# Two-dimensional dynamic topological solitons in ferromagnets

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Bound magnon states with nonzero angular momenta are considered for a two dimensional magnet within the framework of the Landau–Lifshitz equation without damping. It is shown that such dynamic solitons are due to a topological singularity. The topological charge of the soliton $\nu = \pm 1, \pm 2,...$ is introduced. It is observed that if the number of magnons is low enough there exist limiting frequencies of the magnetization-vector precession in the soliton: $\omega = \omega_0/|\nu|$, where $\omega_0$ is the frequency of the homogeneous ferromagnetic resonance. Computer and analytic calculations yield the characteristics of the two-dimensional solitons.

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1. Analysis of essentially nonlinear perturbations in magnets, or magnetic solitons, is attracting much attention of late. A special place among magnetic solitons is occupied by topological solitions whose magnetization field cannot be brought into a homogeneous state by continuous deformation.[1–3]

We note that the two- and three-dimensional solitons investigated in Refs. 1–3 are precessional: the magnetization vector in the soliton precesses at a certain fixed frequency. Their analysis is therefore free of the difficulties connected with the stability of two- and three-dimensional static solitons.[4,5]

The present paper is devoted to the study of the dynamic properties of two-dimensional topological solitons in a uniaxial ferromagnet. We describe the ferromagnet on the basis of the Landau–Lifshitz equations for the magnetization vector $\mathbf{M}(\mathbf{r},t)$. We introduce the angle variables

$$M_z=M_0\cos\theta,\quad M_x+iM_y=M_0\sin\theta\exp(i\varphi),$$

where $M_0$ is the saturation magnetization, and the $z$ axis is chosen along the easy magnetization direction.

Using the polar coordinates $r$ and $\chi$ ($x = r\cos\chi$, $y = r\sin\chi$), we can put for the simplest precession soliton[1]

$$\theta=\theta(r),\quad \varphi=\omega t+\nu\chi+\varphi_0,\qquad \nu=0,\ \pm1,\ \pm2\ldots, \tag{1}$$

where $\omega$ is the precession frequency and the parameter $\nu$ plays the role of the topological charge of the soliton. In the case when $\nu\neq 0$ and

$$\theta(0)=\pi,\quad \theta(\infty)=0, \tag{2}$$

the magnetization field (1) cannot be reduced to the ground state in all of space, and the corresponding soliton is topological.

It is easy to verify that the value of $\nu$ determines also the $z$-projection of the angular momentum of the soliton magnetization field[1]:

$$K_z=-\nu\frac{\hbar M_0}{2\mu_0}\int(1-\cos\theta)\,d^2x=-\hbar\nu N, \tag{3}$$

where

$$N=\frac{M_0}{2\mu_0}\int(1-\cos\theta)\,d^2x. \tag{4}$$

$N$ is the number of spin deviations (number of magnons) per unit length of the soliton, and $\mu_0$ is the Bohr magneton.

A topological soliton corresponds thus to a nonzero projection of the angular momentum of the magnetization field, a projection uniquely connected with the number of spin deviations in the soliton.

The function $\theta(r)$ is defined by the following differential equation[1]:

$$l_0^2\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-\frac{\nu^2}{r^2}\sin\theta\cos\theta\right)-\sin\theta\cos\theta+\frac{\omega}{\omega_0}\sin\theta=0, \tag{5}$$

where $l_0$ is the magnetic length and coincides with the domain-wall thickness in a uniaxial ferromagnet, while $\omega_0 = 2\mu_0\beta M_0/\hbar$ is the frequency of the homogenous ferromagnetic resonance. It can be verified that Eq. (5) has localized solutions only at $\omega < \omega_0$.

In Refs. 1 and 3 this equation was numerically integrated by the "shooting" method (see Ref. 3 for details) for the cases $\nu = 1$ and $\nu = 2$. The obtained magnetization distribution $\theta(r)$ was used to plot the number of $N$ of spin deviations in the soliton against the frequency $\omega$ and the soliton energy $E$ against the number of magnons. The energy $E$ per unit length of the soliton is equal to

$$E=\frac{1}{2}\beta M_0^2\int\left\{l_0^2\left[\left(\frac{d\theta}{dr}\right)^2+\frac{\nu^2}{r^2}\sin^2\theta\right]+\sin^2\theta\right\}d^2x, \tag{6}$$

where $\beta$ is the anisotropy constant ($\beta > 0$).

