Coulomb interaction of compound particles

D. A. Kirzhnits and F. M. Pen'kov

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

(Submitted 31 December 1982)
Zh. Eksp. Teor. Fiz. **85**, 80–93 (July 1983)

The results of an investigation of the interaction between two bound complexes consisting of charged particles—their effective interaction potential, the elastic scattering phase shifts, and the bound state energies—are presented. The simplest case of a system consisting of a charged structureless particle and a complex is considered in detail. It is found that the conventional $1/r^4$ law is valid only for not very light particles. In the opposite case this law holds only for extremely large distances, being replaced at smaller values of r by the $1/r^2$ law, which possesses a number of unusual properties. There occurs in the case of a highly polarizable complex a family of bound states that are similar in their properties to the Efimov levels. Simple analytic formulas expressing the phase and energy level shifts arising as a result of the polarization of the complex in terms of a half-integer moment of its oscillator-strength distribution are presented. The interaction between two complexes corresponds to the ordinary van der Waals forces if certain quite rigid conditions leading to the $1/r^4$ law are not fulfilled. The specific applications pertain to the "charged particle + deuteron" system.

PACS numbers: 03.80. + r, 34.10. + x, 34.20.Fi

I. INTRODUCTION

There exist a number of important atomic- and nuclear-physics problems consisting in the description of the interaction (scattering, bound states) of a light charged particle with a bound complex made up of charged particles. Examples are systems of the type "lepton or pion + atomic nucleus," "electron + muonic atom," "muon + molecule," etc.[1)] The term "light particle" denotes smallness of the particle mass compared with that of the complex's "valence" particle, which determines the size of the complex (in the just cited examples, compared with the masses of the nucleon, the muon, and an intramolecular atom or ion respectively). A more exact condition has the form

$$m \ll (\varepsilon R^2)^{-1}, \qquad (1.1)$$

where m is the particle mass, ε is the characteristic excitation energy of the complex (its binding energy), and R is the radius of the complex; here and below we set $\hbar = 1$.

The method, which suggests itself, of describing systems of the type in question consists in the use of the well-known expression for the polarization potential[1]

$$V(r) = -e^2\alpha(0)/2r^4, \qquad (1.2)$$

where e is the charge of the particle, $\alpha(\omega)$ is the polarizability of the complex, and $r \gg R$. This potential arises as a result of the polarization of the complex by the outer particle, and acts even in the case of a complex that, as a whole, is neutral, when the usual Coulomb interaction between the particle and the complex as a whole does not occur. It turns out, however, that it is precisely for a light particle, i.e., one whose mass satisfies the condition (1.1), that the expression (1.2) ceases to be valid.

Most precisely, the interaction law (1.2) remains valid only when the distance between the particle and the complex is extremely large, i.e., when

$$r \gg (m\varepsilon)^{-1/2}, \qquad (1.3a)$$

it is being replaced in the region

$$R \ll r \ll (m\varepsilon)^{-1/2} \qquad (1.3b)$$

by a law of the $1/r^2$ type. In the case of a slightly polarizable complex

$$m^2e^2\langle d^2\rangle \ll 1, \qquad (1.4)$$

where $\mathbf{d}$ is its dipole moment; the polarizing force in the region (1.3b) acts only in the s state and, like the gravitational force, is proportional to the particle mass. In this region the nature of the polarization potential is itself unusual: what is excited is not internal motion in the complex, as occurs in the region (1.3a), but relative motion of the particle and the complex, the source of the excitation being the zero-point oscillations of the dipole moment of the complex.

The description with the aid of the polarization potential has, as a rule, been carried out for systems of the type "electron + atom," in which the outer and the valence particles have the same mass and the condition (1.1) is clearly not fulfilled. In this case the region (1.3b) does not exist at all, and the usual interaction law (1.2), which is valid for all $r \gg R$, obtains.

The polarization forces acting between a charged particle and a complex (as well as between two complexes) are systematically investigated in Secs. 2–5 of this paper. In Sec. 2 we formulate the basic general relations. Section 3 contains the derivation of the expressions for the polarization potential in the weak-coupling case, when the condition (1.4) is fulfilled. In Sec. 4 we consider the intermediate- and strong-coupling cases as applied to the region (1.3b). It is shown that the interaction law in these cases is also of the $1/r^2$ type, and that there arise at a sufficiently large value of the coupling constant a series of bound states of the particle in the field of

0038-5646/83/070046-09$04.00

the complex that are similar in their properties to the well known Efimov levels.^{2} In Secs. 3 and 4 we consider only systems of the type "particle + complex." Section 5 contains a brief description of systems of the type "complex + complex" and the formulation of the conditions that must be fulfilled in order for the ordinary van der Waals expression to be valid.

In Secs. 6 and 7 we compute the contributions of the polarization effects to the phase of the elastic scattering of a particle by a complex (the particle energy is lower than the breakup threshold ε of the complex) and the bound state energy of a particle in the field of a charged complex. In Sec. 6 we give simple analytic expressions for the phase shift in the case of a neutral complex when the condition (1.4) is fulfilled and in the case of a charged complex when the following more rigid condition is fulfilled:

$$ee_0(m/\varepsilon)^{1/2}\ll 1, \tag{1.5}$$

where e_0 is the charge of the complex. We also give expressions for the energy-level shift, which are valid when the condition (1.5) is fulfilled. In Sec. 7 we give specific formulas pertaining to the interaction between a light particle and a deuteron (or any weakly bound two-particle system when the condition (1.1) is fulfilled).

For simplicity, we consider in the present paper only the case of a maximally symmetric complex having zero mean angular-momentum and multipole-moment values. The relativistic, magnetic, and spin effects are neglected. When the conditions given above are fulfilled, we can also neglect the short-range effects: the nuclear interactions, the exchange effects, etc.$^{2)}$

2. THE POLARIZATION POTENTIAL

It is convenient to carry out the description of the interaction between two complexes, one of which can simply be a charged particle, in terms of the variables $\mathbf{r}$ (the vector joining the centers of mass of the complexes) and $\boldsymbol{\rho}_{ni}$ (the vector connecting the center of mass of the n-th ($n = 1,2$) complex with its i-th particle). Because of the relations

$$\sum_i m_{ni}\boldsymbol{\rho}_{ni}=0$$

(the m's are the masses of the particles of the complex), the number of independent $\boldsymbol{\rho}$ vectors is smaller than the total number of particles. Below we shall denote the product $d\boldsymbol{\rho}_{n1}\,d\boldsymbol{\rho}_{n2}\ldots$ of the independent-vector elements by $d\boldsymbol{\rho}_n$ and the product $d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2$ by $d\boldsymbol{\rho}$.

