Critical phenomena in media with random breeding centers

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The fluctuation shift of the threshold of explosive instability is determined for weak breeding centers and criteria are obtained for the validity of the self-consistent-field approximation for the description of the stationary above-threshold state that sets in when a nonlinear mechanism limits an explosive instability. The instability threshold in a medium is calculated in the case of strong breeding centers with account taken of the contribution from paired breeding-center clusters. The analysis is carried out for media with different dimensionalities.

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We consider here a medium in which an external random action produces for a short time spatial regions that serve as breeding centers for a certain substance. We discuss two fundamental problems, the calculation of the breeding-center critical density at which the threshold of explosive instability is reached in the medium, and the investigation of the fluctuation properties of the above-threshold stationary state that is established in the presence of a nonlinear mechanism that limits the explosive instability.

The system considered by us is far from the state of thermal equilibrium. We assume the large reserve of matter, whose consumption in the course of the breeding reaction can be neglected, is uniformly distributed over the volume of the medium. The energy needed to initiate the reaction in the breeding centers comes from external sources. We note that the appearance of random breeding centers in the medium can be due, e.g., to its irradiation by short irregular laser pulses.[1,2]

It has been repeatedly noted in recent years (see Ref. 3) that a feature of open systems that are in strong disequilibrium includes a special type of critical phenomena, namely effects of qualitative restructuring of the kinetic regime. Phenomena of this type include also the population of the medium above the threshold of explosive instability, when the presence of nonlinear limitation mechanisms in the medium establishes a constant average density level of the breeding matter.

When considering a transition from stationary crystallization of an ordered phase to a growth of a disordered phase at a certain critical deviation from equilibrium, the term "kinetic transition" was used in Ref. 4. We regard it as justified to extend the use of this term and include among the kinetic transitions any qualitative restructuring of the steady-state kinetic regime in open systems that are in strong disequilibrium.

If the kinetic transition is due to external random action, the fluctuating behavior near the transition point can be quite peculiar.[5–7] It will be shown here that for the effect considered by us, the population of a medium, the behavior is reminiscent in many respects of second-order phase transitions. In particular, there is critical slowing-down at the threshold, and the correlation radius of the density fluctuations becomes infinite at this point. At the same time the conditions for validity of the self-consistent-field approximation are different, namely, in a medium of dimensionality d the fluctuation behavior due to external random action is similar to that of a medium of dimensionality $d+2$ during an equilibrium second-order phase transition.

§1. FORMULATION OF MODEL

We consider diffusion of a certain substance X in a medium where decay $X \to P$ and breeding $X \to X + X$ of this substance are possible. We assume that the decay rate α is uniform in space and constant in time, and that the breeding takes place only inside definite breeding centers that appear at random times in random points of the medium, but have the same shape, intensity, and lifetime. The corresponding mathematical model is the equation

$$\dot{n}=-\alpha n+f(\mathbf{r},\ t)n+D\Delta n-\beta n^2, \tag{1}$$

where n is the density of the diffusing substance and D is the diffusion coefficient.

The fluctuating field $f(\mathbf{r},t)$ describes breeding centers that occur randomly independently of one another; this field is given by the sum of the identical pulses $g(\mathbf{r},t)$ located at random points $(\mathbf{r}_j, t_j)$:

$$f(\mathbf{r},t)=\sum_j g(\mathbf{r}-\mathbf{r}_j, t-t_j), \tag{2}$$

and the average number of pulses per unit time and per unit volume is constant and equal to m. Random processes of the type (2) are known as Poisson processes.

The last term in (1) ensures limitation of the explosive instability in the medium. Its origins can vary. We note first of all that such a term appears in the kinetic equation when account is taken of the coalescence reaction $X + X \to X$, which is the inverse of the breeding reaction. It is natural to expect that the coalescence reaction, in contrast to breeding, does not require activation, and therefore occurs every where in the medium. In addition, situations are possible when the decay product increases the decay rate (e.g., for some biochemical fermentation reactions[8]). If it is assumed that the restriction of the density growth is estasblished at sufficiently small values of n, the decay rate can be expanded in powers of the density n and only the linear (αn) and quadratic (βn^2) terms need be retained.

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We emphasize that the model considered by us is quite abstract. Equation (1) can appear in various application, including problems of mathematical ecology. In this respect it is similar to the basic models of percolation theory,[9] which have likewise extensive applications.

We assume that the function $g(\mathbf{r},t)$ that describes an individual breeding center is of the form

$$g(\mathbf{r},t)=\begin{cases} J\chi(\mathbf{r}); & 0\leq t\leq\tau_0,\\ 0; & t<0,\quad t>\tau_0.\end{cases} \tag{3}$$

The quality J characterizes the breeding intensity and τ_0 the lifetime of the center. The function $\chi(\mathbf{r})$, which determines the spatial shape of the center, decreases rapidly to zero at $r>r_0$, so that r_0 gives the spatial dimension of the breeding center; $\chi(0)=1$. It is convenient to introduce the dimensionless concentration of the breeding centers: $c=mr_0^d\tau_0$, where d is the dimensionality of the medium.

