Equations of motion for a Hubbard ferromagnet

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A ferromagnetic metal is considered on the basis of the Hubbard model with strong correlation and with almost half-filled band. Equations of motion are obtained for the interacting vacancion (quantum) and spin (classical) subsystems. The action of the spin subsystem on the vacancy motion recalls the action of an electromagnetic field and is described by a scalar and vector potential; the vacancies influence the spin subsystem via the density and velocity. The equations of motion for the spin density are obtained from the least-action principle.

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1. It is known that one of the possible causes of the onset of magnetic order is the electron correlations in narrow bands, described in the simplest case by the Hubbard model (see, e.g., the review of Khomskiĭ[1]). Thus, in the limit of strong correlation and divisible (alternating) lattices the model yields ferromagnetism for an almost half-filled band (the number of electrons is somewhat larger or smaller than the number of lattice sites for s-type centers). These are precisely the systems dealt with here.

The starting point is the equivalent Hamiltonian obtained in a preceding paper[2] [see Eq. (16) there; equations of that reference will hereafter be designated, e.g., (I.16)]. The equivalent Hamiltonian is of the form

$$\mathcal{H}=-\sum_{nn'} t_{nn'}\alpha_n^{+}\alpha_{n'}\left\{\left(\frac{1}{2}+\hat{S}_{nz}\right)\left(\frac{1}{2}+\hat{S}_{n'z}\right)+\hat{S}_n^{+}\hat{S}_{n'}^{-}\right\}, \tag{1}$$

where α and $\hat{\mathbf{S}}$ are respectively the operators of the vacancy (meaning the shortage of an electron) and the pseudospin. This Hamiltonian is equivalent to the Hubbard Hamiltonian with infinite repulsion by the center under the condition that the eigenvalue of the operator

$$\hat{N}_2=\sum_n \alpha_n^{+}\alpha_n\left(\frac{1}{2}-\hat{S}_{nz}\right) \tag{2}$$

is equal to zero.

In the equivalent Hamiltonian the different degrees of freedom (spin and coordinate) of the electron are separated: the electron is replaced by vacancies (zero-spin Fermi particle) and pseudospins that coincide with the spin at the site occupied by the electron. It is clear already from the form of the equivalent Hamiltonian that an interaction takes place between the coordinate and the spin. This interaction, of course, has nothing in common with the usual spin-orbit interaction. A simplest manifestation of this interaction (the influence of the vacancy flux on the spin wave) was discussed in Ref. 2; we can go farther and deduce from the results of Ref. 2 an expression for the energy of a system of vacancies and magnons:

$$\mathcal{E}_0=\sum_{\mathbf{p}} f_v(\mathbf{p})\left\{\frac{p^2}{2}+\frac{1}{N_0}\sum_{\mathbf{k}} f_s(\mathbf{k})\left[\frac{1}{2}\frac{1-\gamma_0}{1+\gamma_0}k^2-\mathbf{kp}\right]\right\}. \tag{3}$$

The energy is measured here in units of $2t$ and we have left out an inessential constant term that depends only on the total number of vacancies; a quadratic approximation is used for the energies of the vacancy and the magnon; f_v and f_s are the respective distribution functions of the vacancies and the magnons; the interactions of the magnons with one another is not taken into account.

Expression (3) gives some idea of the mutual influence of the magnons and the vacancies; however, for example, the action of a spin wave of finite amplitude (magnon condensate) on the vacancy subsystem cannot be ascertained with the aid of (3) if only because this expression does not take the magnon interaction into account.

Thus, whereas the influence of the vacancies on the spin wave (via the vacancy flux[2]) is understandable in general outline, the reaction of the spin wave on the vacancy motion remains unexplained. The answer to this question is the main task of the present paper. In addition, we refine the equations of motion of the spin subsystems, which are known only for a homogeneous vacancy flux [see (I.42)–(I.44)].

2. We shall be interested exclusively in sufficiently slow spatial changes of the magnetic moment, without restrictions on its deviation from the equilibrium (ordered) value; zero temperature is implied, so that the magnetization is a maximum at each point (all the electron spins in a small volume have the same direction), but its direction varies from point to point. With respect to spin this is a typically classical (not quantum) picture, so that the magnon concept offers little and is not used here.