In terms of the variables introduced, the motion within the complexes, which is described by the equations $(H_n - E_n)\Phi_n = 0$ (H_n, E_n, and Φ_n are respectively the Hamiltonian, the energy, the the normalized—to unity—wave function of the internal motion), is kinematically separable from the relative motion of the complexes. To the latter motion corresponds the Hamiltonian

$$H_R=-\Delta_r/2\mu+U(\mathbf{r},\boldsymbol{\rho}),\quad U=\sum_i e_{1i}e_{2i}/|\mathbf{r}+\boldsymbol{\rho}_{1i}-\boldsymbol{\rho}_{2i}|, \tag{2.1}$$

where e's are the particle charges and μ is the reduced mass of the complexes; here only the long-range Coulomb interaction between the complexes has been taken into account. The Schrödinger equation

$$(H-E)\Psi(\mathbf{r},\boldsymbol{\rho})=0,\quad H=\sum H_n+H_R,\quad E=\sum E_n+E_R, \tag{2.2}$$

where E_R is the energy of the relative motion, forms the basis of the subsequent analysis.

Below the complex-disintegration threshold it is the complex-complex interaction description averaged over the internal motion that makes sense. It amounts to the replacement of the quantity U in (2.1) by some effective potential $V(r)$ that, generally speaking, depends on the momentum operator $\mathbf{p}$ for the relative motion; such a potential should give the same values for the scattering phases, the bound-state energies, etc., as the solution to Eq. (2.2). We shall call the potential V the polarization potential, although its meaning does not directly correspond to this term in all cases.

The internal-motion-averaged wave function $\psi(r)$ of the relative motion of the complexes is evidently equal to the projection of the exact wave function onto the state $\Phi \equiv \Phi_1\Phi_2$:

$$\psi(r)=\int d\boldsymbol{\rho}\,\Phi\Psi.$$

Representing Ψ in the form $\xi\psi$, where ξ is some operator defined by the relation $\int d\boldsymbol{\rho}\bar{\Phi}\xi = 1$, and taking the projection onto Φ of Eq. (2.2), we easily arrive at the Schrödinger equation for the relative motion:

$$[-\Delta/2\mu+V(r)-E_R]\psi=0, \tag{2.3}$$

where the sought polarization potential

$$V(r)=\int d\boldsymbol{\rho}\,\Phi U\xi. \tag{2.4}$$

From (2.1)–(2.3) we can derive an equation for the operator ξ:

$$\Big[-(\Delta_r+2i\mathbf{p}\nabla_r)/2\mu+\sum_n(H_n-E_n)+U-V\Big]\xi=0, \tag{2.5}$$

where $\mathbf{p}$ is the momentum operator, which acts on the wave function ψ. Everywhere below, except in Sec. 4, the structural effects discussed in this paper are assumed to be weak, which corresponds to the closeness of ξ to Φ. Therefore,

$$\xi=\Big\{1+\Big[(\Delta_r+2i\mathbf{p}\nabla_r)/2\mu-\sum_n(\hat{H}_n-E_n)\Big]^{-1}(U-\langle U\rangle)\Big\}\Phi;$$

here and below the angle brackets denote averaging over the internal states of the complexes, i.e., the integration $\int d\boldsymbol{\rho}\bar{\Phi}(\ldots)\Phi$. We find in accordance with (2.4) that

$$V(r)=\langle U\rangle+{\sum_\nu}' U_{0\nu}\Big[(\Delta_r+2i\mathbf{p}\nabla_r)/2\mu-\sum_n(E_{\nu_n}-E_n)\Big]^{-1}U_{\nu 0}, \tag{2.6}$$

where $U_{0\nu}$ is the matrix element of U between the ground Φ and the excited Φ_ν states of the complexes (E_ν is the corresponding energy and $\nu = \nu_1\nu_2$). The prime in the sum (2.6) indicates the omission of the state $\Phi_\nu = \Phi$, and the summation itself encompasses three cases: the case in which only the first complex is excited, the case in which only the second complex is excited, and the case in which both complexes are excited. The formula (2.6) clearly corresponds to the first two orders of perturbation theory in terms of U.

The polarization potential can be expressed in terms of the electric form factors and the generalized polarizabilities of the complexes. These quantities can, in their turn, be expressed in terms of the **r** Fourier component of the charge density:

$$F_n(\mathbf{k},\boldsymbol{\rho})=\sum_i e_{ni}\exp(i\mathbf{k}\boldsymbol{\rho}_{ni}).$$

The form factor $F_n(k)$ of the complex is equal to the mean $\langle F_n(\mathbf{k},\boldsymbol{\rho})\rangle$, and the first term in (2.6), which is denoted below by $V^{(1)}(r)$, has simply the form of the Coulomb interaction energy of the distributed charges:

$$V^{(1)}(r)=\langle U\rangle=4\pi\int\frac{d^3k}{k^2}F_1(k)F_2(k)\exp(i\mathbf{k}\mathbf{r}). \tag{2.7}$$

The generalized polarizability $A_n(\mathbf{k},\mathbf{q},\omega)$ relates the change in the **k**-th Fourier component of the charge density of the complex to the effect of the **q**-th component of the same quantity:

$$A_n(\mathbf{k},\mathbf{q},\omega)$$

$$=2\sum_{\nu_n}{}'(F_n(\mathbf{k},\boldsymbol{\rho}))_{0\nu_n}(F_n(\mathbf{q},\boldsymbol{\rho}))_{\nu_n 0}\omega_{\nu0}(\omega_{\nu0}^2-\omega^2-i\delta)^{-1}, \tag{2.8}$$

where $\omega_{\nu0}=E_{\nu_n}-E_n$. The anti-Hermitian part of (2.8),

$$[A_n(\mathbf{k},\mathbf{q},\omega)-\bar{A}_n(-\mathbf{q},-\mathbf{k},\omega)]/2i,$$

which can simply be denoted by the symbol Im, has, for $\omega>0$, the form

$$\mathrm{Im}\,A_n(\mathbf{k},\mathbf{q},\omega)=\pi\sum_{\nu_n}{}'(F_n(\mathbf{k},\boldsymbol{\rho}))_{0\nu_n}(F_n(\mathbf{q},\boldsymbol{\rho}))_{\nu_n 0}\delta(\omega_{\nu0}-\omega). \tag{2.9}$$

In the long-wave limit, i.e., for **k**, $\mathbf{q}\to0$, the quantity (2.8) goes over into the usual polarizability $\alpha(\omega)$—the function characterizing the response of the "dipole moment **d** to a uniform external electric field:"

$$A_n(\mathbf{k},\mathbf{q},\omega)\to-(\mathbf{k}\mathbf{q})\alpha_n(\omega), \tag{2.10}$$

where α_n is given by the formula (2.8) with F_n replaced by d_z.

