Coherent scattering of high-energy photons in a Coulomb field

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A quasiclassical approach is developed for the description of quantum-electrodynamics processes in a Coulomb field at high energies. An expression for the amplitude of coherent scattering of high-energy photons is obtained in simple form that makes it possible to determine the dependence of the total cross section of the process on the charge of the Coulomb center.

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Coherent scattering of a phonon in a Coulomb field via virtual electron-positron fields (Delbrück scattering[1]) is one of the few nonlinear quantum-electrodynamics processes that are directly observable in experiment (see Refs. 2 and 3). The most favorable is the situation when $\omega \gg m$ (ω is the photon frequency, m is the electron mass, and $\hbar = c = 1$).

A survey of the numerous attempts at a theoretical description of the process can be found in Ref. 2, but only Cheng and Wu[4–6] have made substantial progress in the solution of the problem, by summing in a definite approximation the diagrams of the perturbation theory in terms of the parameter $Z\alpha$ [$Z|e|$ is the charge of the nucleus, $\alpha = e2 = 1/137$ is the fine-structure constant, of the interaction with the Coulomb field. It turned out (see Refs. 4 and 5) that allowance for the corrections in terms of e is the electron charge] changes the result radically at $Z\alpha \sim 1$ compared with the first nonvanishing perturbation-theory approximation.

In the present paper the amplitude of the photon scattering at $\omega \gg m$ is obtained by a method that differs substantially from that developed by Cheng and Wu. The approach is based on the use of an integral representation obtained by us[7] for the Green function of an electron in a Coulomb field, and on an explicit allowance for the classical character of the motion of high-energy charged particles. The method proposed can be used also to solve other problems in a Coulomb field. In the case considered here this approach leads to a much simpler expression for the amplitude of the process and makes it possible to calculate its total cross section.

Let an initial photon with momentum $\mathbf{k}_1 = \omega \mathbf{v}_1$ produce at the point $\mathbf{r}_1$ a pair of virtual particles that is transformed at the point $\mathbf{r}_2$ into a photon with momentum $\mathbf{k}_2 = \omega \mathbf{v}_2$ ($|\mathbf{k}_1| = |\mathbf{k}_2| = \omega$). The main contribution to the amplitude is made by the particle-pair energy $\varepsilon_1 \sim \varepsilon_2 \sim \omega$. We put $\Delta = \mathbf{k}_2 - \mathbf{k}_1$, and then the uncertainty relation gives $\tau \sim \omega(m^2 + \Delta^2)^{-1}$ for the lifetime of the virtual pair (i.e., for the length of the loop, and $\rho = 1/\Delta$ for the characteristic impact parameter. It follows therefore that in the momentum-transfer region

$$m^2/\omega \ll \Delta \ll \omega \qquad (1)$$

we have the ratio $\rho/\tau \ll 1$, i.e., the angles between the vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{r}_2$ and $-\mathbf{r}_1$ are small. The contribution made to the total cross section by momentum transfers that do not satisfy condition (1) is suppressed as m^2/ω^2. It must also be borne in mind that in scattering by atoms the point-charge approximation is valid if $r_c^{-1} \ll \Delta \ll R^{-1}$, where R is the radius of the nucleus, r_c is the screening radius of the nucleus ($r_c \sim (m\alpha)^{-1} Z^{-1/3}$ in the Thomas-Fermi model). This restriction ensures at $\omega \gtrsim 100$ MeV satisfaction of the condition (1), and we shall therefore consider below (except in the Appendix) only this momentum-transfer region. The momentum-transfer region $\Delta \lesssim m^2/\omega$ was investigated in Ref. 6 and called for a special treatment. In our approach we can describe the process in unified manner at all momentum transfers that satisfy the condition. To illustrate this, we calculate in the Appendix the amplitude at $\Delta = 0$.

In the Furry representation, the amplitude of the Delbrück scattering is

$$M = 2i\alpha \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{i(\mathbf{k}_1\mathbf{r}_1 - \mathbf{k}_2\mathbf{r}_2)} \int d\varepsilon_1 d\varepsilon_2 \times \mathrm{Tr}[\hat{e}_1^* G(\mathbf{r}_1, \mathbf{r}_2|\varepsilon_2) \hat{e}_2 G(\mathbf{r}_2, \mathbf{r}_1|\varepsilon_1)] \delta(\omega - \varepsilon_1 + \varepsilon_2), \qquad (2)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the photon polarization vectors, $\hat{e} = e_\mu \gamma_\mu = -(\mathbf{e}\cdot\boldsymbol{\gamma})$, γ^μ are Dirac matrices, and G is the Green function of an electron in a Coulomb field. The integral with respect to ε_1 and ε_2 passes under the real axis in the left half-plane and over it in the right half-plane of the complex variable ε.