Depending on the relative density increment of an individual center, all the breeding centers can be divided into strong and weak. The density increment of a breeding center satisfies the equation

$$-\dot{n}=-D\Delta n-J\chi(\mathbf{r})n,\quad 0\leq t\leq\tau_0, \tag{4}$$

which coincide formally with the Schrödinger equation with imaginary time and potential $U(r)=-J\chi(r)$. Its general solution is

$$n(\mathbf{r},t)=\sum_l C_l e^{\lambda_l t}\varphi_l(\mathbf{r})+\int C_\lambda e^{\lambda t}\varphi_\lambda(\mathbf{r})\,d\lambda, \tag{5}$$

where the summation is over the discrete spectrum and the integration over the continuous spectrum of the linear operator

$$\hat{L}=D\Delta+J\chi(\mathbf{r}). \tag{6}$$

For breeding centers ($J>0$) the eigenvalues λ_i belonging to the discrete spectrum are positive. They correspond (cf. (4)) to negative-energy levels corresponding to bound states in the potential well $U(\mathbf{r})$.

Let λ_0 be the largest of the existing eigenvalues of the discrete spectrum. We define a breeding center as strong if $\lambda_0\tau_0\gg 1$ and weak if $\lambda_0\tau_0\ll 1$. It can be seen from the general solution (5) that the increment of matter on a strong center is exponetially large.[1)] The value of λ_0 can be connected with the parameters J and r_0 that characterize the properties of an individual center. We can use to this end the analogy with the Schrödinger equation and recognize that λ_0 corresponds to the deepest level in the potential well U. It is known (see Ref. 10) that in a deep well $J\gg D/r_0^2$ the lower level is of the order of

$$\lambda_0\sim J,\quad J\gg D/r_0^2,\quad d=1,\ 2,\ 3. \tag{7}$$

In the opposite limiting case when $J\ll D/r_0^2$, corresponding to a shallow potential well, the estimate of λ_0 depends on the dimensionality d of the medium. For a one-dimensional medium

$$\lambda_0\sim J(Jr_0^2/D),\quad J\ll D/r_0^2,\quad d=1; \tag{8a}$$

for a two-dimensional one

$$\lambda_0\sim\frac{D}{r_0^2}\exp\left[-\nu\frac{D}{Jr_0^2}\right],\quad J\ll\frac{D}{r_0^2},\quad \nu\approx 1; \tag{8b}$$

and in a three-dimensional medium such a shallow potential well does not contain any discrete levels at all, i.e., all the eigenvalues λ are negative.

Thus, the short-lived ($\tau_0\ll r_0^2/D$) breeding centers are weak if $J\ll J^*$, where

$$J^*=\tau_0^{-1},\quad d=1,\ 2,\ 3. \tag{9}$$

If, however, the centers are long-lived ($\tau_0\gg r_0^2/D$) they are weak at $J\ll J^*$, where

$$J^*=(D/r_0^2\tau_0)^{1/2},\quad d=1; \tag{10a}$$

$$J^*=D/r_0^2\,[\ln(D\tau_0/r_0^2)]^{-1},\quad d=2; \tag{10b}$$

$$J^*=D/r_0^2,\quad d=3. \tag{10c}$$

In the opposite limiting case $J\gg J^*$ the breeding centers are strong.

§2. DETERMINATION OF THE THRESHOLD OF EXPLOSIVE INSTABILITY FOR WEAK BREEDING CENTERS

The presence of a nonlinear limitation is immaterial for the determination of the threshold of explosive instability, since this threshold is determined by terms of the initial equation (1) that are linear in the density n. If we introduce the quantities averaged over the volume of the medium

$$\bar{n}=\langle n(\mathbf{r},t)\rangle,\quad \bar{f}=\langle f(\mathbf{r},t)\rangle,$$

as well as the corresponding fluctuating components $\delta n=n-\bar{n}$ and $\delta f=f-\bar{f}$, the changes of $\bar{n}$ and δn with time will satisfy the equations

$$\dot{\bar{n}}=-(\alpha-\bar{f})\bar{n}+\langle\delta f\delta n\rangle, \tag{11}$$

$$\delta\dot{n}=-(\alpha-\bar{f})\delta n+D\Delta\delta n+\delta f\bar{n}+(\delta f\delta n-\langle\delta f\delta n\rangle). \tag{12}$$

Since the average concentration $\bar{n}$ increases or decreases with time very smoothly near the critical point compared with the characteristic microscopic times, such as the lifetime τ_0 of an individual center, the quantity $\bar{n}$ in (12) can be regarded as constant. To determine the threshold of the explosive instability, which manifests itself in an exponential growth of the density $\bar{n}$, it is necessary to calculate first the paired correlator $\langle\delta f\delta n\rangle$ that enters in Eq. (11). For this calculation we use the perturbation-theory diagram technique for classical random processes, which was developed in Ref. 11 (see also Ref. 12).

