omega)-\tilde{K}_j\frac{k_y}{k(\omega)}\Big]\Big\}dk_yd\omega. \quad \text{(A.11)}$$

The expression for $U_2(t,y,\tilde{x}_1)$ can be obtained from (A.11) by reversing the signs of $\tilde{\Omega}_j$, $\tilde{K}_j$, and $\tilde{\varphi}_j$ (where $1\leqslant j\leqslant n$); we consider therefore $U_1(t,y,\tilde{x}_1)$. We make in (A.11) the change of variables $\omega^{\text{ir}}=\omega+\tilde{\Omega}_n$ and $k_y^{\text{ir}}=k_y+\tilde{K}_n$ and expand the function in the argument of the cosine and the function in the argument of the exponential in Taylor series about ω^{ir} and k_y^{ir}, retaining in both expansions the terms linear in $\tilde{\Omega}_j$ and $\tilde{K}_j$ ($1\leqslant j\leqslant n$) the superscript "ir" of ω^{ir} and k_y^{ir} will be omitted hereafter). Then

$$U_1(t,y,\tilde{x}_1)$$
$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} C(\omega,k_y,\tilde{x}_{n+1})\exp\{i\omega t-ik_yy-i(\tilde{x}_1-\tilde{x}_{n+1})$$
$$\times[k^2(\omega)-k_y^2]^{1/2}\}\frac{\tilde{m}_n}{2}\exp\Big\{-i[\tilde{\Omega}_nt-\tilde{K}_ny+\tilde{\varphi}_n]$$
$$-i\Big[\Big(-\tilde{\Omega}_nk'(\omega)+\tilde{K}_n\frac{k_y}{k(\omega)}\Big)$$
$$\times\Big(1-\frac{k_y^2}{k^2(\omega)}\Big)^{-1/2}\Big](\tilde{x}_1-\tilde{a}_n)\Big\}\prod_{j=1}^{n}\tilde{m}_j\cos\Big\{[\tilde{\Omega}_jt-\tilde{K}_jy+\tilde{\varphi}_j]$$
$$-(\tilde{x}_1-\tilde{a}_j)$$
$$\times\Big[1-\frac{k_y^2}{k^2(\omega)}\Big]^{-1/2}\Big[\tilde{\Omega}_jk'(\omega)-\tilde{K}_j\frac{k_y}{k(\omega)}\Big]\Big\}dk_yd\omega. \quad \text{(A.12)}$$

Transforming $U_2(t,y,\tilde{x}_1)$ in similar fashion we see easily that relation (A.5) is valid at $N-M+1=n$. Consequently (A.5), meaning also (A.2), describes the behavior of the wave at any number of modulations.

We transform in (A.2) to integration with respect to the angle $\theta=\arcsin[k_y/k(\omega)]$ rather than with respect to k_y:

$$U(t,y,x_N)=\frac{1}{(2\pi)^2}\int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\infty}^{\infty}C(\omega,k(\omega)\sin\theta,x_{M-1})k(\omega)\cos\theta$$

$$\times\exp(i\omega t-i\mathbf{kr})\prod_{i=M}^{N}m_i\cos\left\{[\Omega_i t-K_i y+\varphi_i]-(x_N-a_i)\left(\frac{\Omega_i}{v_{\text{гр}}(\omega)\cos\theta}-K_i\operatorname{tg}\theta\right)\right\}d\theta d\omega, \quad \text{(A.13)}$$

where $\mathbf{r}=x\mathbf{x}^0+y\mathbf{y}^0$, $\mathbf{k}=k_x\mathbf{x}^0+k_y\mathbf{y}^0=k(\omega)(\mathbf{x}^0\cos\theta+\mathbf{y}^0\sin\theta)$, $\mathbf{x}^0$, and $\mathbf{y}^0$ are unit vectors in the directions of the axes x and y.

The region of validity of (A.2) and (A.13) is bounded by the conditions

$$(x_N-x_i)\ll\pi[P(\Omega_i,K_i)\Omega_l+Q(\Omega_i,K_i)K_l]^{-1}, \quad \text{(A.14)}$$

where $M\leqslant i<l\leqslant N$, and $Q(\Omega_i,K_i)$ and $P(\Omega_i,K_i)$ are of the order of

$$P(\Omega_i,K_i)\approx\frac{1}{\omega\cos\theta}\left\{\frac{\Omega_i}{\omega}[1-\operatorname{tg}^2\theta]k(\omega)+K_i\frac{\sin\theta}{\cos^2\theta}\right\},$$

$$Q(\Omega_i,K_i)\approx\frac{1}{\cos^2\theta}\frac{K_i}{k(\omega)}\left\{\frac{\Omega_i}{\omega}\frac{k(\omega)}{K_i}\sin\theta-1\right\}. \quad \text{(A.15)}$$

If the carrier wave $U(t,y,x_0)$ is noise stationary in time and in y, we can obtain an expression for the dispersion of this noise wave in an arbitrary plane x_N behind the modulators. In (A.2) we denote for brevity the argument of the cosine by $\psi_i(\omega,k_y)$. We shall assume that the mean value of the noise at the input is zero, and then the dispersion of the random signal is

$$\langle U^2(t,y,x_N)\rangle=\frac{1}{(2\pi)^4}\int\int\int\int\limits_{-\infty}^{\infty}\langle C(\omega,k_y,x_0)C^*(\omega^1,k_y^1,x_0)\rangle$$

$$\times\exp\{i(\omega-\omega^1)t-i(k_y-k_y^1)y-i(x_N-x_0)[(k^2(\omega)-k_y^2)^{1/2}-(k^2(\omega^1)-(k_y^1)^2)^{1/2}]\}\prod_{i=1}^{N}m_i^2\cos\psi_i(\omega,k_y)$$

$$\times\cos\psi_i(\omega^1,k_y^1)\,dk_y dk_y^1 d\omega d\omega^1. \quad \text{(A.16)}$$

The carrier signal $U(t,y,x_0)$ is a noise stationary in time and in y, so that its spectral components are delta-correlated[13]:

$$\langle C(\omega,k_y,x_0)C^*(\omega^1,k_y^1,x_0)\rangle=G(\omega,k_y,x_0)\delta(\omega^1-\omega)\delta(k_y^1-k_y), \quad \text{(A.17)}$$

where $G(\omega,k_y,x_0)$ is the spectral density of the power of the carrier wave

$$\langle U^2(t,y,x_0)\rangle=\frac{1}{(2\pi)^4}\iint\limits_{-\infty}^{\infty}G(\omega,k_y,x_0)\,dk_y d\omega. \quad \text{(A.18)}$$

Behind the entire system of modulators, as follows from (A.16) and (A.17), the dispersion of such a noise takes the simple form

$$\langle U^2(t,y,x_N)\rangle=\frac{1}{(2\pi)^4}\iint\limits_{-\infty}^{\infty}G(\omega,k_y,x_0)\prod_{i=1}^{N}m_i^2\cos^2\left\{[\Omega_i t-K_i y+\varphi_i]-(x_N-a_i)\left[1-\frac{k_y^2}{k^2(\omega)}\right]^{-1/2}\left[\Omega_i k'(\omega)-K_i\frac{k_y}{k(\omega)}\right]\right\}dk_y d\omega. \quad \text{(A.19)}$$

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