It is convenient to carry out the calculation in terms of helical amplitudes. We choose the polarization vectors in the form $\mathbf{e}_{1,2}^{(\pm)} = ([\boldsymbol{\lambda} \times \mathbf{v}_{1,2}] \pm i\boldsymbol{\lambda})/\sqrt{2}$, where $\boldsymbol{\lambda} = [\mathbf{v}_1 \times \mathbf{v}_2]/|[\mathbf{v}_1 \times \mathbf{v}_2]|$. There exist two independent amplitudes $M_1 = M_{++} = M_{--}$ and $M_2 = M_{+-} = M_{-+}$. In terms of linear polarization, by virtue of parity conservation, the amplitude differs from zero only when the polarization vectors of the initial and final phonon both lie in the scattering plane ($M_\parallel$) or are perpendicular to it ($M_\perp$). In this case $M_1 = \frac{1}{2}(M_\parallel + M_\perp)$, $M_2 = \frac{1}{2}(M_\parallel - M_\perp)$. We represent the δ function in (2) in the form

$$\delta(\omega - \varepsilon_1 + \varepsilon_2) = \frac{i}{2\pi}\left[\frac{1}{\omega - \varepsilon_1 + \varepsilon_2 + i0} - \frac{1}{\omega - \varepsilon_1 + \varepsilon_2 - i0}\right]. \qquad (3)$$

With the first term in (3) it is possible to deform the contour of the integration with respect to ε_1 and ε_2 in (2) by taking the analytic functions of the function G into account (see, e.g., Ref. 7), in such a way that the integrals with respect to ε_1 and ε_2 encircle the right- and left-hand cuts, respectively. In addition, when the contour with respect to ε_1 is deformed the discrete spectrum makes a contribution that can be neglected at $\omega \gg m$. With the second term in (3), the contours of integration with respect to ε_1 and ε_2 will encircle respectively

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the left- and right-hand cuts; the quantity $\omega - \varepsilon_1 + \varepsilon_2$ turns out to be large, and the contribution of this term can be neglected. Carrying out these transformations and making the substitution $\varepsilon_2 \to \varepsilon_2$ we have

$$M = \frac{\alpha}{\pi} \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{i(\mathbf{k}_1\mathbf{r}_1 - \mathbf{k}_2\mathbf{r}_2)} \int_m^\infty \frac{d\varepsilon_1 d\varepsilon_2}{\omega - \varepsilon_1 - \varepsilon_2 + i0} \times \mathrm{Tr}[\hat{e}_1^* \delta G(\mathbf{r}_1, \mathbf{r}_2 | -\varepsilon_2) \hat{e}_2 \delta G(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon_1)]. \tag{4}$$

Here δG is the discontinuity of the Green function on the cut, $\delta G(\varepsilon) = G^{(+)}(\varepsilon) - G^{(-)}(\varepsilon)$, where $G^{(\pm)}(\varepsilon)$ defines the Green function in the upper and lower half-planes of the complex variable ε, respectively. Expression (4) corresponds to the noncovariant-perturbation-theory diagram that makes the main contribution at $\omega \gg m$. By virtue of the momentum conservation law $M(Z=0) \propto \delta(\mathbf{k}_1 - \mathbf{k}_2)$, i.e., for the case considered by us we have $\Delta \neq 0, M(Z=0) = 0$. It is convenient to subtract from the integrand of M in (4) its value at $Z = 0$. It is precisely for this difference that it is correct to state that the angles between the vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{r}_2$ and $-\mathbf{r}_1$, which make the main contribution to the integral, are small. We assume hereafter that this subtraction had been made; we shall carry it out explicitly in the final answer.

The characteristic value of the angular momentum $l \sim p\omega \sim \omega/\Delta$ turns out according to (1) to be large, and the quasiclassical approximation can be used. For the Green function G [see Eq. (19) of Ref. 7] this means that the contribution to the sum over l is made by $l \gg 1$, and the quantity $\nu = [l^2 - (Z\alpha)^2]^{1/2}$ in the expression for G can be expanded in terms of the parameter $(Z\alpha)^2/l^2$. In the region (1) it suffices to use the zeroth approximation, i.e., the substitution $\nu \to l$. The sum over l can then be calculated with the aid of Eq. (24) of Ref. 7, and we obtain an expression for the quasiclassical Green function of an electron in a Coulomb field:

$$G^{(\pm)}(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon) = \pm \frac{i\varkappa^2}{4\pi} \int_0^\infty \frac{dt}{\mathrm{sh}^2 \varkappa t} \exp\{\pm i[2Z\alpha\varepsilon t + \varkappa(r_2 + r_1)\,\mathrm{cth}\,\varkappa t]\} \cdot \times \left\{ \left[\gamma^0\varepsilon + m \pm \frac{\varkappa}{2}(\boldsymbol{\gamma}, \mathbf{n}_1 - \mathbf{n}_2)\,\mathrm{cth}\,\varkappa t \right] J_0(y) + \frac{iJ_1(y)}{y} \left[\left(\frac{\varkappa^2(r_2 - r_1)}{2\,\mathrm{sh}^2 \varkappa t} + Z\alpha m \gamma^0 \right) (\boldsymbol{\gamma}, \mathbf{n}_1 + \mathbf{n}_2) \mp Z\alpha\varkappa\,\mathrm{cth}\,\varkappa t \gamma^0 (1 - (\boldsymbol{\gamma}\mathbf{n}_2)(\boldsymbol{\gamma}\mathbf{n}_1)) \right] \right\}, \tag{5}$$

where

$$\varkappa = (\varepsilon^2 - m^2)^{1/2}, \quad \mathbf{n}_{1,2} = \mathbf{r}_{1,2}/r_{1,2},$$
$$y = \varkappa[2r_1r_2(1 + \mathbf{n}_1\mathbf{n}_2)]^{1/2}/\mathrm{sh}\,\varkappa t,$$