The first step is the use of a trial function. We consider for the system a wave function of the form

$$\Phi=\prod_n\left\{\begin{pmatrix}u_n\\ v_n\end{pmatrix}(1-\alpha_n^{+}\alpha_n)+\begin{pmatrix}1\\ 0\end{pmatrix}\alpha_n^{+}\alpha_n\right\}\Phi_0, \tag{4}$$

where Φ_0 depends only on the vacancy occupation numbers. The function Φ satisfies the additional condition $N_2=0$. At a site free of vacancies the spin direction is specified by the wave function $\binom{u_n}{v_n}$, with u_n and v_n varying, in accord with the foregoing, slowly from site to site. The following normalization condition is implied:

$$|u_n|^2+|v_n|^2=1. \tag{5}$$

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A function of the type (4) cannot be exact, but it turns out that with the aid of the obtained approximate expressions it is easy to obtain correct ones.

Averaging of the Hamiltonian (2) over the state (4) yields

$$(\Phi,\ \mathcal{H}\Phi)=(\Phi_0,\ \mathcal{H}_e\Phi_0), \tag{6}$$

$$\mathcal{H}_e=-\sum_{nn'} t_{nn'}\alpha_n^{+}\alpha_{n'}(u_nu_{n'}^{*}+v_nv_{n'}^{*}). \tag{7}$$

The problem is now simpler—there are no spin operators in the effective Hamiltonian $\mathcal{H}_e$.

Before simplifying (7) further by using the slowness of the functions u and v we consider, in order to understand the end purpose, the simple case of a plane spin wave:

$$u_n=u=\text{const},\qquad v_n=(1-u^2)^{1/2}e^{i\mathbf{k}\mathbf{R}_n}, \tag{8}$$

where u is real. Transforming to the momentum representation using the usual formulas (I.22), we obtain in place of (7) for the case (8) the expression

$$\mathcal{H}_e=\sum_{\mathbf{p}}\{u^2\varepsilon(\mathbf{p})+(1-u^2)\varepsilon(\mathbf{p}-\mathbf{k})\}\alpha_{\mathbf{p}}^{+}\alpha_{\mathbf{p}}, \tag{9}$$

where $\varepsilon(\mathbf{p})$ is the energy of the vacancy in the absence of the spin wave. The energy of the system described by the Hamiltonian (9) can be written in the approximation quadratic in the momenta in the form

$$\tilde{\mathcal{E}}=\sum_{\mathbf{p}} f_e(\mathbf{p})\left\{\frac{1}{2}[\mathbf{p}-(1-u^2)\mathbf{k}]^2+\frac{k^2}{2}u^2(1-u^2)\right\}, \tag{10}$$

where, just as in (3), expression (I.24) is used for $\varepsilon(\mathbf{p})$, the energy is measured in units of $2t$, and an inessential constant has been left out.

The main difference between (10) and (3) is the absence of the factor $(1-\gamma_0)/(1+\gamma_0)$ in the last term. This is due to the approximate character of the trial function. The corrected expression for the energy should be

$$\mathcal{E}=\sum_{\mathbf{p}} f_e(\mathbf{p})\left\{\frac{1}{2}[\mathbf{p}-(1-u^2)\mathbf{k}]^2+\frac{1-\gamma_0}{1+\gamma_0}\frac{k^2}{2}u^2(1-u^2)\right\} \tag{11}$$

and agrees with (3) at small spin-wave amplitude ($u^2\to 1$).

A few words on how to corroborate (11). The trial function (4) yields an incorrect value of the energy (10) because the function (4) fixes the spin of each site; in fact this is not so: when a vacancy jumps from site n to site n', the spin at the site n turns out to be that of site n' prior to the jump, and no other. If all the same we single out some direction in site n by specifying a spin function $\binom{u_n}{v_n}$, it is necessary to take into account in the $n\to n'$ vacancy jump the possibility of spin flip, inasmuch as the spin at site n turns out to have a wave function

$$\begin{pmatrix}u_{n'}\\ v_{n'}\end{pmatrix}=C_1\begin{pmatrix}u_n\\ v_n\end{pmatrix}+C_2\begin{pmatrix}-v_n^{*}\\ u_n^{*}\end{pmatrix}.$$

Inasmuch as the spin directions in neighboring sites differ little, $|C^2|\ll 1$ and the spin flip can be accounted for by perturbation theory. The result is Eq. (11).

The contribution of the second term in (11), which depends only on the total number vacancies but not on their momentum distribution, can be interpreted as the energy of the spin wave. If we express the spin-wave energy in the usual form (I.43) and choose the constant such that the correct small-oscillation frequency (I.40) is obtained, we get from (I.43) for the plane wave (8) the same result.