We first change over in the stochastic differential equation (12) to Fourier components with respect to both the spatial and temporal variables. As a result this equation takes the form

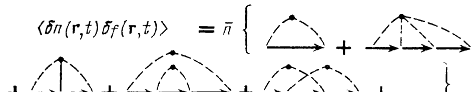
$$\delta n_q=G_q^0\left\{\delta f_q\bar{n}+\int(\delta f_{q-q'}\delta n_{q'}-\langle\delta f_{q-q'}\delta n_{q'}\rangle)\,dq'\right\}. \tag{13}$$

Here δn_q and δf_q are the Fourier transforms of the functions δn and δf, $q\equiv(\omega,\mathbf{k})$, and

$$G_q^0=(-i\omega+\alpha-\bar{f}+D\mathbf{k}^2)^{-1}. \tag{14}$$

A formal solution of the integral equation (13) can be constructed in the form of an infinite iteration series in powers of $\delta f_{q'}$. Multiplying this infinite series by δf_q and carrying out statistical averaging, we can obtain the corresponding expression for the correlator $\langle \delta f \delta n \rangle$, which is also an infinite series. In diagrammatic form, the first few terms of these series are represented by the diagrams

$$\langle \delta n(\mathbf{r},t)\delta f(\mathbf{r},t)\rangle = \bar{n}\Big\{ \qquad\qquad \Big\} \tag{15}$$



A solid line with an arrow denotes the function G_q^0, dashed lines with points on them denote irreducible correlators (cumulants) of the random functions δf_q. In contrast to Gaussian random processes, for which all higher-order correlators break into paired ones, the Poisson random process (2) has nonzero cumulants of all orders. Therefore several dashed lines can converge (be paired) on one point.

Each line in the diagram (15) is set in correspondence with some value of the momentum q. Integration is carried out over the momenta of all the lines. At the vertices at which one dashed and two solid lines converge, the momentum conservation law $q' = q'' + q'''$ is satisfied (q' is the momentum entering the vertex as a solid line, q'' is the momentum of the dashed line, q''' is the momentum going out of the vertex as a solid line). For the extreme left vertex $q' = 0$, and for the extreme right $q''' = 0$. The sum of the momenta of the dashed lines converging at one point is zero.

The distinguishing feature of the series (15) is that it does not contain weakly bound diagrams that decay when one of the lines is cut. These diagrams have already dropped out in the course of the averaging because of the second integrand in (13).

We introduce a function Σ_q defined as the sum of the diagrams of the series (15), under the assumption that a momentum q is introduced in the extreme left vertex of each diagram. The function introduced in this manner satisfies the equation

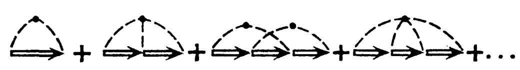
$$\langle \delta n(\mathbf{r},\ t)\delta f(\mathbf{r},\ t)\rangle = \bar{n}\Sigma_0. \tag{16}$$

It is convenient also to define the Green's function G^q that satisfies the Dyson equation[13]

$$G_q^{-1} = G_q^{0\ -1} - \Sigma_q. \tag{17}$$

With the aid of the function G_q, marked on the diagram by double arrows, we can carry out partial summation of the diagrams in the series for Σ_q:

$$\Sigma_q = \qquad\qquad \tag{18}$$



The explosive-instability threshold, as follows from (11) and (16), is defined by the equation

$$\alpha = \bar{f} + \Sigma_0. \tag{19}$$

Using the Dyson equation (17) and Eq. (14) for G_q^0 we obtain an expression for the Green's function

$$G_q = (-i\omega + \alpha - \bar{f} + Dk^2 - \Sigma_q)^{-1}. \tag{20}$$

It can be seen that at the threshold the Green's function has a pole at $q = 0$. Thus, the quantity Σ_0 determines the fluctuation shift of the threshold of the explosive instability. To calculate it we estimate the condition for the convergence of the series (18) at different relations between the parameters and at different dimensionalities of the medium.