The second term in (2.6), which is denoted below by $V^{(2)}(r)$, can easily be expressed in terms of the quantity (2.9):

$$V^{(2)}(r)=-16\pi^2\int\frac{d^3k\,d^3q}{k^2q^2}e^{i(\mathbf{k}+\mathbf{q})\mathbf{r}}\{F_1(k)F_1(q)S_2(\mathbf{k},\mathbf{q})$$

$$+F_2(k)F_2(q)S_1(\mathbf{k},\mathbf{q})+T(\mathbf{k},\mathbf{q})\}, \tag{2.11}$$

where

$$S_n(\mathbf{k},\mathbf{q})=\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Im}\,A_n(\mathbf{k},\mathbf{q},\omega)[\omega+(q^2+2\mathbf{p}\mathbf{q})/2\mu]^{-1}$$

$$T(\mathbf{k},\mathbf{q})=\frac{1}{\pi^2}\int_0^\infty d\omega_1\,d\omega_2\,\mathrm{Im}\,A_1(\mathbf{k},\mathbf{q},\omega)\cdot$$

$$\times\mathrm{Im}\,A_2(\mathbf{k},\mathbf{q},\omega)[\omega_1+\omega_2+(q^2+2\mathbf{p}\mathbf{q})/2\mu]^{-1}.$$

The first term in (2.11) describes the polarization of the second complex by the charge distribution in the first complex; the second term, the inverse process; and the third term, the mutual polarization of the complexes. For a "particle + complex" system, which we shall mainly consider below, the formula (2.11) yields[3]

$$V^{(2)}(r)$$

$$=-16\pi e^2\int_0^\infty d\omega\int\frac{d^3k\,d^3q}{k^2q^2}\mathrm{Im}\,A(\mathbf{k},\mathbf{q},\omega)\frac{\exp(i(\mathbf{k}+\mathbf{q})\mathbf{r})}{[\omega+(q^2+2\mathbf{p}\mathbf{q})/2\mu]}. \tag{2.12}$$

3. THE SYSTEM "PARTICLE + COMPLEX"

In this section the general relations obtained above will be applied to the simplest system in the class of systems considered in this paper, namely, the system consisting of a maximally symmetric complex (see Sec. 1) and a structureless particle with charge e and mass m. We shall seek the expression for the polarization potential in the region of distances r that are large compared to the radius R of the complex. This will allow us to neglect the quantity (2.7), which trivially reduces to the Coulomb interaction, and use the long-wave limit (2.10), which corresponds to the discarding in the potential U of multipole moments of order higher than that of the dipole moment. Accordingly, the formula (2.12) yields

$$V^{(2)}(r)$$

$$=-4e^2\nabla\left(\frac{1}{r}\right)\nabla\int_0^\infty d\omega\,\mathrm{Im}\,\alpha(\omega)\int d^3q\frac{e^{i\mathbf{q}\mathbf{r}}}{[q^2(\omega+(q^2+2\mathbf{p}\mathbf{q})/2\mu)]}. \tag{3.1}$$

Let us compare with each other the terms of the denominator of this expression, the first of which (ω) corresponds to the excitation of the internal motion in the complex, i.e., to the polarization of the complex; the second ($q^2/2\mu+\dots$), to the excitation of the relative motion of the particle and the complex. The first term is of the order of the characteristic excitation energy $\omega_{\nu0}$, denoted in Sec. 1 by ε, of the complex. Since $q\sim1/r$ in (3.1), the second term in that expression is, in order of magnitude, equal to $1/\mu r^2$. Leaving out the case, of little interest, in which the particle is significantly heavier than the complex, we can replace the reduced mass by the particle mass m, and then the estimate for the second term in the denominator of (3.1) assumes the form $1/mr^2$.

It is clear from these estimates that, if the particle mass m is greater than, or of the order of, $1/\varepsilon R^2$, then the second term in the denominator of (3.1) is small compared to the first term in the entire region $r\gg R$ under consideration. Then, using the well known sum rule

$$\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\mathrm{Im}\,\alpha(\omega)=\alpha(0),$$

we easily arrive at the expression (1.2) for the polarization potential (see Sec. 1).

But for a very light particle, when the condition (1.1) is fulfilled, the region $r \gg R$ splits up into two parts. At the distant periphery (1.3a) the second term in (3.1) is, as before, small compared to the first term, and the usual expression (1.2) continues to remain valid there. On the other hand, in the near region (1.3b) the second term in the denominator of (3.1) predominates. Simple calculations with allowance for the sum rule

$$\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Im}\,\alpha(\omega)=\frac{\langle d^2\rangle}{3}$$

(see (2.9), (2.10)) yield for the polarization potential an entirely different expression, which is valid in the region (1.3b):

$$V^{(2)}(r)=-me^2\frac{\langle d^2\rangle O}{3r^2},\qquad O=\frac{\sin(pr)}{pr}\exp(-i\mathbf{pr}). \tag{3.2}$$

And what is more, the mechanism underlying the appearance of the polarization potential in this region is entirely different: It consists not in the polarization of the complex by the outer particle, but in the excitation of the relative motion of the particle and the complex by the zero-point oscillations of the complex's dipole moment (the quantity $\langle d^2\rangle$ in (3.2)). This corresponds to adiabaticity in the particle motion: owing to its small mass, the particle follows the fluctuations in the dipole moment of the complex. The power of r in (3.2) is found in accordance with the mechanism under discussion, namely, in the second-order correction to the energy the square of the matrix element $[\mathbf{d}\nabla(1/r)]^2 \propto 1/r^4$, and the energy denominator is of the order of $1/mr^2$. This is also the cause of the dependence of the potential (3.2) on the particle mass m.