J_0 and J_1 are Bessel functions. We note that in the description of the momentum-transfer region $\Delta \lesssim m^2/\omega$ we must retain in the expression for the Green function the first term of the expansion in terms of the parameter $(Z\alpha)^2/l^2$. The corresponding corrections are given in the Appendix. In expression (5), $\varkappa_1$ and $\varkappa_2$ are real and vary from 0 to ∞. The jump $\delta G(\varepsilon)$ is determined by (5) if we use in it the upper sign and integrate with respect to t from $-\infty$ to ∞. It is convenient to change over to the variable $\varkappa t$ in the integral with respect to t, make the substitution $\mathbf{r}_1 \to -\mathbf{r}_1$ in (4), and change from integration with respect to ε_1 and ε_2 to integratin with respect to $\varkappa_1$ and $\varkappa_2$. We next make in (4) the following change of variables: $r_1 = Rv, r_2 = R/v, \varkappa_{1,2} = p_{1,2}/R$. The integral with respect to R takes then, at the accuracy required, the form

$$\int_0^\infty \frac{dR \exp[-i\omega R(v\mathbf{v}_1\mathbf{n}_1 + \mathbf{v}_2\mathbf{n}_2/v)]}{\omega R - (p_1 + p_2)[1 + m^2(p_1 + p_2)^2/2\omega^2 p_1 p_2] + i0}.$$

The integration with respect to R can be extended here from $-\infty$ to ∞ (the contribution of negative R has a power-law smallness in the parameter m/ω), and the integral can be calculated in elementary fashion. We calculate the trace and expand the integrand in (4) in powers of the small angles. It is convenient here to direct the axis of the spherical system along $\mathbf{v}_1 + \mathbf{v}_2$. Taking the smallness of the angles into account we have $d\Omega_{1,2} \approx \theta_{1,2} d\theta_{1,2} d\varphi_{1,2} = d^2\boldsymbol{\theta}_{1,2}$, with $\boldsymbol{\theta}_{1,2} \cdot (\mathbf{v}_1 + \mathbf{v}_2) = 0$. The Bessel functions depend on the vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ only in the combination $|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|$. We make the change of variables $\boldsymbol{\theta} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1, \boldsymbol{\xi} = \boldsymbol{\theta}_1 v + \boldsymbol{\theta}_2/v$, after which the integral with respect to $d^2\xi$ can be evaluated. An analysis of expression (4) shows that the main contribution to the integral is made by large positive t_1 and t_2 [the integration in the expressions for $\delta G(\varepsilon_1)$ and $\delta G(-\varepsilon_2)$ is carried out with respect to t_1 and t_2, respectively, cf. (5)].

We introduce the variables $x_1 = \frac{1}{2}\exp t_1$, $x_2 = \frac{1}{2}\exp t_2$ and carry out the expansion with allowance for the fact that $x_{1,2} \gg 1$. It is convenient to integrate in the expression for M_2 by parts with respect to x_1 and x_2, so as to leave in the pre-exponential factor only the Bessel functions J_1. We obtain

$$M_{1,2} = -\frac{\alpha}{\pi\omega} \int_0^\infty \frac{dx_1}{x_1^3} \frac{dx_2}{x_2^3} \left(\frac{x_1}{x_2}\right)^{2iZ\alpha} \int_0^\infty \frac{dp_1 dp_2 (p_1p_2)^2}{p_1 + p_2} \int_0^\infty \frac{dv}{1 + v^2} \int d^2\theta e^{i\Phi} T_{1,2},$$
$$\Phi = \frac{1}{2}\left(v + \frac{1}{v}\right)\left[\frac{p_1}{x_1^2} + \frac{p_2}{x_2^2} + \frac{(p_1 + p_2)}{(v + 1/v)^2}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\Delta}}{\omega}\right)^2 - \frac{m^2}{\omega^2}\frac{(p_1 + p_2)^3}{p_1p_2}\right] \tag{6}$$
$$T_1 = J_0(y_1)J_0(y_2)\left[\frac{m^2}{\omega^2}\frac{(p_1 + p_2)^4}{(p_1p_2)^2} + \frac{1}{2}\left(\theta^2 - \frac{\Delta^2}{\omega^2}\right) - \frac{1}{x_1^2} - \frac{1}{x_2^2} + \frac{4iv}{(1 + v^2)(p_1 + p_2)} + \frac{1}{2}\left(\frac{1 - v^2}{1 + v^2}\right)^2\left(\boldsymbol{\theta} - \frac{\boldsymbol{\Delta}}{\omega}\right)^2\right] + i\left(\boldsymbol{\theta}, \boldsymbol{\theta} - \frac{\boldsymbol{\Delta}}{\omega}\right) \cdot \times \left\{\frac{J_0(y_1)J_1(y_2)}{y_2}\left[\frac{p_2(1 - v^2)^2}{2v(1 + v^2)x_2^2} + Z\alpha\right] + \frac{J_0(y_2)J_1(y_1)}{y_1}\left[\frac{p_1(1 - v^2)^2}{2v(1 + v^2)x_1^2} - Z\alpha\right]\right\}, \quad y_{1,2} = \theta\frac{p_{1,2}}{x_{1,2}},$$

$$T_2=\frac{2J_1(y_1)J_1(y_2)}{y_1y_2}\left\{\frac{\Delta^2}{\omega^2}\frac{p_1p_2}{(x_1x_2)^2}+\frac{2vZ\alpha}{1+v^2}\left(\frac{p_1}{x_1^2}-\frac{p_2}{x_2^2}\right)\left(\frac{\Delta}{\omega},\frac{\Delta}{\omega}-\theta\right)+\frac{4v^2(Z\alpha)^2}{(1+v^2)^2}\left[\frac{2}{\Delta^2}[\theta\times\Delta]^2-\left(\theta-\frac{\Delta}{\omega}\right)^2\right]\right\}.$$

