Quasi-equilibrium states of a pinching electron-ion beam

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We consider the known classical Bennett problem of the equilibrium of an electron-ion beam in its magnetic self-field. Objections are raised to the known approach based on consideration of beams of infinite radius. In our opinion this approach leads to the prevalent incorrect conclusion that the particles condense around the axis at arbitrary beam parameters. It is shown that within the framework of Maxwell-Boltzmann statistics, at arbitrary parameters, the beams can have quasi-stationary states with a finite particle density on the axis and with a maximum on the plot of the potential of one of the components vs the radius. The maximum determines in maximum fashion the effective beam radius R_m. Even though the potential well has only a finite depth for this component, a criterion formulated in the paper for an approximate thermodynamic equilibrium is satisfied in a wide range of conditions, so that at distances $\rho < R_m$ the Bennett equations that describe a quasi-equilibrium beam are perfectly applicable. A numerical experiment is described and illustrates how a system of superimposed pinching beams tends to a quasi-stationary state with finite particle density even from a strong disequilibrium state.

The problem of equilibrium of an axisymmetric electron-ion longitudinally uniform electron ion beam in its self-field was formulated by Bennett back in 1934.[1] The system of equations he obtained can be written, in the case when the ion component is at rest, in the form

$$\frac{d}{d\rho}\left(\rho\frac{d\psi_e}{d\rho}\right)=\rho[fe^{-\psi_i}-(1-\beta^2)e^{-\psi_e}],$$
$$\frac{d}{d\rho}\left(\rho\frac{d\psi_i}{d\rho}\right)=\rho(e^{-\psi_e}-fe^{-\psi_i})\tau, \qquad (1)$$

where

$$\rho=r/r_e,\quad r_e^2=kT_\perp/4\pi e^2 n_{e0},\quad \beta^2=v^2/c^2,\quad \tau=T_\perp/T_i,$$
$$T_\perp=T_e(1-\beta^2)^{1/2},\quad f=n_{i0}/n_{e0},$$

n_{e0} and n_{i0} are the densities of the electrons and ion on the beam axis in the lab, r is the distance from the axis, v is the beam-electron velocity, and ψ_e and ψ_i are the potentials of the electromagnetic force fields in which the electrons and ions are respectively located. It is assumed that both the electron and the ion component are in thermodynamic Boltzmann equilibrium with respective temperatures T_e and T_i. The electrons are acted upon by a radial force due both to the electric field and to the magnetic field of the current, while the ions are acted upon only by the radial electric field.

An analytic solution of Eqs. (1) was obtained only in one particular case, when the ion density at any point is proportional to the electron density, i.e., n_i/n_e = const$\equiv f$. The latter occurs under the condition

$$\tau=\beta^2/(1-f)-1>0, \qquad (2)$$

and the Bennett solution itself, with zero boundary conditions

$$\psi_{e,i}(0)=\left.\frac{d\psi_{e,i}}{d\rho}\right|_{\rho=0}=0 \qquad (3)$$

is expressed as

$$\psi_B=2\ln[1+\beta^2\tau\rho^2/8(1+\tau)]. \qquad (4)$$

The next step in the investigation of the equilibrium of two-component beams free of external field was made in Ref. 2. It was shown that self-pinching stationary beams with finite particle density at the axis are possible only at a strictly defined relation between the parameters [the Meierovich–Sukhorukov (MS) condition], a condition that includes also relation (2).

On the other hand, the question of the state of the system in the general case (at arbitrary beam parameters) has so far not been answered to any degree of satisfaction. Moreover, when raising this question, erroneous conclusions, in our opinion, were drawn in Ref. 2 concerning the behavior of this now-classical physical object. Starting from the requirement that the total number of electrons per unit length of an infinitely wide beam ($r\to\infty$) be finite, the authors of Ref. 2 found that at arbitrary parameters this requirement is satisfied only if one admits solutions in which the particle density increases without limit towards the axis. In their opinion, "an appreciable fraction of the total number of particles may turn out to be condensed on the beam axis," and this calls for the use of quantum statistics to describe the degenerate electrons in the axis region. In contrast to the treatment of singular solutions as corresponding to current carrying beams or to charge-carrying wire on the axis (named the "generalized Bennett distribution),[3,4] it is stated in Ref. 2 that the solution with singularity at the origin must be treated as a stationary state of a self-pinched relativistic beam, and the current and charge on the axis, needed for self-consistency, "should be regarded as pertaining to the beam itself being produced by an appropriate number of electrons and ions compressed into the origin, when the collective-interaction energy is not compensated by the energy of the thermal spreading of the