If we retain in (18) only the first term, Σ_q is given by

$$\Sigma_0^{(1)} = \int S(q)\,G_q\,dq, \tag{21}$$

where $S(q)$ is the Fourier transform of the paired correlation function $\langle \delta f(\mathbf{r},t)\delta f(0,0)\rangle$, i.e.,

$$S(q) = (2\pi)^{-d-1}(c/\tau_0 r_0^d)\,D(\mathbf{k},\ \omega)\,D(-\mathbf{k},\ -\omega),$$
$$D(\mathbf{k},\ \omega) = J\tau_0 r_0^d(1-i\omega\tau_0)^{-1}\Psi(\mathbf{k}r_0), \tag{22}$$

where $\Psi(\mathbf{k}r_0)$ is the Fourier transform of $\chi(\mathbf{r})$, and c is the dimensionless concentration of the breeding centers. Substituting in (21) this expression for the paired correlation function and integrating, we get

$$\Sigma_0^{(1)} = \xi c J (J/J^*); \tag{23}$$

the coefficients ξ are given by the following expressions: at $D\tau_0 \ll r_0^2$ we have for $d = 1,2,3$

$$\xi = \frac{1}{2r_0^d}\int \chi^2(\mathbf{r})\,d\mathbf{r}; \tag{24}$$

At $D\tau_0 \gg r_0^2$ we have

$$\xi = \left[\frac{1}{2r_0^d}\int \chi(\mathbf{r})\,d\mathbf{r}\right]^2 \begin{cases} 1, & d=1, \\ (2\pi)^{-1}, & d=2, \\ \pi^{-2}, & d=3. \end{cases} \tag{25}$$

These coefficients are of the order of unity. Expressions (24) and (25) were obtained for a model in which it was assumed that the intensity of an individual breeding center does not remain constant during the time τ_0, but decreases smoothly like $\exp(-t/\tau_0)$ from the instant of creation of the center.

An estimate of the diagrams with irreducible correlators of order higher than second in the series (18) shows that each diagram with a correlator of order m makes a contribution

$$\Sigma_0^{(1)}\,(J/J^*)^{m-2}.$$

Since the condition $J \ll J^*$ is the criterion of the weakness of the breeding centers, such diagrams introduce for weak centers only a small correction to $\Sigma_0^{(1)}$.

The lowest of the diagrams with crossed dashed lines (the third in the series (18)) makes a contribution of order $\gamma^2\tau_0^2$, where

$$\gamma = cJ\tau_0(J/J^*). \tag{26}$$

Diagrams of this type introduce in Σ_0 terms that contain various powers of the breeding-center concentration. We note that the contribution $\Sigma_0^{(1)}$ from the first of the diagrams in the series (18) is of the order $\gamma\tau_0^{-1}$. Thus, the contribution from diagrams with intersection of dashed lines are small when the fluctuation shift of the threshold Σ_0 is itself small, i.e., under the condition $\gamma \ll 1$.

The condition $\gamma \ll 1$ imposes a restriction on the concentration of the breeding centers. On the other hand, the explosive-instability threshold is reached at a perfectly defined value of the concentration c_{cr}. Neglecting the fluctuation shift, the value of c_{cr} is given by the equality $\alpha = \bar{f}$ and is therefore of the order of $c_{cr}^0 \sim \alpha/J$. Stipulating that at this concentration the parameter γ be really small, we obtain the condition

$$(\alpha\tau_0)(J/J^*) \ll 1, \tag{27}$$

satisfaction of which makes the fluctuation shift of the threshold for small centers ($J \ll J^*$) small. In this case, taking into account the first fluctuation correction, the critical value of the breeding-center concentration is of the form

$$c_{cr} = \frac{\alpha}{\zeta_1 J}\left[1 - \xi\frac{J}{J^*}\right]; \quad \zeta_1 = \frac{1}{r_0^d}\int \chi\, d\mathbf{r}, \tag{28}$$

where the numerical coefficients ξ is of the order of unity.

If the inequality (27) is violated, the threshold of the explosive instability is reached at much lower values of the density c. The physical reason for such a strong lowering of the threshold was discussed by us earlier in Ref. 14, where we investigated the case of Gaussian fluctuations of the decay rates and of the breeding. If the condition (27) is violated, the principal role in the formation of the explosive-instability threshold is assumed by sparse clusters of week breeding centers, which can behave like individual strong centers. The calculation of the threshold of explosive instability in this situation calls for summation of an infinite sequence of diagrams in (18), containing intersections of dashed lines, and this is a rather complicated problem. We can only note that there is a preferred concentration value

$$c^* = J^*/J^2\tau_0, \tag{29}$$

at which $\gamma = 1$, and the contribution from the diagrams of all orders become of the same order of magnitude.

§3. FLUCTUATING BEHAVIOR IN A CRITICAL TRANSITION IN THE CASE OF WEAK CENTERS

In the presence of nonlinear-restriction mechanism, a stationary value of the density $\bar{n}$ of the breeding substance is established above the threshold of the explosive instability. We shall call this effect of populating the medium a kinetic transition in the considered system.