Renormalization of the energy of the vacancies in the spin-wave field reduces to a shift in momentum space:

$$\tfrac{1}{2}p^2\to\tfrac{1}{2}[\mathbf{p}-(1-u^2)\mathbf{k}]^2.$$

In our case this is a negligible change; the situation is different in the case (discussed later) of a time-dependent spin-wave amplitude.

We note that formula (11) is trustworthy, obviously, only in the limit of sufficiently large wavelength:

$$kn_v^{-1/3}\ll 1, \tag{12}$$

where n_v is the vacancy density.

We proceed now to the general case of an arbitrary slow change of u_n and v_n. We expand in (7) the differences $u_{n'}-u_n$ and $v_{n'}-v_n$ up to second order in $\mathbf{R}_{n'}-\mathbf{R}_n$, we change over to the momentum representation $\alpha_n\to\alpha_{\mathbf{p}}$, and assume the vacancy momentum small, as before. Leaving out the simple calculations, we present the final result in the form:

$$\mathcal{H}_e\to\sum\hat{\varepsilon}, \tag{13}$$

$$\hat{\varepsilon}=\tfrac{1}{2}(\hat{\mathbf{p}}-\mathbf{A})^2+\bar{U}(\mathbf{r}), \tag{14}$$

$$\mathbf{A}=-\tfrac{1}{2}i[(\psi^{*},\ \nabla\psi)-(\nabla\psi^{*},\ \psi)], \tag{15}$$

$$\bar{U}=\tfrac{1}{2}[(\nabla\psi^{*},\ \nabla\psi)-\mathbf{A}^2]. \tag{16}$$

Here, again, all the energies are in units of $2t$, and an inessential constant is left out of $\mathcal{H}_e$; summation over the vacancies is implied in (13), and the operator-energy part that depends on the coordinates of one vacancy is given by (14). The quantity ψ stands for the spinor $\psi=\binom{u}{v}$, and the matrix element of any operator $\hat{B}$ over the states ψ and ψ' is understood in the standard manner:

$$(\psi^{*},\ \hat{B}\psi')=\sum_{\sigma\sigma'}\psi_\sigma^{*}B_{\sigma\sigma'}\psi_{\sigma'}'$$

[the summation is over the spinor indices; for convenience the matrix element designation is different here from that of in (6)]. The condition (5) now takes the form

$$(\psi^{*},\ \psi)=1. \tag{17}$$

It is easy to verify that $\bar{U}\geqslant 0$.

Just as (10), Eq. (14) is approximate; it can be refined in analogy with (11), so that in place of (14) we have

$$\hat{\varepsilon}=\frac{1}{2}(\hat{\mathbf{p}}-\mathbf{A})^2+\frac{1-\gamma_0}{1+\gamma_0}\tilde{U}(\mathbf{r}). \tag{18}$$

The meaning of this operator is clear from the manner in which it was derived: when summed over all vacancies and averaged over the wave function of the vacancy system, it yields the energy of the entire system.

3. We proceed now to derive the equation of motion for the vacancies. At first glance it may appear that the equation of motion of the vacancy in a given spinor field ψ is simply the Schrödinger equation with the Hamiltonian (18). In fact this is not quite so. The operator (18) has a bearing on the energy of the entire system (and is conserved), whereas we are interested in the Hamiltonian of the vacancies at a specified motion of the spins (the energy of the vacancies is not conserved in the general case). The doubt seems nevertheless somewhat strange; after all, the Hamiltonian of the system is the sum of the operators (18), and it would be natural to assume (18) to play the role of the Hamiltonian of one vacancy. There is, however, another reason why (18) will not do for this role, and causes us to grope for a correct expression. Assume for a minute that (18) is the Hamiltonian of a vacancy. It follows from the form of this operator that the action of the spinor field on the particle recalls the action of an electromagnetic field. Understandably, the effective "electric" and "magnetic" fields can be connected only with the spin direction at each site, and this direction is not changed by the gauge transformation

$$\psi\to e^{i\varphi}\psi, \tag{19}$$

where φ is an arbitrary real function of the coordinates and the time. For example, at $\psi=e^{i\varphi}\binom{1}{0}$ there should be no fields at all. If the effective fields acting on the particle are not to be altered by this transformation, we have no choice but understand the Hamiltonian of the vacancy in the given spinor field to be the following operator:

$$\hat{\varepsilon}_v={}^1/_2(\hat{\mathbf{p}}-\mathbf{A})^2+U(\mathbf{r}),$$

$$U(\mathbf{r})=\frac{1-\gamma_0}{1+\gamma_0}\tilde{U}(\mathbf{r})+\frac{i}{2}\left[\left(\psi^*,\frac{\partial\psi}{\partial t}\right)-\left(\frac{\partial\psi^*}{\partial t},\psi\right)\right], \tag{20}$$

where in contrast to (18) the scalar potential contains an additional term that ensures gauge invariance [we note that $\tilde{U}$ is invariant to the transformation (19)]. This is in fact the sought Hamiltonian of the vacancy in the given spinor field.

Since this conclusion may see to be not fully corroborated, we shall show how the same result can be obtained by another method (all the more since in this way a possibility arises of obtaining the equations of motion also for the spin subsystem). We shall attempt to write down the Lagrangian of the system, regarding the vacancies as classical particles. The Lagrangian L must certainly contain a part

$$L_v=\sum\left\{\frac{1}{2}\dot{\mathbf{r}}^2+\dot{\mathbf{r}}\mathbf{A}-\frac{1-\gamma_0}{1+\gamma_0}\tilde{U}\right\} \tag{21}$$

(the sum is over the particles) that corresponds to (18) in the classical variant. In addition there must be a part that depends substantially on ψ in the sense that it does not vanish together with the number of vacancies. To find it, we consider the behavior of the spins in an external magnetic field $\mathbf{H}$; their energy is

$$\mathscr{E}_s=-\mu\int d^3r(1-\rho)\mathbf{SH},$$

where

$$\mathbf{S}=(\psi^*,\hat{\mathbf{S}}\psi), \tag{22}$$

ρ is the density of the number of vacancies, and it taken into account that the magnetic moment is proportional to the quantity $(1-\rho)\mathbf{S}$. The corresponding Lagrangian can be written in the form

$$L_s=L_s\ -\mathscr{E}_s,$$
$$L_s^{(0)}=\int d^3r(1-\rho)\frac{i}{2}\left[\left(\psi^*,\frac{\partial\psi}{\partial t}\right)-\left(\frac{\partial\psi^*}{\partial t},\psi\right)\right]. \tag{23}$$

By varying the action with respect to ψ^* we obtain for ψ an equation of motion that is used subsequently to calculate $\partial\mathbf{S}/\partial t$; this yields the known precession equation [it must be assumed here that ρ is independent of time, since the vacancy motion cannot be taken into account within the framework (23)]; in addition, from (23) follows also an expression for the energy $\mathscr{E}_s$. All this confirms the correctness of the expression assumed for the Lagrangian (23).

Thus, the Lagrangian of interest to us, of the entire system in the absence of external field, is

$$L=L_v+L_s^{(0)},\quad L=\sum\frac{\dot{\mathbf{r}}^2}{2}+\int d^3r\,\mathscr{L},$$
$$\mathscr{L}=\rho(\mathbf{vA})-\frac{1}{2}\frac{1-\gamma_0}{1+\gamma_0}\rho[(\nabla\psi^*,\nabla\psi)-\mathbf{A}^2]$$
$$+(1-\rho)\frac{i}{2}\left[\left(\psi^*,\frac{\partial\psi}{\partial t}\right)-\left(\frac{\partial\psi^*}{\partial t},\psi^*\right)\right], \tag{24}$$

where $\mathbf{v}$ is the average velocity of the vacancies; we used the slowness (12) of the spatial changes of ψ. It is easy to verify that (24) leads to a classical equivalent of (18); if we are interested in the equation of motion of a vacancy in a given field, it suffices to separate from (24) the part that depends on the vacancy coordinates; such a Lagrangian leads precisely to the Hamiltonian (20) in the classical variant, thereby confirming the conclusion drawn above concerning the meaning of (20).

4. We proceed now to derive the equations of motion of the spin subsystem. We use here the least-action principle, which led in the simplest case (23) to a correct result. It will be seen that the equations obtained are valid also in certain other cases.