It was found that as $N\to\infty$ the precession frequency $\omega$ tends to zero, and the soliton energy $E$ is proportional to $N^{1/2}$. In the other limiting case ($N\to 0$) the functions $\omega = \omega(N)$ and $E = E(N)$ are essentially governed by the value of $\nu$. If $\nu = 1$, it turns out[1] that as $N\to 0$

$$\omega(N)\to\omega_0,\quad E=4\pi\alpha M_0^2,$$

where $\alpha$ is the inhomogeneous exchange constant. If $\nu = 2$ localized solutions are observed[3] only at $\omega < \frac{1}{2}\omega_0$ and the maximum soliton energy is $E = 8\pi\alpha M_0^2 (N\to 0)$.

On the basis of these results, and also of a preliminary numerical analysis of solitons with $\nu = 3$ and $\nu = 4$, it was suggested in the review[6] that, first, the magnetic solitons of type (1) are possible only at frequencies $\omega < \omega_0/|\nu|$ and second, that in the limit as $N\to 0$ the soliton energy is

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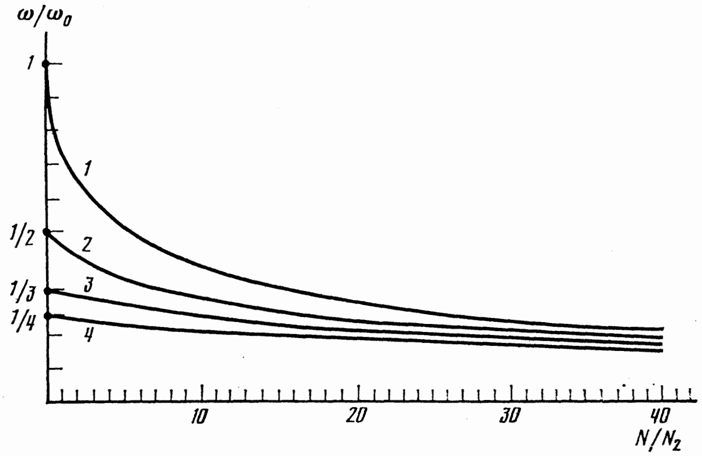


FIG. 1. Results of analysis of the function $\omega(N)$. The number of the curve coincides with the value of the parameter $\nu$.

$E=4\pi|\nu|\alpha M_0^2$. In the present paper we justify this suggestion analytically.

2. To determine the behavior of the soliton maximum frequency and maximum energy, we have integrated numerically Eq. (5) at $\nu=3$ and $\nu=4$. The results of an analysis of the function $\omega(N)$ are shown in Fig. 1. It is easily seen that the law $\omega<\omega_0|\nu|$ remains valid also at $\nu=3,4$.

In addition, calculation of the soliton energy confirms the limiting formula $E=4\pi|\nu|\alpha M_0^2$ for the cases $\nu=3$ and $\nu=4$. Plots of $E(N)$ as $N\to 0$ for different values of $\nu$ are shown in Fig. 2.

Notice must be taken, however, of the following circumstance. In the computer calculations of both Ref. 3 and of the present paper, the function $\omega(N)$ tended to the value $\omega_0/|\nu|$ as $N\to 0$ only up to $N/N_2\gtrsim 10^{-3}$, where $N_2=M_0l_0^2/2\mu_0\gg 1/a$. The quantity $aN_2$ is the characteristic number of magnons in a two-dimensional magnetic soliton and is of the order of the maximum number of magnons on an area with dimension of the order of $l_0$. At small values of $N$ the function $\omega(N)$ decreases with decreasing $N$ (see Fig. 3). But in this region of $N$ the numerical analysis is difficult, since the derivatives of the function $\theta(r)$ becomes large as $r\to 0$. Therefore the question of the behavior of $\omega(N)$ at extremely small $N$ calls for an additional analysis.

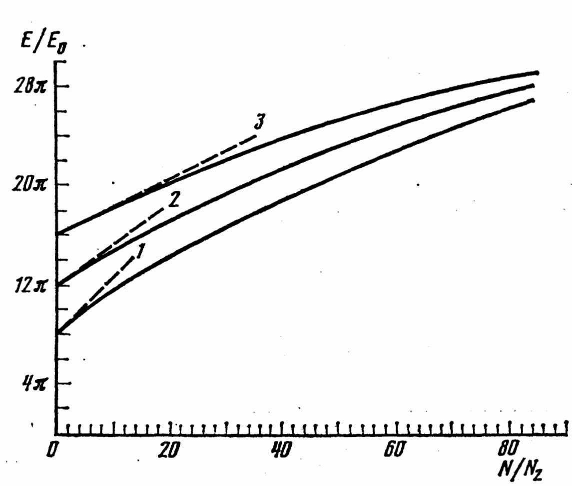


FIG. 2. Plot of $E(N)$ at different $\nu$: curve 1—$\nu=2$, curve 2—$\nu=3$, curve 3—$\nu=4$, $E_0=aM_0^2$.