We make the change of variables

$$p_{1,2}\to p_{1,2}\left(v+\frac{1}{v}\right)\Big/\theta^2,\quad x_{1,2}\to x_{1,2}\left(v+\frac{1}{v}\right)\Big/\theta,$$

and carry out inversion in the space of the vectors θ ($\theta\to 1/\theta$), after which the Bessel functions cease to depend on θ and the integral with respect to $d^2\theta$ is easily evaluated. In the subsequent calculation of the amplitude M_1 we make the substitutions

$$p_1=E^2t,\quad p_2=E^2/t,\quad x_1=xEe^{\tau/2},\quad x_2=xEe^{-\tau/2},$$

after which we integrate first with respect to E and then with respect to x, using the relations (Ref. 8, pp. 732 and 744 of Russ. original)

$$\int_0^\infty dx\,xe^{icx^2}J_\nu(ax)J_\nu(bx)=\frac{ie^{i\pi\nu/2}}{2c}J_\nu\left(\frac{ab}{2c}\right)\exp\left[-\frac{i(a^2+b^2)}{4c}\right],$$

$$\int_0^\infty dx\,e^{icx}J_0(x)=\frac{i\,\mathrm{sign}\,c}{(c^2-1)^{1/2}},\tag{7}$$

$$\int_0^\infty dx\,e^{icx}J_1(x)=1-\frac{|c|}{(c^2-1)^{1/2}},\quad |c|>1$$

and the relations obtained from them by differentiation with respect to the parameter. In the calculation of the amplitude M_2 we make the change of variables

$$x_1=e^{\tau/2}/x,\quad x_2=e^{-\tau/2}/x,\quad p_1=Et,\quad p_2=E/t$$

and integrate with respect to x and then with respect to E, using (7). We obtain ultimately for the amplitudes

$$M_{1,2}=i\frac{32\alpha\omega}{\Delta^2}\int_0^\infty d\tau\int_0^\infty\frac{dv\,v}{(1+v^2)^2}\int_0^\infty\frac{dt}{t}S_{1,2},$$

$$S_1=\sin^2(Z\alpha\tau)\frac{B}{D^3}\left\{(1-B)\,\mathrm{ch}\,\tau-d+2B\left[1-\frac{t^2}{(1+t^2)^2}\right]\left[\frac{3(1-B)d}{D^2}+2\,\mathrm{ch}\,\tau\right]\right\},$$

$$S_2=\frac{2(Z\alpha)^2\cos(2Z\alpha\tau)}{d(t+1/t)^2}\left[e^{-\tau}-\frac{2B}{R}-\frac{gB^2}{RD(D+d)}\right]$$

$$+\frac{3B^2\sin^2(Z\alpha\tau)}{dD^5}(g+d\,\mathrm{ch}\,\tau)+\left[\frac{2\sin^2(Z\alpha\tau)}{d^3}-\frac{Z\alpha\sin(2Z\alpha\tau)}{d^2}\frac{te^{-\tau}+e^{\tau}/t}{t+1/t}\right]\left[\frac{1}{R}+\frac{B}{D}\left(1+\frac{\mathrm{ch}\,\tau}{R}\right)+\frac{gB^2}{D^3}\right],$$

$$d=\mathrm{ch}\tau+\frac{1}{2}(t^2e^{-\tau}+e^{\tau}t^{-2}),\quad B=\frac{m^2}{\Delta^2}\left(v+\frac{1}{v}\right)^2\left(t+\frac{1}{t}\right)^2,\tag{8}$$

$$g=(1-B)\,\mathrm{sh}^2\,\tau-2d\,\mathrm{ch}\,\tau,\quad D=[d^2+g(1-B)]^{1/2},\quad R=D+d-(1-B)\,\mathrm{ch}\,\tau.$$

We obtain now the asymptotic form of the amplitudes M_1 and M_2 at $m\ll\Delta\ll\omega$. In this limit, the main contribution to M_1 is made by the region of integration with respect to t in (8) near $t_0=\exp(\tau/2)$. Assuming $t_0=\exp(\tau/2)(1+x)$, we have $D\approx 2[x^2+B\cosh^2(\tau/2)]^{1/2}$. In the remaining functions we can put $t=60$, after which the integrals with respect to x and v are calculated. We obtain

$$M_1(\Delta\gg m)\approx i\frac{2\alpha\omega}{\Delta^2}\int_0^\infty\frac{d\tau\sin^2(Z\alpha\tau)}{\mathrm{ch}^4(\tau/2)}\mathrm{ch}\,\tau$$

$$=-i\frac{8\alpha\omega}{3\Delta^2}\left\{\frac{2\pi Z\alpha}{\mathrm{sh}(2\pi Z\alpha)}[1-2(Z\alpha)^2]-1\right\}.\tag{9}$$

