The functional equation method in the theory of exactly soluble quantum systems

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We propose a new method for evaluating the eigenvalues of the transfer matrix in exactly soluble quantum systems on a finite chain. We demonstrate the method for the *XXZ* Heisenberg model as an example and apply it to a spin system connected with the Zhiber-Shabat-Mikhaĭlov quantum field model and to a system with $O(n)$ symmetry. The latter is of special interest in connection with attempts to find an exact solution of the Gross-Neveu quantum model and of the **n**-field model.

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§1. INTRODUCTION

There are in the many-body system quantum theory well-known models for which the many-particle state wave functions can be evaluated exactly using the so-called Bethe coordinate Ansatz. Examples are the one-dimensional Heisenberg magnet,[1,2] the one-dimensional Bose gas with δ-function interactions,[3] the Thirring model,[4,5] and so on.

The advent of the quantum inverse scattering method[6,7] was an important stage in the development of the theory of exactly soluble quantum models. Using this method one found a deep connection between the solutions of the Yang-Baxter equation, integrable quantum systems, and exactly soluble models of statistical mechanics on a plane lattice.

The Yang-Baxter equation has the following form:

$$\sum_{k_1k_2k_3} R^{i_1i_2}_{k_1k_2}(u) R^{k_1i_3}_{j_1k_3}(u+v) R^{k_2k_3}_{j_2j_3}(v) = \sum_{k_1k_2k_3} R^{i_2i_3}_{k_2k_3}(v) R^{i_1k_3}_{k_1j_3}(u+v) R^{k_1k_2}_{j_1j_2}(u). \tag{1}$$

This is a functional equation for the matrix $R^{ij}_{kl}(u)$ which acts in the direct product of two spaces and which depends on the spectral parameter u. Equation (1) arose in the study of quantum systems with a factorized scattering.[9,10] It is the condition for the self-consistency of the factorization of the three-particle S-matrix in a product of two-particle ones; $R(u)$ then plays the role of the two-particle scattering matrix.

One can connect with the matrix $R(u)$ a model of statistical physics with states on the edges of a plane lattice. $R^{ij}_{kl}(u)$ must then be interpreted as the Boltzmann weights matrix corresponding to the site on which the edges with the states i, j, k, l meet. The partition function of such a model on an $N\times M$ square lattice with periodic boundary conditions can be expressed simply in terms of the transfer matrix:

$$\begin{gathered} Z=\operatorname{Tr}T(u)^M, \\ T(u)^{i_1\ldots i_N}_{j_1\ldots j_N}=\sum_{k_1\ldots k_N} R^{k_1i_1}_{k_2j_1}(u)\ldots R^{k_Ni_N}_{k_1j_N}(u) \\ \equiv(\operatorname{tr}_a R_{a1}(u)\ldots R_{aN}(u))^{i_1\ldots i_N}_{j_1\ldots j_N}. \end{gathered} \tag{2}$$

Here V_a is the space of the states on the horizontal edges (indexes $k_1\cdots k_N$), V_k is the space of the states on the vertical edges which go to the k th vertex (indexes i_k, j_k). The matrix $R_{ak}(u)$ is nontrivial only in the direct product of V_a by V_k:

$$R_{ak}(u)^{i_ai_1\ldots i_N}_{j_aj_1\ldots j_N}=R^{i_ai_k}_{j_aj_k}(u)\delta^{i_1}_{j_1}\ldots\delta^{i_{k-1}}_{j_{k-1}}\delta^{i_{k+1}}_{j_{k+1}}\ldots\delta^{i_N}_{j_N}. \tag{3}$$

The trace Tr is taken in the direct product of all spaces V_k with $k=1,\cdots,N$.

Baxter has shown for the 8-vertex model that Eq. (1) is the condition for exact solubility; using (3) it can be written in more compact form:

$$R_{12}(u)R_{13}(u+v)R_{23}(v)=R_{23}(v)R_{13}(u+v)R_{12}(u); \tag{4}$$

$T(u)$ then forms a commuting set:

$$[T(u),\ T(v)]=0. \tag{5}$$

The existence of this commutability enables us to consider $T(u)$ as a generating function for the Hamiltonians of quantum systems on a chain with an infinite number of conservation laws and with a space of states, V_k, on the k th vertex of the chain. Thus, if $R(u)$ is the matrix of the weights of the 8-vertex model, the corresponding quantum system is the Heisenberg spin-$\frac{1}{2}$ *XYZ* magnet.[11]

Knowing the eigenvalues of the transfer matrix $T(u)$ we can evaluate the partition function (2) in the thermodynamic limit, the asymptotic behavior of the corresponding correlation functions,[12] and also the spectrum of the integrable quantum system connected with $T(u)$.