The steady-state value of the average density $\bar{n}$ is determined by the stationary solution of the equation

$$\dot{\bar{n}} = -\alpha\bar{n} + \bar{f}\bar{n} - \beta\bar{n}^2 + \langle\delta f \delta n\rangle - \beta\langle\delta n^2\rangle. \tag{30}$$

Neglecting the fluctuation corrections, i.e., the last two terms of (30), this equation has a solution

$$\bar{n} = \begin{cases} 0, & c < c_{cr}^0, \\ \dfrac{\alpha}{\beta}\left(\dfrac{c}{c_{cr}^0} - 1\right), & c \geqslant c_{cr}^0, \end{cases} \tag{31}$$

where c_{cr}^0 is obtained from the condition $\alpha = \bar{f}$.

As shown in §2, allowance for the fluctuations lowers the explosive-instability threshold c_{cr} compared with the value c_{cr}^0, which remains small when condition (27) is satisfied. In the present section we investigate the behavior of the density fluctuations $\langle \delta n^2 \rangle$ near the threshold and determine by the same token the region of validity of the self-consistent-field approximation for the considered kinetic transition.

Neglect of the density fluctuations, to which the self-consistent-field approximation reduces, is justified under the condition

$$[\langle\delta n^2\rangle]^{1/2} \ll \bar{n}. \tag{32}$$

It is known that in the theory of second-order phase transitions[15] this condition is always violated sufficiently close to the transition point if the dimensionality of the space is less than four. We shall see below that in a kinetic transition the situation is different. We note that since the density fluctuations δn are caused in our problem by an action external to the system, the correlator $\langle \delta n(\mathbf{r}) \delta n(0) \rangle$ need not diverge if the arguments are equal.

The density fluctuations δn satisfy the stochastic differential equation

$$\delta\dot{n} = -(\alpha + 2\beta\bar{n} - \bar{f})\delta n + D\Delta\delta n + \delta f(\mathbf{r}, t)\bar{n} + (\delta f \delta n - \langle\delta f \delta n\rangle) - \beta(\delta n^2 - \langle\delta n^2\rangle). \tag{33}$$

To find the value of $\langle \delta n^2 \rangle$ with the aid of this equation we can again use the Wyld diagram technique.[11]

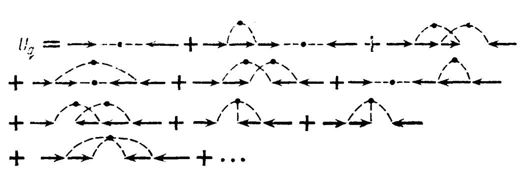
We introduce a function U_q defined by the relation

$$\langle\delta n_q \delta n_{q'}\rangle = U_q \delta(q + q')\bar{n}^2, \tag{34}$$

so that the following equality is satisfied

$$\langle\delta n^2\rangle = \bar{n}^2 \int U_q dq. \tag{35}$$

Iterating successively in the equation for δn_q obtained by a Fourier transformation of (33), from which we discard on the basis of the inequality (32) the terms in the last parentheses we can construct formally a solution for δn_q in the form of an infinite series in powers of the random force $\delta f_{q'}$. Multiplying two such infinite series for δn_q and carrying out the procedure of statistical averaging, we arrive at a formal series for the function U_q, the first diagrams of which are the following:

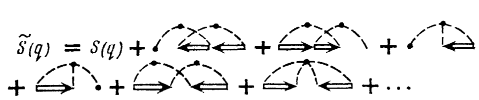


(36)

The summation in this series of weakly coupled diagram leads to the Wyld integral equation

$$U_q = |G_q|^2 \left\{ \tilde{S}(q) + \int S(q - q') U_{q'} dq' \right\}. \tag{37}$$

The function $\tilde{S}(q)$ in (37) is given by the diagram series



(38)

We leave out of (38) all the diagram terms and put approximately $\tilde{S}(q) = S(q)$. The, if we introduce a new function Z 1 defined by the relation $U_q = |G_q|^2 Z_q$, it will satisfy the integral equation

$$Z_q=S(q)+\int S(q-q')|G_{q'}|^2 Z_{q'}dq'. \qquad (39)$$

Since the Green's function G_q has at the threshold a pole at $q=0$, the main contribution to the integral near the threshold is made by the region of small q. Taking $S(q)$ outside the integral we obtain, after simple algebraic transformations, Z_q and then $\langle\delta n^2\rangle$:

$$\langle\delta n^2\rangle=\bar{n}^2\mu/(1-\mu), \qquad (40)$$

where

$$\mu=\int |G_q|^2 S(q)dq. \qquad (41)$$

The criterion (32), which ensures validity of the self-consistent-field approximation, is thus satisfied if $\mu\ll 1$. After performing the required calculations we can arrive at the following expressions for μ in the case of media of different dimensionality:

a) long-lived breeding centers, i.e., $\tau_0\gg r_0^2/D$, or $l\gg r_0$, where $l=(D\tau_0)^{1/2}$:

$$\mu\propto\begin{cases}\gamma(r_0/l), & d=3,\\ \gamma\ln(r_c/l)/\ln(l/r_0), & d=2,\\ \gamma(r_c/l), & d=1;\end{cases} \qquad (42)$$

b) short-lived breeding centers, i.e., $l\ll r_0$:

$$\mu\propto\begin{cases}\gamma(r_0/l)^2, & d=3,\\ \gamma(r_0/l)^2\ln(r_c/r_0), & d=2,\\ \gamma(r_0/l)^2(r_c/r_0), & d=1.\end{cases} \qquad (43)$$