The calculation of the contribution made to the asymptotic form of the amplitude M_2 by the term proportinal to D^{-5} in S_2 [Eq. (8)] is carried out similarly. In the remaining terms of S_2 we assume $B=0$. After straightforward but awkward calculations we get

$$M_2(\Delta\gg m)\approx-i\frac{8\alpha\omega}{\Delta^2}(Z\alpha)^2\int_0^\infty d\tau\frac{\cos(2Z\alpha\tau)}{\mathrm{sh}\,\tau}\left(\mathrm{ch}\,\tau-\frac{\tau}{\mathrm{sh}\,\tau}\right).\tag{10}$$

The integral with respect to τ in (10) is calculated with the aid of the relation (Ref. 8, p. 376 of Russ. orig.)

$$\int_0^\infty dx\,e^{-\beta x}\left(\frac{1}{x}-\mathrm{cth}\,x\right)=\psi\left(\frac{\beta}{2}\right)-\ln\frac{\beta}{2}+\frac{1}{\beta},\tag{11}$$

where $\psi(x)=d\ln\Gamma(x)/dx$. We obtain ultimately

$$M_2(\Delta\gg m)=-i\frac{8\alpha\omega}{\Delta^2}(Z\alpha)^2[(Z\alpha)\,\mathrm{Im}\,\psi'(1-iZ\alpha)-1].\tag{12}$$

The asymptotic forms (9) and (12) agree with the result of Cheng and Wu (see Ref. 4).

We obtain now the asymptotic forms of the amplitudes M_1 and M_2 at $m^2/\omega\ll\Delta\ll m$. To calculate the asymptotic form of M_1 we divide the region of integration with respect to τ into two: from 0 to τ_0 (M_{11}) and from τ_0 to ∞ (M_{12}), where

τ_0 is such that $B^{-1/2} \ll \tau_0 \ll 1$. We then assume $D \approx B \sinh\tau$ in the integrand of M_{12} and obtain after elementary transformations

$$M_{12}=i\frac{28\alpha\omega}{9m^2}\int\limits_{\tau_0}^{\infty}\frac{d\tau\,\mathrm{ch}\,\tau}{\mathrm{sh}^3\,\tau}\sin^2(Z\alpha\tau). \tag{13}$$

Using (11) we have

$$M_{12}=-i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\mathrm{Re}\,\psi(1-iZ\alpha)+C+\ln 2\tau_0-\frac{3}{2}\right], \tag{14}$$

where $C = 0.577...$is the Euler constant. We expand in the expression for M_{11} with allowance for the smallness of τ. In this case

$$D\approx[B(t+1/t)^2+B^2\tau^2]^{1/2}.$$

After integration we obtain

$$M_{11}=i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln\frac{2\tau_0 m}{\Delta}-\frac{11}{21}\right]. \tag{15}$$

Adding (14) and (15), we ultimately obtain

$$M_1(\Delta\ll m)=i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln\frac{m}{\Delta}+\frac{41}{42}-C-\mathrm{Re}\,\psi(1-iZ\alpha)\right]. \tag{16}$$

In the limit considered, the contribution to the amplitude M_2 is made by small $\tau \sim B^{-1/2}$, and the principal term is the first one [$\propto (Z\alpha)^2$] in S_2. Expanding in powers of τ and transforming to the variable $x = \tau B^{1/2}/(t + 1/t)$, the integrals with respect to v and t become elementary and we have

$$M_2(\Delta\ll m)\approx -i\frac{8\alpha\omega}{9m^2}(Z\alpha)^2\int\limits_0^{\infty}dx\left[x-(x^2+1)^{1/2}+\frac{1}{2(x^2+1)^{1/2}}\right]=i\frac{2\alpha\omega}{9m^2}(Z\alpha)^2. \tag{17}$$

The asymptotic forms (16) and (17) agree with the results of Cheng and Wu (see Refs. 5 and 6).

For an unpolarized initial photon the differential scattering cross section is

$$\frac{d\sigma}{dx}=\frac{m^2}{16\pi\omega^2}(|M_1|^2+|M_2|^2),\quad x=\frac{\Delta^2}{m^2}$$

and does not depend on the photon frequency. Expression (8) for the amplitudes M_1 and M_2 was found to be very convenient for numerical calculations, since all the integrals in it converge rapidly. Figure 1 shows a plot of $\sigma_0^{-1}d\sigma/dx$ ($\sigma_0 = (Z\alpha)^4 r_e^2/16\pi, r_e = \alpha/m, r_e^2/16\pi \approx 1.58$ mb) vs the momentum transfer Δ for $Z = 1$ (curve 1), $Z = 47$ (curve 2), and $Z = 92$ (curve 3). The muon contribution can become noticeable only at large momentum transfers. The corresponding amplitudes are obtained from (8) by the substitution $m \to m_\mu$. At $Z = 92$ this contribution increases the differential cross section for $\Delta = 10$, 20, and 30 MeV by 2, 6.5, and 12.4% respectively. The differential cross section obtained by us agrees with the results of the numerical calculations carried out by Willutzki[3] using the equations of Ref. 4 for uranium and gold, and with the experimental values obtained for