3. We proceed to an analytic description of the properties of topological solitons. We prove first that in the two-dimensional case localized solutions of (5) can exist only at $\omega>0$. To this end it suffices to multiply (5) by $r^2d\theta/dr$ and integrate with respect to $r$ from 0 to $\infty$:

$$l_0^2\int_0^\infty\left[r^2\frac{d\theta}{dr}\frac{d^2\theta}{dr^2}+r\left(\frac{d\theta}{dr}\right)^2\right]dr-l_0^2\nu^2\int_0^\infty\sin\theta\cos\theta\frac{d\theta}{dr}\,dr$$

$$=\int_0^\infty\sin\theta\cos\theta\frac{d\theta}{dr}r^2\,dr-\frac{\omega}{\omega_0}\int_0^\infty\sin\theta\frac{d\theta}{dr}r^2\,dr. \qquad (7)$$

Carrying out the integration in the left-hand side of (7) and taking (2) into account we easily find that the integrals in it vanish identically. Therefore

$$\frac{1}{2}\int_0^\infty\sin^2\theta r\,dr=\frac{\omega}{\omega_0}\int_0^\infty(1-\cos\theta)r\,dr. \qquad (8)$$

Since both integrals in (8) are positive, it follows from this equation that the frequency of a two-dimensional soliton cannot be negative. This property of two-dimensional solitons distinguishes them from one-dimensional ones, whose precession frequency can be negative.[6]

We proceed to an analysis of the dependences of $\omega$ and of $E$ on the number $N$ of bound excitons. At large $N$ ($N\gg N_2$) the solitons with all the values of $\nu$ recall a cylindrical magnetic bubble: in the cylindrical part of radius $R\approx l_0(N/N_2)^{1/2}\gg l_0$ the density of the spin deviations reaches practically the maximum value $M_0/\mu_0$, after which the magnon density in a layer of thickness $l_0$ decreases exponentially to zero. Therefore

$$E\approx 2\pi RE_B\approx 2\pi l_0E_B(N/2N_2)^{1/2}, \qquad (9)$$

where $E_B=2\beta l_0M_0^2$ is the energy of a Bloch domain wall per unit area.

The precession frequency of the magnetization vector in the soliton is determined with the aid of the relation[6]

$$\hbar\omega=dE/dN. \qquad (10)$$

From (9), (10), and the definition of $N_2$ it follows that

$$\omega\approx\omega_0(2N_2/N)^{1/2},\quad N\gg N_2. \qquad (11)$$

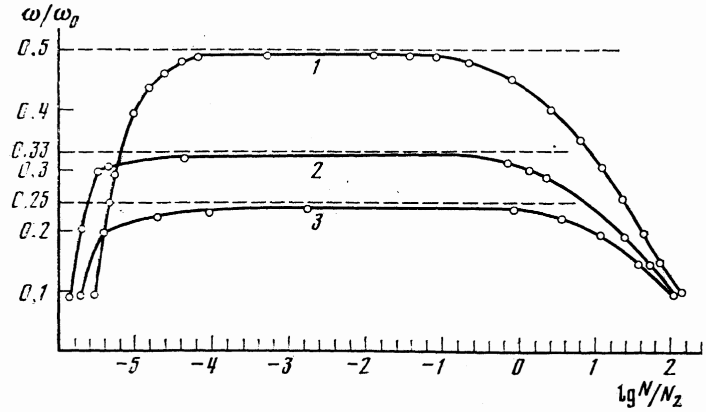


FIG. 3. Behavior of the function $\omega = \omega(N)$ in a large interval of $N$: curve 1—$\nu = 2$, curve 2—$\nu = 3$, curve 3—$\nu = 4$.

The distinguishing features of topological solitons manifest themselves at small $N$ ($N \ll N_2$). Small $N$ correspond to a small localization radius $R$ of the soliton ($R \ll l_0$). Therefore a small parameter $(R/l_0)^2 \ll 1$ appears in the theory and permits an analytic investigation of the properties of the magnetization field.