Thus, taking (24) as the Lagrangian and varying the corresponding action with respect to ψ^* we obtain the equation

$$(1-\rho)i\frac{\partial\psi}{\partial\tau}=\frac{i}{2}\frac{\partial\rho}{\partial\tau}\psi$$
$$+\frac{1}{2}(i\nabla+\mathbf{A}+\mathbf{V})\{\rho(i\nabla+\mathbf{A}+\mathbf{V})\psi\}-\frac{1}{2}\rho(\mathbf{A}+\mathbf{V})^2\psi, \tag{25}$$

where

$$\tau=\frac{1-\gamma_0}{1+\gamma_0}t,\quad \mathbf{V}=\frac{1+\gamma_0}{1-\gamma_0}\mathbf{v}. \tag{26}$$

We are interested only in those solutions of (25) which satisfy the condition (17). Such solutions indeed exist. To verify this it suffices to consider the equation obtained in the usual manner for the probability density (ψ^*, ψ):

$$\begin{aligned}&\frac{\partial}{\partial\tau}[(1-\rho)(\psi^*,\psi)]+\operatorname{div}\mathbf{J}=0,\\ &\mathbf{J}={}^1/_2\rho[(\psi^*,(-i\nabla-\mathbf{A}-\mathbf{V})\psi)+\text{c.c.}].\end{aligned} \tag{27}$$

Recalling the definition (15) of **A**, we transform **J** into

$$\mathbf{J}=\rho\mathbf{A}[1-(\psi^*,\psi)]-\rho\mathbf{V}(\psi^*,\psi).$$

From the condition (17) on (27) we obtain

$$\partial\rho/\partial\tau+\operatorname{div}(\rho\mathbf{V})=0,$$

i.e., simply the continuity equation for a vacancy liquid. This necessary condition turns out to be also sufficient. We verify by the same token that an aggregate of the solutions of interest to us is contained among the total set of solutions of (25).

We show now how to obtain an equation for a physical quantity, the average spin **S** (22). Differentiating (22) with respect to time and using (25) we obtain for the component S_k.

$$\begin{aligned}&\frac{\partial}{\partial\tau}[(1-\rho)S_k]+\operatorname{div}\mathbf{I}_k=0,\\ &\mathbf{I}_k=-\rho S_k(\mathbf{A}+\mathbf{V})-{}^1/_2i\rho[(\psi^*,\hat{s}_k\nabla\psi)-\text{c.c.}].\end{aligned} \tag{28}$$

We note that this equation reflects the conservation of the total spin of the system, i.e., of the integral of the quantity $(1-\rho)\mathbf{S}$.

Using the completeness of the orthonormalized functions

$$\psi=\begin{pmatrix}u\\ v\end{pmatrix},\quad \bar{\psi}=\begin{pmatrix}-v^*\\ u^*\end{pmatrix},$$

we transform the matrix element $\hat{S}_k$:

$$(\psi^*,\hat{s}_k\nabla\psi)=S_k(\psi^*,\nabla\psi)+(\psi^*,\hat{s}_k\bar{\psi})(\bar{\psi}^*,\nabla\psi),$$

after which we get

$$\mathbf{I}_k=-\rho S_k\mathbf{V}-{}^1/_2i\rho[(\psi^*,\hat{s}_k\bar{\psi})(\bar{\psi}^*,\nabla\psi)-\text{c.c.}].$$

We now replace $\hat{S}_k$ by the operator identity

$$\hat{s}_k=-ie_{klm}\hat{s}_l\hat{S}_m$$

and write out fully the matrix element of the product of operators in the usual manner (we again use the completeness of $\psi, \bar{\psi}$),
obtaining

$$\mathbf{I}_k=-\rho S_k\mathbf{V}-\rho e_{klm}S_l[(\psi^*,\hat{S}_m\bar{\psi})(\bar{\psi}^*,\nabla\psi)+\text{c.c.}].$$

Finally, adding in the square bracket the zero-value expression

$$S_m[(\psi^*,\nabla\psi)+(\nabla\psi^*,\psi)],$$

we verify that

$$\mathbf{I}_k=-\rho\{S_k\mathbf{V}+e_{klm}S_l\nabla S_m\}. \tag{29}$$

Equation (28) with allowance for (29) is in fact the sought equation of motion of the spin system. It can be rewritten in the form

$$(1-\rho)\partial S_k/\partial\tau-\rho(\mathbf{V}\nabla)S_k=\operatorname{div}[\rho e_{klm}S_l\nabla S_m] \tag{30}$$

(retention of $\rho\ll1$ in the first term seems an exaggeration of the accuracy).