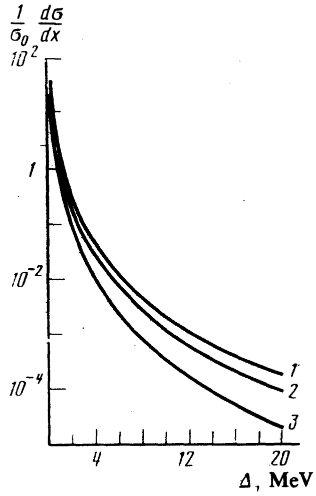

FIG. 1. Differential scattering cross section vs Δ at $Z = 1$ (curve 1), $Z = 47$ (curve 2), and $Z = 02$ (curve 3).

these cross sections in Ref. 3. Figure 2 is a plot of the total cross section (in units of σ_0) vs the charge of the Coulomb center. The only result obtained so far for the total cross section of the process at $\omega \gg m$ is that of Ref. 9 [see Eq. (29) there] and is of the form

$$\sigma=2^5(7/9)^2\sigma_0\approx 19.4\sigma_0.$$

Since the Born approximation was used in Ref. 9, this result must be compared with that shown in Fig. 2 at the point $Z = 1$, where $\sigma \approx 54\sigma_0$. This discrepancy is due to the fact, indicated in Ref. 5 (see also Ref. 2), that the approach used in Ref. 9 is incorrect. It follows from Figs. 1 and 2 that the Coulomb corrections decrease the cross section substantially. If the initial photon is not polarized, partial polarization of the final photon in the scattering plane is produced. The corresponding Stokes parameters are

$$\xi_3=2\,\mathrm{Re}\,(M_1M_2^*)/(|M_1|^2+|M_2|^2),\quad \xi_1=\xi_2=0.$$

Figure 2 shows the dependence of the Stokes parameter ξ_3 on Δ. It can be seen that the degree of polarization increases with increasing Δ and approaches an asymptotic value obtained by substituting in the definition of ξ_3 the expressions for M_1 and M_2 from Eqs. (9) and (12). At a fixed value of Δ, the value of ξ_3 increases with increasing Z.

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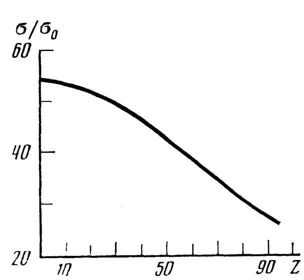

FIG. 2. Dependence of the total scattering cross section on the Coulomb-center charge.

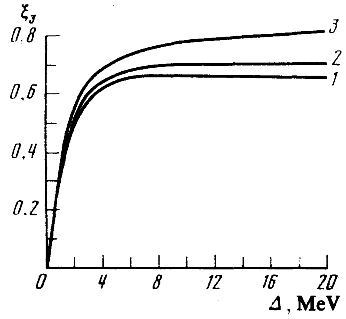

FIG. 3. Dependence of the Stokes parameter on Δ at $Z=1$ (curve 1), $Z=47$ (curve 2), and $Z=92$ (curve 3).

APPENDIX

In the momentum transfer region $\Delta \lesssim m^2/\omega$ it is necessary to take into account the corrections to the quasiclassical Green function (5). The exact Green function in a Coulomb field [see Eqs. (19) and (20) of Ref. 7] contains sums over l, in the form

$$S_A=\sum_{l=1}^{\infty} l e^{-i\pi\nu} J_{2\nu}(y)\,[P_l'(\mathbf{n}_1\mathbf{n}_2)+P_{l-1}'(\mathbf{n}_1\mathbf{n}_2)],$$

$$S_B=\sum_{l=1}^{\infty} e^{-i\pi\nu} J_{2\nu}(y)\,[P_l'(\mathbf{n}_1\mathbf{n}_2)-P_{l-1}'(\mathbf{n}_1\mathbf{n}_2)], \tag{A1}$$

where

$$\nu=[l^2-(Z\alpha)^2]^{1/2},\quad y=2\varkappa R/\mathrm{sh}\,\varkappa t,\quad \varkappa=(\varepsilon^2-m^2)^{1/2},\quad R=(r_1r_2)^{1/2},$$

and P_l' are derivatives of Legendre polynomials. The transition to the quasiclassical approach consists of an approximate calculation of these sums with allowance for the fact that the contribution is made by $l \gg 1$. Substituting $\nu \to l$ in (A1) we obtain [with the aid of Eq. (24) of Ref. 7] in the zeroth approximation

$$S_A^{(0)}=-\frac{y^2}{8}J_0\left(y\cos\frac{\varphi}{2}\right),$$

$$S_B^{(0)}=-yJ_1\left(y\cos\frac{\varphi}{2}\right)\Big/4\cos\frac{\varphi}{2}. \tag{A2}$$

Here φ is the angle between n_1 and n_2: $\cos(\varphi/2)=[(1+\mathbf{n}_1\mathbf{n}_2)/2]^{1/2}$. In the momentum-transfer region defined by the condition (1), the main contribution was made by small angles between $\mathbf{n}_2$ and $-\mathbf{n}_1$, i.e., $\psi\equiv(\pi-\varphi)\ll 1$. For such angles ψ, the Legendre-polynomial combination $P_l'(\mathbf{n}_1\cdot n_2)+P_{l-1}'(\mathbf{n}_1\cdot\mathbf{n}_2)$ is approximately equal to $(-1)^{l+1} lJ_0(l\psi)$. Using the integral representation for $J_{2l}(y)$ we obtain

$$S_A^{(0)}=-\int_0^\infty dl\,l^2J_0(l\psi)\frac{1}{\pi}\int_0^\pi d\theta\cos(2l\theta-y\sin\theta).$$