If it is assumed that $R \ll l_0$, it is easy to verify that the anisotropy energy is less significant in the analysis of the soliton structure than the exchange energy. To verify this, we estimate the corresponding terms in Eq. (5). The exchange energy gives rise to the first term of (5), the term with the parentheses. Its order of magnitude is estimated at

$$l_0^2 \frac{d^2\theta}{dr^2} \sim l_0^2 \frac{1}{r}\frac{d\theta}{dr} \sim \frac{l_0^2 \nu_2}{r^2} \sin\theta\cos\theta \sim \left(\frac{l_0}{R}\right)^2 .$$

The anisotropy energy gives rise to the second term of (5), with unity order of magnitude. This is also the maximum order of magnitude of the last term of (5). Consequently, in the principal approximation in terms of the small parameter $(R/l_0)^2$ Eq. (5) can be repaced by

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{\nu^2}{r^2} \sin\theta\cos\theta = 0. \tag{12}$$

Equation (12) is gauge invariant: if the function $\theta = f(r)$ is a solution of (12), the function $\theta = f(Ar)$, where $A$ is a constant, is also a solution of the same equation.

This equation describes formally the static ($\omega = 0$) field of the magnetization in an isotropic ferromagnet. The soliton solutions of this equation were investigated in Refs. 7 and 8. Its solution $\theta = \theta_0(r)$, which satisfies the required boundary conditions, is known and can be written in the form

$$\operatorname{tg}\frac{\theta_0}{2} = \left(\frac{R}{r}\right)^{|\nu|}, \qquad R = \text{const}. \tag{13}$$

Here $R$ plays the role of the soliton dimension.

The function (13) decreases at large distances ($r \gg R$) in power-law fashion:

$$\theta_0 \infty (R/r)^{|\nu|}. \tag{14}$$

The solution of (5) at $r \gg l_0$ decreases exponentially:

$$\theta \infty \frac{1}{r^{1/2}} \exp\left(-\frac{r}{l_0}\left(1-\frac{\omega}{\omega_0}\right)^{1/2}\right) .$$

Thus, at $r \gtrsim l_0$ the last terms of Eq. (5), which were left out in the course of its reduction and of the transition to Eq. (13), become significant. At $r \gg l_0$, however, the value of $\theta$ is small and Eq. (5) that defines it can be linearized:

$$l_0^2 \left[\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{\nu^2}{r^2}\theta\right] - \left(1 - \frac{\omega}{\omega_0}\right)\theta = 0. \tag{15}$$

A solution of this equation, which decreases as $r \to \infty$, is expressed in terms of a Macdonald function

$$\theta = CK_\nu\left(\frac{r}{l_0}\left(1-\frac{\omega}{\omega_0}\right)^{1/2}\right) . \tag{16}$$

We note that at $r \ll l_0$ the function (16) behaves at $\nu \neq 0$ like

$$K_\nu \infty (l_0/r(1-\omega/\omega_0)^{1/2})^{|\nu|}. \tag{17}$$

The relation (17) appears, generally speaking, outside the region where (15) is valid. However, a comparison of (17) and (14) leads us to the following conclusion. If we define the parameter $R$ by the condition $R \ll l_0$ we can, by an appropriate choice of the constant $C$ in the solution (16), make the functions (13) and (16) equal in the interval $R \ll r \ll l_0$ with accuracy equal to that of Eqs. (14) and (17). This choice of the constant $C$ leads us to the equation

$$\operatorname{tg}\frac{\theta}{2} = \frac{2}{(\nu-1)!}\left[\frac{R(1-\omega/\omega_0)^{1/2}}{2l_0}\right]^{|\nu|} K_\nu\left(\frac{r(1-\omega/\omega_0)^{1/2}}{l_0}\right). \tag{18}$$

We see first of all that in the interval $R \ll r \ll l_0$, where the limiting expressions (14) and (17) are "joined together," $\tan(\theta/2) \ll 1$ and Eq. (16) coincides with the solution of Eq. (5) in practically the entire interval $r \gg R$. The asymptotic form (17) is thus fully justified. We obtain thus Eq. (18), which describes with good accuracy, relative to the parameter $R/l_0$, the localized solution of the nonlinear equation (5) at all values of $r$.