The force acting on the particle and corresponding to (3.2) is proportional to the particle mass. This leads to a situation in which all the particles for which the condition (1.1) is fulfilled move in identical fashion in the field of the complex. We can therefore speak of a "principle of equivalence" of the forces under consideration and the inertial forces. We are convinced that this property is not at all an exclusive attribute of the gravitational forces, as is often assumed, but is also a characteristic of simple systems with the Coulomb interaction.

Another characteristic of the potential (3.2) is that it acts only in the s state:

$$O=\delta_{l,0}. \tag{3.3}$$

This can easily be verified by letting the operator O act on the wave function

$$\psi(\mathbf{r})=\int d^3p\,\psi(\mathbf{p})\exp(i\mathbf{pr})$$

and expanding $\psi(\mathbf{p})$ in terms of the spherical harmonics. Thus, when the condition (1.1) is fulfilled, the polarization potential in the states with $l\neq 0$ acts only in the region (1.3a): to within higher-order effects the polarization interaction does not occur at all in the region (1.3b).

4. STRONG COUPLING

The analysis in Sec. 3 encompasses only the case of weak coupling pertaining to the interaction of a particle with a rigid, slightly polarizable complex. The corresponding dimensionless coupling constant for the region $r \gg R$ can be found by making in (2.1) the substitution

$$U\to e\mathbf{d}\mathbf{r}/r^3,\qquad \mathbf{d}=\sum_i e_i\boldsymbol{\rho}_i, \tag{4.1}$$

where $\mathbf{d}$ is dipole-moment operator for the complex (we have, for simplicity, dropped the trivial Coulomb term in the expression for the particle—charged complex interaction). Comparison of (4.1) with the kinetic energy $1/\mu r^2$, i.e., their ratio, yields the dimensionless coupling constant, which it is convenient to choose in the form

$$g=2\mu e(\langle d^2\rangle)^{1/2}. \tag{4.2}$$

In the weak coupling case $g \ll 1$.

The same dependence on r of the interaction (4.1) and the kinetic energy allows us to describe the case of the intermediate and strong coupling $g \gtrsim 1$ as well if we limit ourselves to the consideration of the region (1.3b) (the condition (1.1) is assumed to be fulfilled) and the s state. After making the substitution (4.1) in Eq. (2.5) and dropping the small quantity $\Sigma_n(H_n - E_n)$, we can seek the solution to the equation in the form $\xi = f(x)\Phi$, where x is the cosine of the angle between the vectors $\mathbf{d}$ and $\mathbf{r}$; it is important here that in the case of the s state the operator $\mathbf{p}\cdot\nabla_r$ does not act on the angle variable. Going over to the dimensionless variable $\delta = \mathbf{d}/(\langle d^2\rangle)^{1/2}$, we obtain with allowance for (4.2) the equation

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}-(g\delta x+\sigma)\right]f=0. \tag{4.3}$$

The quantity σ in this equation is given by the relation $\int d\mathbf{p}\bar{\Phi}\xi = 1$ (see Sec. 2), and is equal to

$$\sigma=-g\langle\!\langle\delta\overline{xf}\rangle\!\rangle,\qquad \langle\!\langle\bar{f}\rangle\!\rangle=1. \tag{4.4}$$

Here and below in this section the averaging over the internal states of the complex has been split into two parts: angle averaging, which is denoted by a bar, i.e.,

$$\overline{(\dots)}=\frac{1}{2}\int_{-1}^{1}dx(\dots),$$

and averaging, denoted by double angle brackets, over the absolute value of δ, i.e., the dipole moment.

The polarization potential is given by the expression (2.4):

$$V^{(2)}(r)=-\sigma/2mr^2. \tag{4.5}$$

It can be seen that the $1/r^2$ law holds in the region (1.3b) for any value of the coupling constant.

Let us proceed to solve Eq. (4.3). It the quantity δ were a fixed number, the condition (4.4) (without the brackets) would have determined σ and, above all, ensured the regularity of $f(x)$ at the points ± 1. But in fact we have $A = \overline{(g\delta x + \sigma)f} \neq 0$ (only the result of the δ averaging, i.e., the quantity $\langle\langle A\rangle\rangle$, is equal to zero on account of (4.4)). This leads to the appearance in $f(x)$ of a logarithmic irregular part, and, after expanding the remaining regular part in a series in Legendre polynomials, we obtain

$$f(x)=1+A\left[\frac{1}{2}\ln(1-x^2)+\sum_{l=1}^{\infty}a_lP_l(x)\right], \tag{4.6}$$

where the a_l are unknown quantities to be determined. The substitution of (4.6) in (4.3) with the use of the well-known properties of the Legendre polynomials leads to the system

$$\sum_{l'=0}^{\infty}C_{ll'}a_{l'}=b_l,$$

where $a_0\equiv1/A$, the nonzero C have the form

$$C_{ll}=-[\sigma+l(l+1)], \qquad C_{l-1,\,l}=g\delta l/(2l+1),$$

$$C_{l,\,l-1}=g\delta l/(2l-1),$$

and the right-hand sides of the equations are given by

$$b_0=1-\sigma(1-\ln 2), \qquad b_1=(4-3\ln 2)g\delta/3,$$
$$b_{2l}=-\sigma(4l+1)/[2l(2l+1)], \qquad b_{2l+1}=g\delta(4l+3)/[2l(2l+3)].$$