We have left out of the expressions for μ numerical factors of the order of unity. These expressions contain (except for a three-dimensional medium) the correlation radius of the density fluctuations

$$r_c=[D/\alpha(c/c_{cr}-1)]^{1/2}, \qquad (44)$$

which becomes infinite at $c=c_{cr}$, i.e., at the threshold of the explosive instability. As a result, at $d=1$ and $d=2$ the value of μ increases when the critical point is approached, whereas in a three-dimensional medium ($d=3$) μ remains constant in the limit as $c\to c_{cr}$.

We note also that the parameter γ in expressions (42) and (43) is small ($\gamma\ll 1$) if the condition (27) is satisfied; it characterizes the fluctuation shift of the critical point (cf. (23) and (26)).

According to the terminology used in the theory of equilibrium phase transitions, the region near the critical point, where the self-consistent field approximation is not valid, is called a fluctuation region. As seen from (42), for long-lived breeding centers, when condition (27) is satisfied, there is no fluctuation region at all in the case of a three-dimensional medium. For a two-dimensional region it is exponentially narrow:

$$\left|\frac{c}{c_{cr}}-1\right|\lesssim(\alpha\tau_0)^{-1}\exp\left[-2\frac{\ln(l/r_0)}{\gamma}\right], \qquad (45)$$

and only in the case of a one-dimensional medium is the narrowness of such a region proportional to a power (namely, the square) of the small parameter γ:

$$\left|\frac{c}{c_{cr}}-1\right|\lesssim(\alpha\tau_0)^{-1}\gamma^2. \qquad (46)$$

For short-lived breeding centers ($l\ll r_0$) the expressions for μ contain in place of the parameter γ the combination $\gamma(r_0/l)^2$. This quantity is small under the condition

$$(\alpha r_0^2/D)(J/J^*)\ll 1, \qquad (47)$$

which is more stringent than (27). If the inequality (47) is satisfied, the fluctuation region does not exist at $d=3$ and is exponentially narrow at $d=2$:

$$\left|\frac{c}{c_{cr}}-1\right|\lesssim(\alpha r_0^2/D)^{-1}\exp\left[-\frac{2}{\gamma}\left(\frac{l}{r_0}\right)^2\right], \qquad (48)$$

and has a power-law narrowness at $d=1$.

$$\left|\frac{c}{c_{cr}}-1\right|\lesssim(\alpha r_0^2/D)^{-1}\gamma^2\left(\frac{r_0}{l}\right)^4. \qquad (49$$

Comparing these results with the conclusions of the theory of equilibrium second-order phase transitions, it can be noted that the fluctuations become significant for the considered kinetic transition at a lower dimensionality of the region. A fluctuation region exists in fact only in the one-dimensional case, and not starting with $d=3$.

Estimating the contributions from the different diagrams in the series (38), we can show that the contributions from diagrams with irreducible correlators of higher order, namely third, fourth, and sixth in the series (38), are small in the parameter J/J^*. For long-lived breeding centers ($l\gg r_0$) the smallness of the contributions from diagrams with crossing dashed lines (the first, second, and fifth diagram in the series (38)) is ensured by satisfaction of the condition (27), and for short-lived ones ($l\ll r_0$) by satisfaction of condition (47).

In our earlier paper[14] we studied an analogous kinetic transition in the case when the fluctuations in the decay rates and breeding were Gaussian. An erroneous statement was made there that the correlation radius of the density fluctuations remains finite at the transition point.[2)] Actually, however, the correlation radius diverges at the critical point also for Gaussian fluctuations

$$r_c=[D/\alpha(S/S_{cr}-1)]^{1/2}. \qquad (50)$$

Taking this circumstance into account, one more conclusion of Ref. 14 must be altered. For one-dimensional media, the self-consistent-field approximation remains applicable in the Gaussian situation not up to the transition point, but only outside the fluctuation region:

$$\begin{aligned}&\left(\frac{S}{S_{cr}}-1\right)\lesssim(l/r_{diff})^2, \quad r_0\ll l\ll r_{diff};\\ &\left(\frac{S}{S_{cr}}-1\right)\lesssim(r_0/r_{diff})^2, \quad l\ll r_0\ll r_{diff},\end{aligned} \qquad (51)$$

where $r_{diff}=(D/\alpha)^{1/2}$. The conclusion of Ref. 14 concerning two-and three-dimensional systems remain valid when account is taken of the divergence of the correlation radius. Although a fluctuation region does appear formally at $d=2$, it is exponentially narrow and can be neglected.