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particles"; this compensation is uniquely related by the authors to satisfaction of the condition obtained by them.

Without denying, in principle, the feasibility of producing very dense beams (see, e.g., Ref. 5) in which quantum-mechanical effects begin to manifest themselves, we assume that the existence of such beams is in no way connected with satisfaction or nonsatisfaction of the MS condition. A cylindrical plasma is not acted upon at the axis by the peripheral beam particles, and its state does not depend at all on whether their number is finite or infinite. In particular, the condensation problem should be considered on the basis of local equations. The MS relation, on the other hand, is of "global" character. Indeed, it is the consequence of the requirement that the integrals

$$\nu_{e,i}=\int_0^\infty n_{e,i}\rho\, d\rho$$

converge at infinity, a requirement equivalent to the following asymptotic behavior of the potentials:

$$\psi_e\approx[\nu_i f-(1-\beta^2)\nu_e]\ln\rho,\quad \psi_i\approx[\nu_e-f\nu_i]\tau\ln\rho,\quad \rho\to\infty,$$
$$\nu_i f-(1-\beta^2)\nu_e>2,\quad (\nu_e-f\nu_i)\tau>2.$$

From the first integral of the system (1), which is of the form

$$\frac{\tau}{2}\left(\rho\frac{d\psi_e}{d\rho}-2\right)^2+\frac{(1-\beta^2)}{2\tau}\left(\rho\frac{d\psi_i}{d\rho}-2\right)^2+\left(\rho\frac{d\psi_e}{d\rho}-2\right)$$
$$\times\left(\rho\frac{d\psi_i}{d\rho}-2\right)+\rho^2(\tau e^{-\psi_e}+fe^{-\psi_i})\beta^2=\text{const},$$

and from the boundary conditions (3) we then obtain an algebraic relation between f, τ, β^2, and $\nu_{e,i}=N_{e,i}/2\pi n_{e,i0}$, where $N_{e,i}$ is the total number of the corresponding particles in a system with infinite radius:

$$[f\nu_i-(1-\beta^2)\nu_e]^2+(1-\beta^2)(\nu_e-f\nu_i)^2$$
$$+2[f\nu_i-(1-\beta^2)\nu_e](\nu_e-f\nu_i)-4\beta^2(\nu_e+f\nu_i)=0. \qquad (5)$$

This relation generalizes the MS condition at arbitrary $\beta^2<1$ and goes over into it at $\beta^2=1$. At $\nu_e=\nu_i=4(1+\tau)/\beta^2\tau$ relation (5) goes over into the condition (2), which corresponds to the Bennett solution (4). However, the convergence of the integrals

$$\int_0^\infty n_{e,i}\rho\, d\rho$$

is not dictated by physical considerations, for it makes sense to consider only beams of finite thickness. Moreover, a stationary current always has a corresponding counter current, which can be naturally regarded in the considered axisymmetric problem as flowing somewhere at large ρ. If, however, we do not require that $\nu_{e,i}$ converge at infinity, the condition (5) is not obligatory. The role of the particle number should thus be assumed by the integrals

$$\int_0^{R_{\text{eff}}} n\rho\, d\rho,$$

where R_{eff} is a certain effective beam radius to be determined. The boundary conditions in the form (3), on the other hand, must be regarded as satisfied beforehand in the absence of extraneous fields.