Consequently, $a_0=D_0/D$, here $D=\det|C|$, and D_0 is obtained from D by substituting the b_l for the zeroth column. The condition $\langle\langle A\rangle\rangle=0$ yields an equation for the σ as a function of the coupling constant g:

$$\langle\langle D/D_0\rangle\rangle=0. \tag{4.7}$$

When g is small, σ is also small. This yields

$$D=2\sigma-g^2\delta^2/3, \qquad D_0=-2, \qquad \sigma=g^2/6,$$

which takes us back to the formula (3.2) (for the s state). As $g\to\infty$, we also have $\sigma\to\infty$ at the same time. In this limit the system of equations for the a_l and the left-hand side of (4.7) are homogeneous in σ and g; whence

$$\sigma\to Kg, \tag{4.8}$$

where the numerical constant K depends on the specific structure of the complex. From this it follows that the value $\sigma=\frac{1}{4}$, which corresponds to the onset of the "fall-to-the-center" regime, is attained at some $g=g_0$ (see (4.5)).[1] Of course, an actual fall does not occur because of the presence of the left boundary of the range (1.3b) of action of the law (4.5). The quantity g_0 is itself determined by Eq. (4.7), in which we can use for D and D_0 the following expressions, which are accurate to within 3–5%:

$$D={}^{27}/_{16}-g_0{}^2\delta^2, \qquad D_0={}^{81}/_{16}(3+\ln 2)+(4-3\ln 2)g_0{}^2\delta^2.$$

This yields

$$\left\langle\left\langle\left[1+\frac{4-3\ln 2}{13}\left(g_0{}^2\delta^2-\frac{27}{16}\right)\right]^{-1}\right\rangle\right\rangle=1 \tag{4.9}$$

(by definition $\langle\langle\delta^2\rangle\rangle=1$). The quantity $g_0\sim1$, but depends on the structure of the complex.

The most important property of the system "particle + complex" when $g>g_0$ consists in the occurrence of a series of bound states that arise as a result of the action of the polarization potential. The corresponding wave function has the form[1]

$$\psi(r)\infty\cos\left[(\sigma-{}^1/_4)^{1/2}\ln(r/R)+\text{const}\right],$$

from which we can find the number N of such bound states as the number of zeros of ψ occurring within the limits of the region (1.3b):

$$N=\frac{(\sigma-{}^1/_4)^{1/2}}{4\pi}\ln\left(\frac{1}{m\varepsilon R^2}\right). \tag{4.10}$$

The level energies obey also the scaling law

$$\frac{E_{n+1}}{E_n}=\exp\left[\frac{\pi}{(\sigma-{}^1/_4)^{1/2}}\right] \tag{4.11}$$

(n is the level number), which is obtained by substituting (4.5) into the Bohr-Sommerfeld quantization rules for the radial motion.

The bound states in question are very similar in their properties (4.10) and (4.11) to the Efimov levels,[2] which occur in a system of three particles resonantly interacting with each other (the two-particle scattering length a is large compared to the range r_0 of the forces). And in this case there appears in the region $a\gg\mathfrak{R}\gg r_0$, where $\mathfrak{R}$ is the mean dimension of the system, a potential of the type $1/\mathfrak{R}^2$, which is the cause of the appearance of the set of bound states. The foregoing shows that an analogue of the Efimov levels can occur also in systems of the "particle + complex" type when the particle mass is small and the polarizability of the complex is sufficiently high.

5. THE "COMPLEX + COMPLEX" SYSTEM

To complete the investigation of the polarization potential, we turn in this section to a system that consists of two complexes containing charged particles. In the general formula (2.11), which corresponds to the weak-coupling case, we shall be most interested in the last term, which describes the mutual polarization of the complexes (the potential (2.7) does not merit a special discussion, and the first two terms of (2.11) yield potentials that, in the case of charged complexes, are similar to those considered in the preceding sections and, in the case of neutral ones, fall off rapidly with distance). This term has a form similar to that of the potential (3.1):

$$V^{(2)}(r)=-\frac{4}{\pi}\nabla_i\nabla_j\left(\frac{1}{r}\right)\nabla_i\nabla_j\int_0^{\infty}d\omega_1\,d\omega_2$$
$$\times\operatorname{Im}\alpha_1(\omega_1)\operatorname{Im}\alpha_2(\omega_2)\int\frac{d^3q\exp(iqr)}{q^2[\omega_1+\omega_2+(q^2+2pq)/2\mu]}. \tag{5.1}$$

Let us compare the sum of the first two terms of the denominator in (5.1) with the last term, considering the region of large distances between the complexes, i.e., the region $r\gg R_1+R_2$. This amounts to the comparison of the quantity $\varepsilon_1+\varepsilon_2$ with

$$1/\mu r^2\ll1/[\mu(R_1+R_2)^2].$$

Normally, the characteristic excitation energy $\varepsilon\sim1/m_0R^2$, where m_0 is the mass of the "valence" particle, which determines the size, of the complex; that is the way it is with an atom or an atom-like compact object. Then, for all the values of r under consideration, the ratio of $\varepsilon_1+\varepsilon_2$ to the last term in the denominator of (5.1) is greater than the quantity

$$\left[\frac{1}{m_{01}\xi^2}+\frac{1}{m_{02}(1-\xi)^2}\right]\Big/\left[\frac{1}{M_1}+\frac{1}{M_2}\right],\qquad \xi=\frac{R_1}{R_1+R_2}<1,$$

which, in turn, is large compared to unity, since the mass of the valence particle contained in the complex is clearly smaller than the mass M of the complex.[4)] Consequently, neglecting the last term in the denominator in (5.1), we arrive at the well-known expression for the van der Waals potential with the London coefficient:

$$V^{(2)}(r)=-\frac{6}{\pi^2 r^6}\int\limits_0 \frac{d\omega_1\, d\omega_2}{\omega_1+\omega_2}\operatorname{Im}\alpha_1(\omega)\operatorname{Im}\alpha_2(\omega). \tag{5.2}$$

The opposite situation, in which we can neglect the quantity $\varepsilon_1+\varepsilon_2$ in (5.1), and we have the excitation of the relative motion of the complexes to deal with, is, in contrast to the case of the "particle + complex" system, realized only when the quite rigid conditions $\varepsilon_n \ll 1/m_{0n}R_n^2$ are fulfilled. Such conditions could, in principle, be realized in the case when the complex is, for example, a diatomic molecule with an unusually flat interatomic potential energy. In this case the characteristic excitation energy ε will be determined by an anomalously small vibrational quantum when the oscillator strength is sufficiently great. Although we are not in a position to give actual examples of this type, we shall give the result.