We have replaced the summation over l by integration. Since $y\gg 1$, contributions are made by $l\sim y$ *and* $\theta\ll 1$, and the integral with respect to θ can be replaced by $\delta(2l-y)$. We obtain $S_A^{(0)}\approx -\frac{1}{8}y^2J_0(y\psi/2)$, which agrees with $(A/2)$ at $\psi\ll 1$. The calculation of $S_B^{(0)}$ at these angles is similar.

If $\cos(\varphi/2)\sim 1$, the argument of the Bessel function in (A2) is large and we can use the corresponding asymptotic expansion. We now obtain these asymptotes by the method used by us the corrections to $S_A^{(0)}$ and $S_B^{(0)}$. The principal term of the expansion of $P_l(\cos\varphi)$ at $\sin\varphi\sim 1$ and $l\gg 1$ is of the form

$$P_l(\cos\varphi)\approx\left(\frac{2}{\pi l\sin\varphi}\right)^{1/2}\sin\left[\left(l+\frac{1}{2}\right)\varphi+\frac{\pi}{4}\right]. \tag{A3}$$

Using also the integral representation for $J_{2l}(y)$, we get

$$S_A^{(0)}\approx\frac{[\mathrm{ctg}(\varphi/2)]^{1/2}}{\sin\varphi}\int_0^\infty dl\left(\frac{l}{\pi}\right)^{3/2}\int_0^\pi d\theta\sin\left[l(2\theta-\psi)-y\sin\theta-\frac{\pi}{4}\right]. \tag{A4}$$

Here $\psi=\pi-\varphi$, and we have retained only the term in which cancellation in the argument of the sine function is possible. Summation over l was replaced with integration. We put $\theta=\frac{1}{2}\psi+x$ and expand the argument of the sine function in powers of x, retaining the quadratic terms. After integration with respect to x we have

$$S_A^{(0)}\approx-\frac{1}{\pi\sin\varphi}\left[\frac{2}{y\sin(\varphi/2)}\right]^{1/2}\int_0^\infty dl\,l^{3/2}$$
$$\times\sin\left[y\sin\frac{\psi}{2}+\frac{[2l-y\cos(\psi/2)]^2}{2y\sin(\psi/2)}\right]. \tag{A5}$$

It can be seen that a contribution is made by $l\approx\frac{1}{2}y\cos(\psi/2)$ in the interval $\Delta l\sim[y\sin(\psi/2)]^{1/2}$. We ultimately get

$$S_A^{(0)}\approx-\frac{y^2}{8}\left[\frac{2}{\pi y\cos(\varphi/2)}\right]^{1/2}\sin\left(y\cos\frac{\varphi}{2}+\frac{\pi}{4}\right). \tag{A6}$$

Obviously, Eq. (A6) is the leading term in the expansion of $S_A^{(0)}$ from (A2) at y $\cos(\varphi/2)\gg 1$. The corrections to $S_A^{(0)}$ and $S_B^{(0)}$ are similarly calculated, and it is necessary to retain the next terms of the expansion in $(1/l)$ in (A3) and in $\exp(-i\pi\nu)J_{2\nu}(y)$. If we put $S_A=S_A^{(0)}+S_A^{(1)}+\ldots$ and $S_B=S_A^{(0)}$ and $S_B=S_B^{(0)}+S_B^{(1)}+\ldots$, we obtain at $y\cos(\varphi/2)\gg 1$

$$S_A^{(1)}\approx\frac{(Z\alpha)^2}{\sin\varphi}\left[\frac{y\cos(\varphi/2)}{8\pi}\right]^{1/2}\left[\varphi\cos\beta-\pi e^{i\beta}+\frac{(\varphi\sin\beta+i\pi e^{i\beta})[1+3\cos^2(\varphi/2)]}{4y\sin\varphi\sin(\varphi/2)}-\frac{\sin\beta}{y\sin(\varphi/2)}\right],$$

$$S_B^{(1)}\approx\frac{(Z\alpha)^2}{\sin\varphi}\left[2\pi y\cos\frac{\varphi}{2}\right]^{-1/2}\left[\varphi\sin\beta+i\pi e^{i\beta}+\frac{(\varphi\cos\beta-\pi e^{i\beta})[3+\cos^2(\varphi/2)]}{4y\sin\varphi\sin(\varphi/2)}-\frac{\cos\beta}{y\sin(\varphi/2)}\right], \tag{A7}$$

where $\beta=y\cos(\varphi/2)+\pi/4$. Substituting the corrected values of S_A and S_B in Eqs. (19) and (20) of Ref. 7 we obtain the quasiclassical Green function with the corrections taken into account.