The noted deviations of the fluctuation behavior from the predictions of the theory of second-order phase transitions are far from accidental. It can be easily seen that in kinetic transitions such as population of a medium the fluctuation phenomena can never coincide with those observed in second-order phase transitions. Indeed, for a second-order transition only the average value of the order parameter $\langle\eta\rangle$ vanishes at the critical point, whereas the average fluctuation $\langle\delta\eta^2\rangle$ differs from zero at this point. In a kinetic transition such as population of a medium, the quantity n, which has the meaning of density, is nonnegative. It follows therefore from the vanishing of the average density n at the critical point that all the fluctuation densities n vanish at the same point, including the quantity $\langle\delta n^2\rangle$. In our opinion, this positiveness of the order parameter in a kinetic transition is in fact responsible for this peculiar fluctuation behavior.

§4. DETERMINATION OF THE EXPLOSIVE-INSTABILITY THRESHOLD FOR STRONG BREEDING CENTERS

The total increment of a breeding substance in an individual strong center during its lifetime is

$$\Delta N=n_0 r_1^d \exp(\lambda_0\tau_0), \tag{52}$$

where n_0 is the initial density level and is constant in space, while r_1 is the localization region of the eigenfunction $\varphi_0(\mathbf{r})$ of the operator (6) and corresponds to the maximum positive eigenvalue λ_0:

$$r_1=\left(\int\varphi_0(\mathbf{r})\,d\mathbf{r}\right)^{2/d},\quad \int\varphi_0^2(\mathbf{r})\,d\mathbf{r}=1. \tag{53}$$

The following estimates of r_1 are valid: $r_1\sim r_0$ at $J\gg D/r_0^2$ and $r_1=(D/\lambda_0)^{1/2}$ at $J\lesssim D/r_0^2$.

The explosive instability threshold is determined from the condition that the increment of matter in the breeding centers be equal to its decay per unit volume and per unit time. If the mutual influence of the centers is neglected, this condition can be represented in the form $m\Delta N=\alpha n_0$, so that the critical concentration m_{cr} of the centers is given by the simple expression

$$m_{cr}^{(0)}=(\alpha/r_1^d)\exp[-\lambda_0\tau_0]. \tag{54}$$

Expression (54) does not take correlation effects into account. Actually, if two centers turn out to be accidentally close enough to each other in space and in time, there appears, beside the individual increments in the two centers, and additional paired increment due to the fact that the exponential growth of the density in the second center does not start from the space-averaged density n_0 at that instant of time, but from a higher level, namely the density spot preserved from the preceding first center. More complicated additional contributions, due to clusters of three, four, and more centers, are also possible.

For a cluster of two breeding centers that appear at the instants t_1 and t_2 at the points $\mathbf{r}_1$ and $\mathbf{r}_2$, the value of the additional pair density increment is

$$\Delta N_{1,2}(\mathbf{r}_2-\mathbf{r}_1,t_2-t_1)=n_0 r_1^d\exp(2\lambda_0\tau_0)\int\varphi_0(\mathbf{r}-\mathbf{r}_2)\times G_d(\mathbf{r}-\mathbf{r}',t_2-t_1-\tau_0)\varphi_0(\mathbf{r}'-\mathbf{r}_1)\,d\mathbf{r}\,d\mathbf{r}', \tag{55}$$

where G_d is the Green's function for the diffusion problem in a space of dimensionality d:

$$G_d(\rho,\tau)=(4\pi D\tau)^{-d/2}\exp(-\rho^2/4D\tau). \tag{56}$$

In the derivation of (55) it was assumed that the two centers do not overlap in time, i.e., $t_2>t_1+\tau_0$.

The average contribution from the additional increment in paired clusters per unit volume and per unit time is

$$\overline{\Delta N_{1,2}}=\frac{m}{2}\int\Delta N_{1,2}(\rho,\tau)\,p(\rho,\tau)\,d\rho\,d\tau, \tag{57}$$

where $p(\rho,\tau)$ is the probability density for the appearance of a second center at a distance ρ and at a time τ after the first. It is known that for independently produced centers this distribution is of the form (see Ref. 16)

$$p(\rho,\tau)=m\exp[-mV_d\tau], \tag{58}$$

where V_d is the volume of a sphere of radius ρ in a space of dimensionality d. The quantity $\overline{\Delta N}_{1,2}$ must be compared with the average increment $\overline{\Delta N}_1=m\Delta N$ in single centers.