Assuming that the condition

$$\mu\ll[(\varepsilon_1+\varepsilon_2)(R_1+R_2)^2]^{-1}, \tag{5.3}$$

is fulfilled, we can easily verify that the ordinary van der Waals expression (5.2) remains valid at extremely large distances, i.e., for

$$r\gg[\mu(\varepsilon_1+\varepsilon_2)]^{-1/2}. \tag{5.4a}$$

Straightforward but tedious calculations yield for the potential $V^{(2)}$ in the region

$$R_1+R_2\ll r\ll[\mu(\varepsilon_1+\varepsilon_2)]^{-1/2}, \tag{5.4b}$$

of small distances, where we can neglect the quantity $\varepsilon_1+\varepsilon_2$ in (5.1), the expression

$$V^{(2)}(r)=-\frac{2\mu}{9r^4}\langle d_1^2\rangle\langle d_2^2\rangle O. \tag{5.5}$$

And here we encounter a decrease—by two—of the power of r in the denominator.

As in Sec. 3, the quantity O is a momentum-dependent factor:

$$O=\frac{1}{2}\left[\frac{3\sin(pr)}{pr}-\cos(pr)+3ipr\frac{\sin(pr)}{pr}\right]\exp(-ipr).$$

In the present case it is nonzero not only in the s, but also in the p state, and does not, moreover, reduce simply to projection operators. Expanding the wave function on which O acts in terms of the spherical harmonics, we find that

$$O=\left(1-\frac{r}{2}\frac{\partial}{\partial r}\right)\delta_{l,0}+\left(1+\frac{r}{2}\frac{\partial}{\partial r}\right)\delta_{l,1}. \tag{5.6}$$

Because the r dependences of the polarization potential and the kinetic energy are different, the question of the limits of applicability of the weak-coupling approximation requires a special discussion. In any case, it is necessary that the potential be small compared to the kinetic energy at the right boundary of the region (5.4b):

$$\mu^3(\varepsilon_1+\varepsilon_2)\langle d_1^2\rangle\langle d_2^2\rangle\ll 1. \tag{5.7}$$

And for the expression (5.5) to be applicable in the entire region (5.4b) the following more rigid condition should be fulfilled:

$$\mu^2\langle d_1^2\rangle\langle d_2^2\rangle\ll(R_1+R_2)^2. \tag{5.8}$$

6. SCATTERING AND THE BOUND STATES OF COMPLEXES

In this section we shall apply the above results to the description of the scattering by each other, and the bound states, of complexes. Not being interested in the bound states that arise as a result of the presence of the polarization potential (see Sec. 4), we shall limit ourselves to the computation of the corresponding level shift in a bound system of oppositely charged complexes.

The basic problem will be the problem of two complexes interacting through the potential $V^{(1)}$ (see (2.7)) that constitutes the Coulomb interaction of fixed charges distributed over the volumes of the complexes. The corresponding scattering problem with the relative momentum k is described by the equation

$$(-\Delta/2\mu+V^{(1)}-k^2/2\mu)\psi_0(r)=0, \tag{6.1}$$

where it is convenient to normalize the wave function ψ_0 in accordance with the following condition at infinity:

$$\psi_0\to\sin(kr+\delta_0)/kr. \tag{6.2}$$

Here δ_0 is the scattering phase shift (for simplicity, we consider only the s state). The solution to the complete equation (2.2) with $E_R=k^2/2\mu$ has the asymptotic form

$$\Psi\to\Phi\sin(kr+\delta)/kr, \tag{6.3}$$

where δ is the exact scattering phase shift; its determination is our aim.

There exists a simple relation connecting δ and δ_0:

$$\sin(\delta-\delta_0)=-\frac{\mu k\operatorname{Re}}{2\pi}\int d\mathbf{r}\, d\boldsymbol{\rho}\,\bar{\Phi}\bar{\psi}_0(U-V^{(1)})\Psi. \tag{6.4}$$

To derive it, we must multiply (2.2) on the left by $\bar{\Phi}\bar{\psi}_0$, the complex conjugate of Eq. (6.1) on the right by Ψ, subtract the resulting equations from each other, use the Green formula, and take the asymptotic forms (6.2) and (6.3) into account. Using (2.4) and the definition of the wave function ψ (see (2.3)), we have

$$\sin(\delta-\delta_0)=-\frac{\mu k}{2\pi}\int d\mathbf{r}\psi_0(U-V^{(1)})\psi, \tag{6.4a}$$

where the wave functions are assumed to be real. Similarly, replacing E_R in (2.2) by the exact energy value E_n and $k^2/2\mu$ in (6.1) by E_n^0, the energy value corresponding to the potential $V^{(1)}$, and carrying out the same operations performed above, we find a relation for the level shift:

$$E_n-E_n^0=\int d\mathbf{r}\psi_{0n}(V-V^{(1)})\psi_n\Big/\int d\mathbf{r}\psi_{0n}\psi_n. \tag{6.5}$$

Below we shall limit ourselves to the consideration of the system "particle + complex," assuming the condition

(1.1) to be fulfilled. Let us, limiting ourselves to the analysis of only the weak-coupling case, rewrite the corresponding criterion (see Sec. 4) in the form

$$me(\langle d^2\rangle)^{1/2}\sim R/a_0\ll 1. \tag{6.6}$$

Here $a_0 = 1/me\bar{e}$ is the Bohr radius and $\bar{e} = (\langle d^2\rangle)^{1/2}/R$ is the effective charge of the complex. It is assumed, for simplicity, that, if the complex is charged, its actual charge is of the order of $\bar{e}$. Hence the subsequent formulas will contain only the quantity a_0, which, in relations pertaining to a charged complex, should be taken to be $1/mee_0$, where e_0 is the total charge of the complex.

In the weak-coupling approximation we can replace the sine in (6.4a) by its argument and ψ by the quantity ψ_0; we can also replace in Eq. (6.5) ψ_n by the bound-state wave function ψ_{0n} normalized to unity. As a result, we obtain

$$\delta-\delta_0=-\frac{mk}{2\pi}\int d\mathbf{r}\psi_0V^{(2)}\psi_0,\qquad E_n-E_n{}^0=\int d\mathbf{r}\psi_{0n}V^{(2)}\psi_{0n}, \tag{6.7}$$

where $V^{(2)}$ is taken in the form (2.12). In fact, owing to the condition (1.1), we can go over in this last expression to the long-wave limit: the characteristic values of q and k are of the order of $(m\varepsilon)^{1/2}$, while the characteristic momentum of the internal motion in the complex is $\sim 1/R$, and is significantly higher. Let us also note that, when the condition (6.6) is fulfilled, the purely structural effects, which are responsible for the deviation of $V^{(1)}$ from the Coulomb potential (or from zero in the case of a neutral complex), are weak, which allows us to replace ψ_0 by the Coulomb (or, respectively, the free) wave function; the same thing applies to the function ψ_{0n} (see below).