At $\rho=n_v=\text{const}$ and $\mathbf{V}=0$ we obtain simply the Landau-Lifshitz equation with only the exchange interaction (I.43) taken into account. In an arbitrary case the equation is generalized in trivial fashion to include an exchange interaction that depends on the coordinates and the time ($\rho\neq\text{const}$) and, in addition there appears a supplementary term, determined by the vacancy flux ($\mathbf{V}\neq0$), the meaning of which was discussed in Ref. 2.

We note that perhaps the most interesting is not so much the final result (28)–(30) as the method of obtaining it from the least-action principle and the intermediate result, namely Eq. (25) for the spinor ψ. It is also of interest that the motion of the spin liquid [see the first term in (29)] is obtained in natural fashion as a consequence of correct allowance for the interaction between the vacancies and the spins. Incidentally, the fact that the interaction depends on the velocity justifies the appearance of a vector-potential in the theory.

5. Thus, we have solved the problem and obtained the equations for two interacting subsystems (vacancy ion and spin). The cause of the interaction is obvious—the jump of the vacancy is accompanied by a jump of the spin; when the vacancy moves along a closed contour the spin configuration is changed, i.e., the energy is changed, thus indicating that an interaction takes place and that the interaction cannot be determined by the vacancy coordinate alone (hence the need for a vector potential). These arguments pertain in general to those ferromagnetic metals in which the spontaneous magnetic moment is produced by mobile particles; one can therefore expect the results obtained here within the framework of a simple model to remain basically in force also under more realistic conditions.

Let us emphasize the basic aspects of the paper. A Hamiltonian was used to obtain the approximate expression (14). The transformation to the exact expressions (18), (20) and (24) was practically unique with the aid of general considerations and by comparison with known results. The general requirements on the theory are invariance to independent rotations in the spin and coordinate spaces and gauge invariance; in addition to normalization condition (17) must be satisfied and only the lower-order derivatives are to be taken into account [the long-wave limit (12)]. These require-

ments make it possible to construct out of the spinor ψ a single vector **A** (15) in coordinate space and a single invariant, in all respects, $\tilde{U}$ (16). There are also the scalars

$$(\nabla\psi^*, \nabla\psi), \quad \mathbf{A}^2, \operatorname{div}\mathbf{A},$$

which are not gauge-invariant separately. Were any of them to appear in (18) [on going from (14) to (18)], it would appear also in the vacancy Hamiltonian (30), and this is inadmissible.

Concluding remarks. The spin subsystem acts on the vacancies via a scalar potential and a vector potential, which cannot be expressed in terms of the vector **S** and its derivatives; this is why we spoke throughout of interactions of vacancies with a spinor field and not with a field of constant-length vectors [so that Eq. (25) for the spinor ψ is in no way superfluous]. The effective magnetic and electric fields are expressed in terms of **S** in the following manner:

$$\mathbf{H}_e=\operatorname{rot}\mathbf{A}=2e_{klm}S_k[\nabla S_l\nabla S_m],$$
$$\mathbf{E}_e=-\frac{\partial\mathbf{A}}{\partial t}-\nabla U=-4e_{klm}S_k\frac{\partial S_l}{\partial t}\nabla S_m-\frac{1-\gamma_0}{1+\gamma_0}\nabla\tilde{U},$$
$$\tilde{U}=\frac{1}{2}\frac{\partial S_k}{\partial x_m}\frac{\partial S_k}{\partial x_m}.$$

We see therefore that the effective magnetic field can be quite strong—at the limit $kn_v^{-\frac{1}{3}}\sim 1$ of the applicability of the theory the magnetic field is of the order of the Fermi energy (all the quantities here are dimensionless).

As for the effective electric field, we note only one consequence of the theory—the appearance of a drawing electric field when the spin wave is pumped and relaxed. Indeed, for the plane wave (8) the vector potential is

$$\mathbf{A}=(1-u^2)\mathbf{k},$$

and the corresponding electric field is

$$\mathbf{E}_e=\mathbf{k}du^2/dt.$$

The introduction of true electromagnetic fields into the theory entails no difficulty. We point out in this connection only the appearance, in the nonuniform magnetic field, of an additional force acting on the vacancy:

$$\mathbf{E}_e=-\mu S_k\nabla H_k,$$

due to allowance for the energy $\mathscr{E}_s$ [see its definition ahead of Eq. (22)] and having a lucid physical meaning; an important role in the calculation of this force is played by the contributions of **A** and of the additional term in U, see (20).

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