The solution (18) is characterized by a free parameter $R$ that should be fixed by the value of one of the integrals of the motion. The corresponding integral is the number of magnons $N$, which we now proceed to calculate. It is easy to see that in the principal approximation in $R/l_0$ the number $N$ is proportional to $R^2$. If $|\nu| > 1$, we can calculate it in fact by using the solution (13), since the difference between (13) and (28) is small in terms of the parameter $R/l_0$. As a result we obtain

$$N=\frac{\pi R^2 M_0}{\mu_0}\frac{\pi}{\nu\sin(\pi/\nu)},\qquad |\nu|>1. \tag{19}$$

At $|\nu| = 1$ Eq. (13) is valid only over distances $r$ smaller than the magnetic length $l_0$. At these distances we can neglect the magnetic anisotropy and use Eq. (12). At distances $r \gg l_0$ the inhomogeneity of the magnetization is described by Eq. (5) and falls off exponentially. To calculate $N$ at $|\nu| = 1$ we can therefore use Eq. (18). Calculation yields

$$N=\frac{\pi R^2 M_0}{\mu_0}\ln\left[\frac{4l_0^2\omega_0}{e\gamma^2(\omega_0-\omega)R^2}\right],\qquad |\nu|=1, \tag{20}$$

where $e$ is the base of the natural logarithms and $\gamma \approx 1.78$ is Euler's constant.

It follows from (19) and (20) that the condition $R^2 \ll l_0^2$ is equivalent to the requirement $N \ll N_2$. But since we are interested in the limit as $n \to 0$, the condition obtained is quite satisfactory to us.

Analyzing (9) and (10), we arrive at the conclusion that at $|\nu| > 1$ the parameter $R$ is indeed determined uniquely by the number of magnons, with $n \propto R^2$. At $|\nu| = 1$ the function $N(R)$ is more complicated, since Eq. (20) contains the frequency $\omega$, whose dependence on $N$ or $R$ must be determined from a few additional conditions. By way of such a condition we can use, e.g., the identity (8).

Having the solution (18), we proceed to calculate the soliton energy $E$ and the precession frequency $\omega$ as functions of $N$. In the principal approximation in the small parameter $R/l_0$ the contribution to the total energy of the soliton is made only by the inhomogeneous-exchange energy, for the calculation of which it suffices to use the solution (13):

$$E=4\pi\alpha M_0^2|\nu|=4\hbar\omega_0 N_2|\nu|. \tag{21}$$

Since (21) does not depend on $N$, it is valid also at $N = 0$, and coincides with the known formula for the limiting energy of a singular soliton in a two-dimensional uniaxial ferromagnet.[7,9]

To find the $\omega(N)$ dependence it suffices to use relation (8) which we rewrite, taking the definition (4) into account, in the form

$$\frac{1}{2}\int_0^\infty \sin^2\theta\cdot 2\pi r\,dr=\frac{2\mu_0\omega}{\omega_0 M_0}N. \tag{22}$$

If now we use the solution (18) to calculate the integral in the left-hand side of (22), it turns out that in the principal approximation in $R/l_0$ this integral is also proportional to $N$:

$$\frac{1}{2}\int_0^\infty \sin^2\theta\cdot 2\pi r\,dr=\frac{2\mu_0}{M_0}\frac{N}{|\nu|}. \tag{23}$$

Comparing (22) and (23), we obtain the sought $\omega(N)$ dependence at $N \ll N_2$:

$$\omega=\omega_0/|\nu|. \tag{24}$$

Knowing the limiting energy (21) of the soliton and using (10), we obtain

$$E(N)=\hbar\omega_0[4N_2+N/|\nu|]. \tag{25}$$

This formula expresses the soliton energy as a function of the number of magnons at sufficiently small $N$, accurate to terms of order $(N/N_2)^2$ or $(R/l_0)^4$.

Equations (24) and (25) answer the question of how the distinguishing features of topological solitons manifest themselves as $N \to 0$.

There remains, however, the question of the anomalous behavior of the curve $\omega = \omega(N)$, obtained by numerical calculation, at extremely small $N$ (Fig. 3). It follows from the numerical calculation that as $N \to 0$ the precession frequency $\omega$ vanishes if $|\nu| \neq 1$. The magnetization distribution that ensures a nonzero energy $E = 4\pi\alpha|\nu|M_0^2$, is transformed into a singular function as $N \to 0$. This may possibly indicate the existence of static singular solutions of Eq. (5) for $|\nu| > 1$, mentioned in Ref. 9. If such a singular solution ($N = 0$, $\omega = 0$, $E = 4\pi\alpha|\nu|M_0^2$) exists, it cannot be, by virtue of the obtained relations (24) and (25), the limiting one with respect to solitions of a magnetization-field with finite values of $N$ and $\omega$.


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