The formulas (6.7) correspond, of course, simply to a perturbation theory in terms of the polarization potential. For an uncharged complex the first of them gives, after the substitution $\psi_0 = \sin(kr)/kr$, the following difference:

$$\delta-\delta_0=\frac{m^{3/2}e^2k}{2^{1/2}\pi}\int\limits_{k^2/2m}^{\infty}\frac{d\omega}{\omega^{1/2}}\operatorname{Im}\alpha(\omega)\varphi\left(\frac{k}{(2m\omega)^{1/2}}\right), \tag{6.8}$$

where

$$\varphi(x)=(1-x^2)^{-1/2}\left\{1+\frac{\operatorname{arctg}[2x(1-x^2)^{1/2}(1-2x^2)^{-1}]}{2x(1-x^2)^{1/2}}\right\}^2.$$

It turns out that, when the condition

$$1/(m\varepsilon)^{1/2}a_0\ll 1, \tag{6.9}$$

which is more rigid than (6.6), is fulfilled the formula (6.8) is valid in the case of a charged complex as well, if we further introduce in its right-hand side the amplification factor

$$A^2=\psi_0{}^2(0)=z/(e^z-1),\qquad z=2\pi/ka_0,$$

which is equal to the ratio of the probabilities of finding the charged particles to be coincident in the presence and absence of the Coulomb interaction. The point is that the characteristic value of r in (6.7) is, as can be seen from (2.12), of the order of $(m\varepsilon)^{-1/2}$, and this quantity[5)] is, when (6.9) is fulfilled, small compared to a_0, i.e., lies in the region where $\psi_0 = A\sin(kr)/kr$. For small values of k

$$\delta-\delta_0=\frac{2\sqrt{2}}{\pi}m^{3/2}e^2A^2k\int\limits_0^{\infty}\frac{d\omega}{\omega^{1/2}}\operatorname{Im}\alpha(\omega). \tag{6.10}$$

Turning to the second formula in (6.7), i.e., the formula for the level shift, and assuming that the condition (6.9) is fulfilled, we easily arrive at the expression

$$E_n-E_n{}^0=4(2m)^{1/2}e^2\psi_{0n}{}^2(0)\int\limits_0^{\infty}\frac{d\omega}{\omega^{1/2}}\operatorname{Im}\alpha(\omega), \tag{6.11}$$

which is valid for levels with $E_n^0\ll\varepsilon$. The last condition follows directly from (6.9). Notice that the last two relations contain a moment of the quantity $\operatorname{Im}\alpha$ (or, which is almost the same thing, a moment of the oscillator-strength density) lying between the moments entering into the polarization potentials in the regions (1.3a) and (1.3b) (see Sec. 3). This is due to the fact that the quantities (6.10) and (6.11) have the character of space integrals.

Speaking of scattering by the polarization potential, we should note that this potential, in behaving like $1/r^2$ in the region (1.3b), has the features of a long-range potential from the point of view of potential scattering theory.[1] Thus, the total particle-complex scattering cross section, which diverges logarithmically in the case when the law $1/r^2$ obtains in all space, has in our case a singularity of the type $\ln(1/m\varepsilon R^2)$.

In conclusion of this section, let us briefly discuss the quantities δ_0 and E_n^0 corresponding to the potential $V^{(1)}$ (see (6.1)). It is easy to verify that the inequalities assumed above lead to a situation in which the deviations of these quantities from the purely Coulomb values are small and can be computed with the aid of perturbation theory in terms of the difference $V^{(1)} - V^c$ (the index c denotes the Coulomb interaction). In fact, the situation reduces to one in which, if the form factor (see Sec. 2) for small k is

$$F(k)=e_0-k^2Q/6+\dots,\qquad Q=\sum_i e_i\langle\rho_i{}^2\rangle,$$

then the corresponding corrections for the structure of the complex have the form

$$\delta_0-\delta^c=kmeA^2\dot{Q}/3,\qquad E_n{}^0-E_n{}^c=(2\pi eQ/3)\psi_{0n}{}^2(0), \tag{6.12}$$

which is well known in the literature (see, for example, Ref. 4).

7. INTERACTION OF A CHARGED PARTICLE WITH A DEUTERON

In this final section we consider as a specific example the system "particle + deuteron," where by particle we mean any lepton or meson (electron, muon, pion, etc.) lighter than the nucleon. The results presented below equally apply to systems in which the deuteron is replaced by a weakly bound two-particle complex (when, of course, all the necessary conditions, i.e., (1.1), (6.6), and (6.9), are fulfilled). For the deuteron itself, these conditions are fulfilled quite well if we consider it in combination with the above-enumerated particles.

The internal dynamics of the deuteron can be described with the aid of the Hamiltonian $H_d = -\Delta_\rho/4M$, which corresponds to free motion, with energy $E_d = -\varkappa^2/M$ when the following boundary condition is imposed at $\rho = 0$:

$$(\rho\Phi)'/\rho\Phi=-2\varkappa,$$

where M is the nucleon mass.[6] the quantity $\varkappa = 0.23\ \mathrm{fm}^{-1}$ serves as the only characteristic of the internal structure of the deuteron, determining its radius $R\sim\varkappa^{-1}$, the characteristic excitation energy $\varepsilon\sim\varkappa^2/M$, etc. The deuteron polarizability $\alpha(\omega)$ can be found from the long-wave limit of the formula (2.8) by writing it in the operator form

$$e_0{}^2\langle\rho_z[(H_d-E_d-\omega-i\delta)^{-1}+(H_d-E_d+\omega+i\delta)^{-1}]\rho_z\rangle$$

and using it to average the deuteron wave function

$$\Phi(\rho)=\frac{1}{2}\left(\frac{\varkappa}{2\pi}\right)^{1/2}\frac{\exp(-2\varkappa\rho)}{\rho},$$

satisfying the above boundary condition. Let us leave out the unwieldy formula for the polarizability itself, and give the expression for its imaginary part:

$$\operatorname{Im}\alpha(\omega)=\frac{2e_0{}^2\varkappa}{3M^3\omega^4}(M\omega-\varkappa^2)^{3/2}\theta(M\omega-\varkappa^2). \qquad (7.1)$$