In the present paper we demonstrate on the basis of such an approach the possible existence of quasi-stationary beams with finite particle density at the axis, and the range of the admissible parameters of the system is larger than that defined by the MS relation. When considering finite-thickness beams we start from the only known analytic solution of Eq. (1), which we write in the form

$$\rho\frac{d\psi_e}{d\rho}=\int_0^\rho \rho[fe^{-\psi_i}-(1-\beta^2)e^{-\psi_e}]d\rho, \qquad (6a)$$

$$\rho\frac{d\psi_i}{d\rho}=\int_0^\rho \rho[e^{-\psi_e}-fe^{-\psi_i}]\tau\, d\rho. \qquad (6b)$$

We label the corresponding equilibrium parameters that satisfy relation (2) by the letter B and consider qualitatively the possible state of the system (which is already different!) if condition (2) is violated by adding to one of the parameters, say τ_B, a small increment $\delta\tau$ but keep the densities of the beams at the axis fixed.

For the Bennett solution (4), $d\psi_e/d\rho$ and $d\psi_i/d\rho$ are always positive, but when T_i is decreased, i.e., at $\delta\tau>0$, the ion density, which is proportional to $\exp(-\psi_i)$ (when fixed on the axis), decreases everywhere, the uncompensated charge of the electrons increases, and consequently the electronic component will be broader in this new beam, i.e., $\exp(-\psi_e)$ increases. It can then be seen from (6a) that $d\psi_e/d\rho$ can go through zero and then become negative. This means that the potential ψ_e of the electrons reaches a maximum and begins to decrease, whereas the ion potential ψ_e, as can be seen from (6b), increases even more steeply than the Bennett potential. It follows similarly from (6) that ψ_i can reach a maximum at $\delta\tau<0$, and the electronic component becomes even more strongly pinched.

These qualitative considerations are confirmed by numerical integration of Eqs. (1) at different $\delta\tau$ (Fig. 1), and the

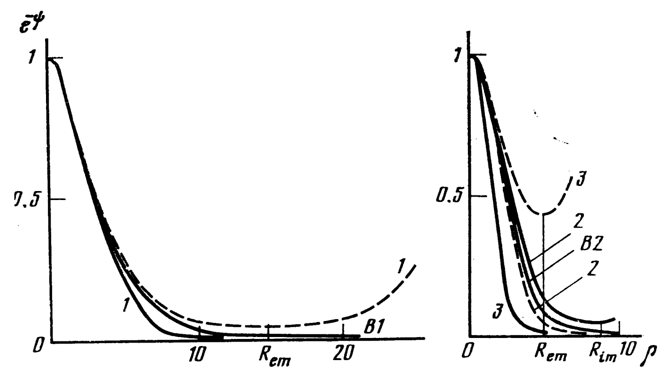

FIG. 1. Radial distribution of the particle density $n/n_0=e^{-\psi}$. The curves are numbered in accord with the table. Electrons—dashed line, ions—solid. Curves B 1, and B 2 are Bennett distributions with respective parameters $\beta_0^2=0.4, f_0=0.9, \tau_0=3$ and $\beta_0^2=0.8, f_0=0.68, \tau_0=1.5$.

TABLE I.

№	β_0^2	f_0	τ_0	δ	$R_m(r_e)$	$R_m e^{-\psi(R_m)}$	η
1	0.4 0.4 0.4003	0.9 0.90003 0.9	3 3.003 3.003	$\delta\tau=3\cdot10^{-3}$ $\delta f=3\cdot10^{-5}$ $\delta\beta^2=-3\cdot10^{-4}$	$R_{em}=15$	0.75	$\eta_e=6.9\cdot10^4$
2	0.8 0.8 0.768	0.68 0.693 0.68	1.5 1.4 1.4	$\delta\tau=-0.1$ $\delta f=-1.3\cdot10^{-2}$ $\delta\beta^2=3.2\cdot10^{-2}$	$R_{im}=9$	1	$\eta_i=1.2\cdot10^5$
3	0.8 0.8	0.68 0.584	1.5 2.5	$\delta\tau=1$ $\sigma f=9.6\cdot10^{-2}$	$R_{em}=4.9$	1.94	$\eta_e=2.4\cdot10^5$