To calculate the average increment $\overline{\Delta N}_{1,2}$ it is necessary generally speaking, to know the explicit form of the eigenfunction $\varphi_0(\mathbf{r})$, a form determined by the concrete type of the breeding centers. We, however, are primarily interested in the limit of sufficiently low concentrations, when the centers in a paired cluster are far from each other in time. In this case it can be assumed that during the time of the diffusion spreading after the end of the action of the first center, the spot that remains from it extends to a region of space of size much larger than the initial radius r_1. When calculating the additional paired increment it is then possible to neglect the density inhomogeneity in the region where the next second center appears. As a result, the expression for $\overline{\Delta N}_{1,2}$ becomes much simpler:

$$\overline{\Delta N_{1,2}}={}^1/_2 n_0 m^2 r_1^{2d}\exp(2\lambda_0\tau_0)\times\int G_d(\rho,\tau)\exp[-mV_d(\rho)\tau]\,d\rho\,d\tau. \tag{59}$$

Calculating the integral for media of different dimensionality, it can be shown that, in order of magnitude

$$\overline{\Delta N_{1,2}}\sim n_0 r_1^{2d} m e^{2\lambda_0\tau_0}(m/D)^{d/(d+2)}. \tag{60}$$

Thus, the additional average increment in the paired clusters is small compared with the average increment on single centers if the following condition is satisfied:

$$\left[\frac{mr_1^{2+d}}{D}\right]^{d/(d+2)} e^{\lambda_0\tau_0}\ll 1. \tag{61}$$

This condition can be obtained also by stipulating that the volume of the space-time region, which is strongly perturbed because of the increment of the density in the breeding center, by much smaller than the volume per center. If the inverse condition is satisfied, the indicated regions corresponding to different centers overlap strongly and the independent-center approximation is patently invalid. On going through the concentration value

$$m^*=\frac{D}{r_1^{d+2}}\exp\left[-\left(1+\frac{2}{d}\right)\lambda_0\tau_0\right] \tag{62}$$

the contributions from the clusters made up of several breeding centers increase rapidly, and the increase of matter in the medium increases strongly at the same time.

The purpose of our analysis was to calculate the critical concentration of the breeding centers. Comparing (61) with (54) we see that the clusters do not play a decisive role in the formation of the explosive-instability threshold if the relation $\alpha \ll \alpha^*$ is satisfied, where

$$\alpha^* = (D/r_1^2) \exp\left(-\frac{2}{d}\lambda_0\tau_0\right). \quad (63)$$

With increasing decay rate α, the contribution from clusters consisting of two centers begins to increase. The expression for the explosive-instability threshold, with account taken for the correction due to paired clusters, takes at $\alpha \ll \alpha^*$ the form

$$m_{\text{cr}} = m_{\text{cr}}^0 \left\{1 - \gamma_d \left(\frac{\alpha}{\alpha^*}\right)^{d/(d+2)}\right\}, \quad (64)$$

where

$$\gamma_d = \begin{cases} (1/6)\Gamma(1/3), & d=1, \\ \sqrt{\pi}/8, & d=2, \\ {}^2/_5\Gamma(4/5)\,(3\sqrt{2}/5)^{2/5}, & d=3, \end{cases}$$

and $\Gamma(x)$ is the gamma function.

If $\alpha \gtrsim \alpha^*$, the threshold drops substantially compared with $m_{\text{cr}}^{(0)}$. The principal role in its determination is played then by clusters made up of a large number of centers.

In conclusion we wish to emphasize that the explosive-instability threshold for fluctuating media can depend substantially on the statistical characteristics of the random fluctuating configuration of the breeding centers. From the mathematical viewpoint the problems that arise here are no less interesting than the traditional problems of the theory of disordered media.[16] In the present paper we have classified for the model (1) the principal qualitatively different types of critical behavior, and indicated solutions for several simplest cases. Among the more difficult and unsolved problems is the calculation of the explosive-instability threshold when the condition (27) is violated for weak centers, as well as for strong centers at large decay rates when $\alpha \gtrsim \alpha^*$. We obtained criteria for the applicability of the self-consistent-field approximation to the description of a kinetic transition such as population of a medium. Its applicability to weak centers is restricted by the condition (27) or (47). Even if these conditions are satisfied, points where the fluctuations are strong exists in the one dimensional case a narrow fluctuation region near the transition. This fluctuation region can apparently be investigated by renormalization-group methods. What remains unclear is the character of the critical behavior in a kinetic transition for the case of strong centers, and also for weak centers when condition (27) or (47) is violated.

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[1)] We note that since $\lambda_0\tau_0$ is contained in the argument of the exponential, the centers are strong even at $\lambda_0\tau_0 \sim 3$ to 4.

[2)] The error in Ref. 14 is due to the fact that the correlation radius was calculated there from the behavior of the paired correlator of the density fluctuations, and in the calculation of this correlator no due account was taken of the terms that ensure the shift of the critical point. As a result, the correlation radius was actually estimated at a point where the correlator had no singularities whatever.

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