Hence we find with the aid of the sum rules given in Sec. 3 that

$$\alpha(0)=Me_0{}^2/32\varkappa^4, \qquad \langle d^2\rangle=e_0{}^2/8\varkappa^2. \qquad (7.2)$$

Let us further give the expression for the form factor of the deuteron:

$$F(k)=\frac{2e_0\varkappa}{k}\ \operatorname{arctg}\left(\frac{k}{2\varkappa}\right).$$

Let us go over to the expressions for the light particle-deuteron interaction potential. The potential $V^{(1)}(r)$ is obtained from (2.7) by replacing one of the form factors by the particle charge, and has the form

$$V^{(1)}(r)=\frac{ee_0}{r}-\frac{ee_0}{r}[\exp(-2\varkappa r)-2\varkappa rE_i(2\varkappa r)]. \qquad (7.3)$$

Here the first term represents the Coulomb interaction of the particle with the deuteron as a whole; the second factor is the structural correction, which reflects the charge distribution over the deuteron; and E_i is an exponential integral function. When the condition (1.1), which has the simple form $m\ll M$ (m is the particle mass) in the case under consideration, is fulfilled, the polarization potential $V^{(2)}(r)$ is given in the region $r\gg(M/m)^{1/2}/\varkappa$ by the formula (1.2), i.e.,

$$V^{(2)}(r)=-Me^2e_0{}^2/64\varkappa^4r^4, \qquad (7.4)$$

and in the region $1/\varkappa\ll r\ll(M/m)^{1/2}/\varkappa$ by the formula (3.2):

$$V^{(2)}(r)=-me^2e_0{}^2O/24\varkappa^2r^2. \qquad (7.5)$$

We further give, for completeness, the interaction potential for two deuterons:

$$V^{(2)}(r)=-Me_0{}^4/32\varkappa^4r^4-0.0012Me_0{}^4/\varkappa^6r^6, \qquad (7.6)$$

where the first term corresponds to the polarization of each of the deuterons by the charge of the other deuteron and the second term has the ordinary van der Waals form.

Further, let us give the expressions for the polarization corrections to the particle-deuteron scattering phase shift (for small momenta) and the ground state energy of the corresponding atom. The substitution of (7.1) into (6.10) and (6.11) yields

$$\delta-\delta_0=\frac{32}{105\pi}\left(\frac{M}{2m}\right)^{1/2}\frac{kA^2}{a_0{}^2\varkappa^3},$$

$$E_n-E_n{}^0=\frac{64}{105\pi}\left(\frac{M}{2m}\right)^{1/2}\frac{e^2}{a_0{}^4\varkappa^3}. \qquad (7.7)$$

For the systems (μd), (πd), etc., the contributions of these expressions are quite substantial. Let us again give the expressions (6.12), which give the quantities δ_0 and E_n^0 themselves:

$$\delta_0-\delta_c=-kA^2/24\varkappa^2a_0, \qquad E_n{}^0-E_{nc}=e^2/12a_0{}^3\varkappa^2. \qquad (7.8)$$

Notice that the corrections of second order in the potential $V^{(1)}$ can be neglected, since they are smaller than the quantities (7.7) by a factor of the order of $(M/m)^{1/2}$.

In conclusion, we give without derivation the expressions for the polarization corrections to the phase shift and the energy in the general case when the ratio of the particle mass to the deuteron mass is arbitrary (but the condition (6.9) is fulfilled):

$$\delta-\delta_0=kA^2\chi(\sigma)/16\pi a_0{}^2\varkappa^3, \qquad E_n-E_n{}^0=e^2\chi(\sigma)/8\pi a_0{}^4\varkappa^3, \qquad (7.9)$$

where $\sigma=(m/2M)^{1/2}$ and

$$\chi(\sigma)=$$

$$(3+\sigma^4)(1+\sigma^2)^2\operatorname{arctg}\sigma/\sigma^8-(15+25\sigma^2+13\sigma^4-5\sigma^6)/5\sigma^7-\pi/2.$$

In deriving these formulas we used the general expression (2.12), which contains the generalized polarizability, i.e., multipoles of all orders. Applying (7.9) to proton scattering by the deuteron, we see that the corresponding polarization correction to the scattering phase shift is extremely small. In fact, the structural correction due to the potential $V^{(1)}$ is also small (see Ref. 3).

We are grateful to V. L. Ginzburg and the participants of the seminar conducted by him, as well as to Yu. N. Demkov, L. I. Ponomarev, and I. I. Sobel'man, for useful discussions.

---

[1] Of special interest are systems in which the role of the complex is played by a quark system: a nucleon, pion, quarkonium, etc.

[2] Concerning the interference of the Coulomb and short-range interactions between the complexes, see Ref. 3.

[3] In the formula (2.12) e is the particle charge. Below we shall, in considering the system "particle + complex," drop the indices 1 and 2.

[4] This does not apply to quark systems.

[5] Notice that this characteristic r value is, on account of (1.1), large compared to the radius of the complex. Therefore, the formulas given are insensitive to short-range interactions such as nuclear forces, exchange effects, etc.

[6] The occurrence of the "superfluous" powers of two in these formulas and those given below is due to the fact that the coordinate describing the internal structure of the deuteron is, in accordance with the variables adopted in Sec. 2, not the distance between the nucleons, as is usually the case, but the distance from the center of mass of the deuteron to the nucleon, which distance is smaller than the internucleon separation by a factor of two.

[1] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Nauka, Moscow, 1974 (Eng. Transl., Pergamon, Oxford, 1977).

[2] V. N. Efimov, Yad. Fiz. **12**, 1080 (1970) [Sov. J. Nucl. Phys. **12**, 589 (1971)].

[3] D. A. Kirzhnits and F. M. Pen'kov, Zh. Eksp. Teor. Fiz. **82**, 657 (1982) [Sov. Phys. JETP **55**, 393 (1982)].

[4] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Systems, Springer Verlag, Berlin, 1958 (Russ. Transl., Fizmatgiz, Moscow, 1960).

Translated by A. K. Agyei