larger $|\delta\tau|$ the smaller the $\rho = R_m$ at which an inflection appears on the plot of the electrons at $\delta\tau > 0$ and of the ions at $\delta\tau < 0$. It is easily seen that when f and β^2 deviate from f_B and β_B^2 the deviation of the solution of the system (1) from the corresponding ψ_B has the same character as when τ deviates from τ_B. Moreover, the same solution of the system (1) can be obtained from the corresponding ψ_B by a definite deviation of any parameter from its Bennett value, as illustrated in the table.

In the region past the inflection ($\rho > R_m$) particles of the same species drop down from the potential hill, and since there is no reflecting wall on the periphery, there are no conditions here for thermodynamic equilibrium. The solutions of Eq. (1) in this region are purely formal, since the equations themselves are inapplicable. Particles from one of the beams, when landing there, are freely accelerated from the system axis. In the region $\rho < R_m$, however, both components are present in the potential wells. To be sure, for one of them the well depth is finite and the corresponding particles cannot be in a state of true thermodynamic equilibrium. Nonetheless, if the number of particles that surmount the potential barrier is small, it can be assumed that the system is in thermal quasi-equilibrium, and the Boltzmann distribution can be used. Let us formulate a criterion for the applicability of the idea of such an approximate equilibrium. We assume that the gas in the potential well is in a quasi-stationary state if the number of particles in the considered component does not change substantially within the time of the thermodynamic relaxation. In our case this means that the number N of the collisions between particles must greatly exceed the flux I of the particles through the potential barrier, i.e., the number of collisions with a "wall" of radius R_m is

$$N \gg I. \tag{7}$$

To calculate the total number N of the collisions of electrons having a distribution function $f(\mathbf{r}, \mathbf{v})$ in an inhomogeneous system we use for the number of collisions at a point $\mathbf{r}$ a formula from which N is obtained by integration over the system volume V:

$$N = \int_V d^3\mathbf{r} \iint_{-\infty}^{\infty} d^3\mathbf{v}_1\, d^3\mathbf{v}_2 \int_\Omega d\omega \lambda f(\mathbf{r}, \mathbf{v}_1) f(\mathbf{r}, \mathbf{v}_2).$$

Here λ depends on the parameters of the Coulomb collisions and is independent of r. In our case $f(\mathbf{r},\mathbf{v}) = n_0 \tilde{f}(\mathbf{r})\varphi(\mathbf{v})$, so that the integration with respect to $\mathbf{v}$ and $\mathbf{r}$ is separated in the integral:

$$N = n_0^2 \int_V f^2(\mathbf{r})\, d^3\mathbf{r} \iint_{-\infty}^{\infty} d^3\mathbf{v}_1\, d^3\mathbf{v}_2 \int_\Omega d\omega \lambda \varphi(\mathbf{v}_1)\varphi(\mathbf{v}_2) = n_0^2 W \int_V f^2(\mathbf{r})\, d^3\mathbf{r}, \tag{8}$$

where W is the interval over the velocities and the angle variables. On the other hand, for a homogeneous system with density n_0, the number of Coulomb collisions N_{hom} was calculated without using Eq. (8) and is equal to $n_0 \nu_{\text{Coul}} V/2$, where

$$\nu_{\text{Coul}} = 4\pi e^4 n_0 \Lambda / m^{1/2} (kT)^{3/2}$$

is the frequency of the Coulomb collisions in a homogeneous system with temperature T, and Λ is the Coulomb logarithm. The same number N_{hom} can be obtained from Eq. (8), which yields $W = \nu_{\text{Coul}}/2n_0$, and for the number of collisions in an inhomogeneous system we obtain ultimately

$$N = \frac{n_0 \nu_{\text{Coul}}}{2} \int_V d^3\mathbf{r} f^2(\mathbf{r}). \tag{9}$$