As already noted, the contributions to the sum are made by the large $l\approx\frac{1}{2}y\cos(\psi/2)$, with $\Delta l/l\ll 1$. Introducing the impact parameter $\rho=l/\varkappa$ and denoting $\varkappa_1t_1$ and $\varkappa_2t_2$ by t_1 and t_2 [these are the integratin variables in the expressions for the Green functions, see (5)], we have

$$\rho_{1,2}\approx R\cos(\psi/2)/\mathrm{sh}\,t_{1,2}. \qquad (A8)$$

At momentum transfers of the order of $\Delta \lesssim m$ the value of R is of the order of the length of the loop: $R \sim \omega/m^2$ and $\delta\rho = |\rho_2 - \rho_1| \sim 1/m$. It is clear from geometric considerations that $\rho \sim R\psi$. Comparing the last relation with (A8) we have $\psi \sim 1/\sinh t$. For the quantity $\delta\rho$ from (A8) we obtain $\delta\rho \sim \rho|t_2 - t_1|$. It follows from the foregoing relations that $\psi|t_2 - t_1| \sim m/\omega$, i.e., at $\psi \sim 1$ contributions are made by the values $|t_2 - t_1| \sim m/\omega$ and $\sinh t \sim 1$. For angles $\psi \ll 1$ we have $\sinh t \sim 1/\psi \gg 1$, a fact used in the calculation of the amplitudes in the region (1).

We proceed now to calculate the amplitude M_1 at zero momentum transfer ($\Delta = 0$), which we denote by M_0. The amplitide M_2 vanishes at $\Delta = 0$ by virtue of the conservation of the angular momentum projection along the direction of motion of the initial photon. We subtract from M_0 the terms that do not depend on $Z\alpha$. For the case considered by us this corresponds to renormalization of the amplitude M_0. The expression for M_0 is obtained from (4) by the substitution $\mathbf{k}_2 \to \mathbf{k}_1$; in addition, e_2^μ can be replaced by e_1^μ and averaging carried out over the polarizations:

$$\langle e_1^{\mu *} e^\nu \rangle = -{}^1/_2 g^{\mu\nu}.$$

The trace is then easily calculated. We make the change of variables $r_1 = Rv, r_2 = R/v$ and $\varkappa_{1,2} = \rho_{1,2}/R$, and then the integral with respect to R is calculated just as before. It is proportional to $(\mathbf{v}_1(\mathbf{n}_2/v - \mathbf{n}_1 v))$. The meaning of this function is that when the z axis is chosen along the direction of the incident photon we have $z_2 - z_1 > 0$, where z_1 (z_2) is the coordinate of the point of pair creation (annihilation); this is equivalent of time-ordering of these events. At the angles $\psi \ll 1$ that contributed to the transfer region (1), this ϑ function separated the contributions of the positive values of t_1 and t_2. According to the foregoing physical picture, it is convenient, when calculating M_0, to divide the region of integration with respect to ψ in two. We choose ψ_0 such that $m/\omega \ll \psi_0 \ll 1$. In the region $\psi > \psi_0$ the integrals with respect to t_1, t_2, and v and with respect to the azimuthal angle between the projections of $\mathbf{n}_1$ and $\mathbf{n}_2$ on a plane perpendicular to $\mathbf{k}_1$ are evaluated by the stationary-phase method. Inasmuch as $\psi > \psi_0$ the contributions are made by $|t_2 - t_1| \ll 1$, while the parameter $Z\alpha$ enters in the phase of the integrand for M in the combination $Z\alpha(t_1 - t_2)$, the contribution to this region is proportional to $(Z\alpha)^2$, which corresponds to the Born diagram. The corrections to the Green function (5) need be taken into account only in this region. After simple calculations we obtain for the contribution of the angles $\psi > \psi_0$

$$M_0^{(1)} = -i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln\frac{\psi_0}{2} + \frac{11}{21}\right]. \qquad (A9)$$

For small angles ($\psi < \psi_0$) it is convenient, just as in the calculation of the asymptotic form of the amplitude M_1 at $m^2/\omega \ll \Delta \ll m$, to divide the region of integration with respect to $\tau = t_1 - t_2$ into two: $|\tau| < \tau_0$ and $|\tau| > \tau_0$, where $m/\omega\psi_0 \ll \tau_0 \ll 1$. In the region $|\tau| > \tau_0$ the calculations are carried out exactly as in the derivation of (14). For this contribution we have

$$M_0^{(2)} = -i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln 2\tau_0 - \frac{39}{28} + C + \mathrm{Re}\,\psi(1 - iZ\alpha)\right]. \qquad (A10)$$

In the region $|\tau| < \tau_0$ the integrals with respect to $(t_1 + t_2)/2$, v, and the azimuthal angle are calculated by the stationary-phase method. For this contribution we obtain

$$M_0^{(3)} = -i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln\frac{m}{2\omega\psi_0\tau_0} + \frac{i\pi}{2} + \frac{97}{28}\right]. \qquad (A11)$$

Adding $M_0^{(1)}$, $M_0^{(2)}$, and $M_0^{(3)}$ we obtain for M_0

$$M_0 = i\frac{28\alpha\omega}{9m^2}(Z\alpha)^2\left[\ln\frac{2\omega}{m} - \frac{109}{42} - C - \mathrm{Re}\,\psi(1 - iZ\alpha) - i\frac{\pi}{2}\right]. \qquad (A12)$$

Expression (A12) agrees with the result of Ref. 10. We emphasize that the method described here can be used to calculate the amplitudes in the entire momentum-transfer region $\Delta \lesssim m^2/\omega$.

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