The flux of particles with mass m and density $n(\rho)$ to a wall of radius R_m is

$$I = R_m n(R_m) (2\pi kT/m)^{1/2}, \tag{10}$$

where n and T for particles of a given species should be taken in the proper frame of the species. In the integral

$$\int_V f^2\, d^3\mathbf{r}$$

we can replace approximately $\exp(-\psi)$ by $\exp(\psi_B)$ and extend the integration with respect to ρ to infinity. Substituting the expression obtained in this manner in (9), we obtain for the quasi-equilibrium criterion (7)

$$\Lambda_e \frac{n_{e0}^{\prime\, 3/2}}{T_e^{5/2}} \gg \eta_e \equiv 2.2\cdot10^5 \frac{\beta^2 \tau R_{em} \exp\{-\psi_e(R_{em})\}}{(1+\tau)}, \tag{11a}$$

$$\Lambda_i \frac{n_{i0}^{3/2}}{T_i^{5/2}} \gg \eta_i \equiv 2.2\cdot10^5 \frac{\beta^2 f^{1/2} \tau^{3/2} R_{im} \exp\{-\psi_i(R_{im})\}}{1+\tau}. \tag{11b}$$

The condition (11a) pertains to the maximum of the electronic potential at the point R_{em}, and (11b) to the maximum of the ionic potential at the point R_{im}. The Coulomb logarithms

$$\Lambda_{e,i}=\ln\frac{3}{\sqrt{8\pi e^3}}\left(\frac{T_{e,i}^{3/2}}{n'^{1/2}_{e,i0}}\right)$$

are calculated using the densities on the axis, and this only strengthens the quasi-equilibrium criterion.

Here R_m is measured in units of r_e and T in degrees, n'_{e0} is the density of the particles of the corresponding component on the axis in their proper reference frame. If relation (11) is satisfied, the Boltzmann-distribution assumption is justified, with R_m in essence the radius R_{eff} of the quasistationary beam. In the opposite case Eqs. (1) are not valid even approximately. It can be seen from the data in the table that all the cited examples can represent a real physical situation at sufficiently large $\Lambda N_0^{3/2}/T^{5/2}$.

We note that inequalities (11) should be satisfied also if the beam parameters lie in the range defined by (5), since the equilibrium of real beams, which always have a finite diameter, can be only approximate. In this case R_m must be taken to mean the effective beam aperture, i.e., the radius of the region in which Eqs. (1) are valid.

Thus, both in the case when the parameters of the pinching beam satisfy relation (5) and when this relation does not hold, quasistationary states with finite particle densities on the axis are perfectly feasible and are satisfactorily described by the classical equations (1).

Since the number of particles of one of the components decreases continuously in the considered quasistationary beams with nonmonotonic potential, it is legitimate to raise the question of the consequences of this spilling of the particles out of the potential well. It can be easily seen that the species of the particles leaving the beam is precisely that for which the depth of the well for the remaining species increases with increasing number of the departing particles, i.e., the system should evolve self-consistently, by losing the "excess" particles, into a state in which both components are well retained. Moreover, the numerical example described below shows that beams tend to such a quasi-Bennett state even when their initial state is far from equilibrium.

The purpose of our computer experiment was to track, using a known simulation method,[6] the spatial evolution of a stationary electron beam with electron and ion longitudinal velocities v_e, v_i, where $v_e \gg v_i$. Each beam is represented by a large number of concentric tubes of like charge, and the charge en per unit length of the tube is set equal to the corresponding charge of the ion tube. Interacting with one another via their electric and magnetic self-fields, the tubes change in radius and penetrate freely through one another. At a sufficiently large axial velocity of the particles, it can be approximately assumed that only radial forces are present in the system. In this case, the shape of each tube, $r_{ik}(z)$ and $r_{ek}(z)$ $(k = 1,2,\ldots,K)$, can be described by the following system of equations in the dimensionless variables $z' = z\omega_e/\sqrt{2}v_e$, $r' = r/r_0$, where $\omega_e^2 = 4\pi e^2\bar{n}/m_e$, $\bar{n} = nK/\pi r_0^2$ and r_0 can be arbitrary:

$$\frac{d^2r_{ik}'}{dz'^2}=\frac{m_e v_e^2}{Kr_{ik}' m_i v_i^2}\left[\left(1-\frac{v_i^2}{c^2}\right)\sum_{s=1}^{K}F_{iks}(r_{ik}'-r_{is}')\right.$$
$$\left.-\left(1-\frac{v_i v_e}{c^2}\right)\sum_{s=1}^{K}F_{iks}(r_{ik}'-r_{es}')\right],$$
$$\frac{d^2r_{ek}'}{dz'^2}=\frac{1}{Kr_{ek}'}\left[\left(1-\frac{v_e^2}{c^2}\right)\sum_{s=1}^{K}F_{eks}(r_{ek}'-r_{es}')\right.$$
$$\left.-\left(1-\frac{v_e v_i}{c^2}\right)\sum_{s=1}^{K}F_{eks}(r_{ek}'-r_{is}')\right], \tag{12}$$

$F(x)=1$ at $x>0$, $F(x)=1/2$ at $x=0$, $F(x)=0$ at $x<0$.

The system (12) was solved numerically with boundary conditions that specified the coordinates $r_{e,ik}$ of all the tubes at $z = 0$ and

$$\left.\frac{dr_{ik}}{dz}\right|_{z=0}=\left.\frac{dr_{ek}}{dz}\right|_{z=0}=0.$$

The chosen boundary conditions correspond to cold beams of electrons and ions that are somewhate separated at the point $z = 0$. To circumvent the difficulties in the calculation of the tube contour, due to the finite tube charge and to the singularity as $r \to 0$, the trajectories were specularly reflected from a cylinder with a certain small radius h, a procedure justified by the axial symmetry of the system.

The computer-calculated particle trajectories are shown in Fig. 2. It can be seen that the absence of beam equilibrium at $z = 0$ leads to oscillations of the tube diameters, along z, under the influence of the electric field, the spatial periods of the electron and ion tubes being proportional to the electron and ion energies. Owing to the difference between the phases and amplitudes the oscillations, the trajectories become entangled. The tubes are particularly rapidly randomized if the magnetic pinching concentrates them at the system axis and the beams change from tubular

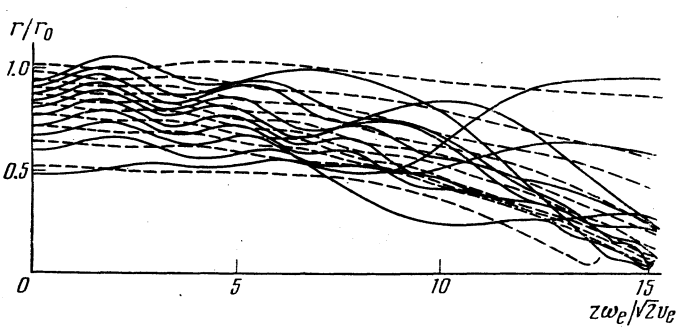

FIG. 2. Particle trajectories in superimposed cold beams. Solid lines— electrons, dashed—ions.

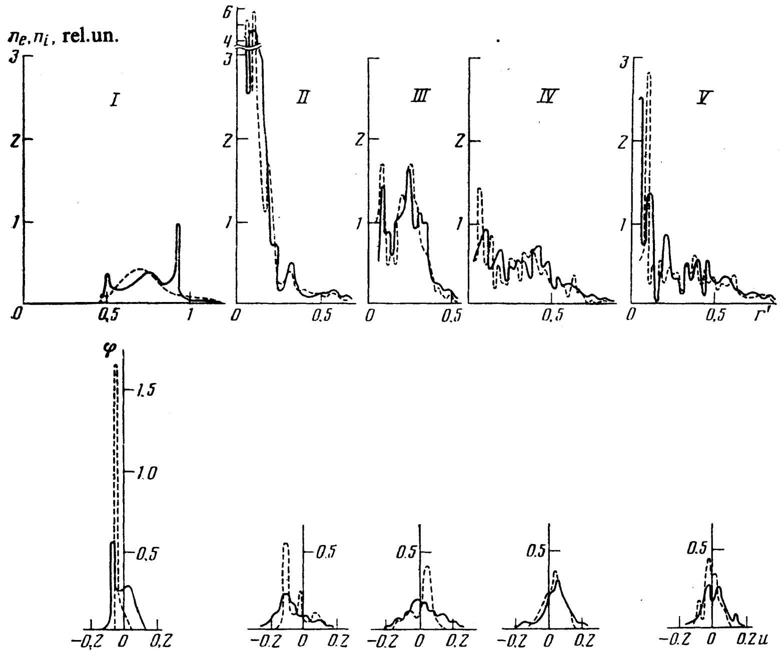

FIG. 3. Spatial evolution of radial dependences of the densities n_e and n_i, and of the particle distribution function φ averaged over the cross section, with respect to the radial velocities $u = \sqrt{2}v_r/\omega_e r_0$ for different z':I—$z' = 5$, II—15, III—19, IV—23, V—27; $K = 250$. Solid lines—electrons, dashed—ions.

to solid. The evolution of the integral characteristics of the beams is illustrated in Fig. 3, which shows the radial distributions of the density of each component and the particle distribution functions in the transverse velocities at different z. It is noteworthy that the oscillations of the distribution function attenuate with increasing z and at the end of the considered spatial integral the distribution function, just as for quasi-equilibrium beams, is close to symmetric about the point $dr/dz = 0$, while the density distributions are similar to those of Bennett (except for the artificially produced forbidden region $0 < r' < h$). We note also that if the ratio $\bar{f}$ of the positive charge contained in a circle of some radius r' to the corresponding negative charge varies strongly along z because of the disequilibrium of the beam, and furthermore differently for different r' (Fig. 4), at the end of the computation these oscillations attenuate. The ratios indicated then become nearly equal for different r' and smaller than unity, as should be the case in quasi-equilibrium beams.

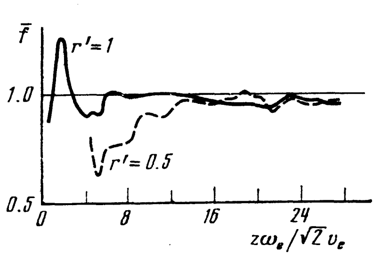

FIG. 4. Variation, with distance, of the ratio $\bar{f}$ of the positive charge in a circle of radius r' to the corresponding negative charge.

Summarizing the results of the described numerical experiments and the deductions of the preceding part of the paper, we can conclude that quasi-equilibrium beams are perfectly realizable physical systems. Their states are stable in the sense that when one such state is disturbed the system tends to go into another state of the same form. Of course, the question of the effect of possible two-stream instability on the investigated system calls for a special study.

[1]W. H. Bennett, Phys. Rev. **45**, 890 (1934).
[2]B. É. Meĭerovich and S. T. Sukhorukov, Zh. Eksp. Teor. Fiz. **68**, 1783 (1975) [Sov. Phys. JETP **41**, 895 (1975)].
[3]G. Benford and D. L. Book, Relativistic Beam Equilibria, in: Adv. in Plasma Physics, A. Simon and W. B. Thompson, eds., Interscience, 1971, p. 125.
[4]A. A. Rukhadze, L. S. Bogdankevich, S. E. Rosinskiĭ, and V. G. Rukhlin, Fizika sil'notochnykh relyativistkikh elektronnykh puchkov (Physics of High-Current Relativistic Electron Beams), Atomizdat, 1980.
[5]E. D. Korop, B. É. Meĭerovich, Yu. V. Sidel'nikov, and S. T. Sukhorukov, Usp. Fiz. Nauk **129**, 87 (1979) [Sov. Phys. Usp. **22**, 727 (1979)].
[6]A. S. Roshal', Modelirovanie zaryzhennykh puchkov (Simulation of Charged Beams), Atomizdat, 1979.

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