The problem of the calculation of the spectrum of $T(u)$ is solved for an R-matrix with a special structure. To construct the eigenvectors of $T(u)$ in such models we use the algebraic Bethe Ansatz.[7] Unfortunately, if the R-matrix has a complicated structure there arise serious difficulties of a combinatorial nature in the construction of the many-particle states. The question of the applicability of the algebraic Bethe Ansatz to the majority of models connected with the solutions of Eq. (1) still remains open.

However, it was shown in Refs. 13 and 14 that the problem of evaluating the largest eigenvalue of $T(u)$ and those close to it can be solved in the thermodynamic limit as $N\to\infty$ without using the Bethe Ansatz by the "inverse transfer matrix" method.

In the present paper this method is extended to the evaluation of all eigenvalues of $T(u)$ for a chain of finite length N. We show that the eigenvalues of the transfer matrix for finite N satisfy a set of functional equations which one can solve by

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using the property that $R(u)$ is analytic in the spectral parameter. We use the Heisenberg XXZ model as an example to demonstrate the method in §2. Then, in §3 we find the spectrum of the transfer matrix, using the R-matrix of the Zhiber-Shabat-Mikhaĭlov quantum-field model to construct it. In §4 we consider a system with $O(n)$ symmetry.

§2. THE *XXZ* MODEL

The Hamiltonian of the Heisenberg XXZ model describes an interacting chain of spin-$\frac{1}{2}$ atoms:

$$H=\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z), \tag{6}$$

where N is the number of atoms (sites) in the chain, Δ the anisotropy parameter, σ_n^x, σ_n^y, σ_n^z are the spin operators (Pauli matrices) acting in the space of states of the nth site of the chain.

It is well known that the XXZ model is connected with the 6-vertex model for a ferroelectric. The Hamiltonian (6) is the logarithmic derivative of the transfer matrix of the 6-vertex model:

$$H=2(\Delta^2-1)^{1/2}T'(0)T(0)^{-1}-\Delta N, \tag{7}$$

where $T(u)$ is constructed, using (2). The Boltzmann weight matrix of the 6-vertex model has the following form:

$$R_{kl}^{ij}(u)=\operatorname{sh}u\,\delta_k^i\delta_l^j+\operatorname{sh}\eta\,\delta_l^i\delta_k^j+[\operatorname{sh}(u+\eta)-\operatorname{sh}u]\,\delta_k^i\delta_l^j(\delta_1^i\delta_1^j+\delta_2^i\delta_2^j). \tag{8}$$

The parameter η is connected with the anisotropy parameter in (6): $\Delta=\cosh\eta$.

In this section we shall demonstrate how we can use the analyticity properties and the degeneracy points of the matrix (8) to obtain explicitly the eigenvalues of $T(u)$. The Heisenberg model is sufficiently well studied by transitional methods[7] so that the present section is more of a methodological nature.

The matrix $R(u)$ is skew-symmetric:

$$R_{12}^{t_2}(u)=-\sigma_1^yR_{12}(-u-\eta)\sigma_1^y, \tag{9}$$

and automorphic in u[16]:

$$R_{12}(u+i\pi)=-\sigma_1^zR_{12}(u)\sigma_1^z. \tag{10}$$

Here σ_1^i is the ith Pauli matrix in the first space and t_2 the transposition in the second space.

In the point $u=-\eta$ the matrix $R_{12}(u)$ is proportional to the antisymmetrization operator P_{12}^-. It follows from Eq. (1) that

$$P_{12}^-R_{13}(u)R_{23}(u+\eta)=P_{12}^-R_{13}(u)R_{23}(u+\eta)P_{12}^-,$$

whence

$$P_{12}^-R_{13}(u)R_{23}(u+\eta)P_{12}^+=0. \tag{11}$$

[P_{12}^+ is the symmetrization operator; $P_{12}^{\pm}=(1\pm P_{12})/2$; P_{12} is the permutation operator].

From (11) we get a block triangularity for the product:

$$R_{13}(u)R_{23}(u+\eta)=\begin{pmatrix}\operatorname{sh}u\operatorname{sh}(u+2\eta) & 0\\ * & \operatorname{sh}(u+\eta)\tilde{R}_{(12)3}(u)\end{pmatrix}. \tag{12}$$

The matrix on the right-hand side is a 4×4 matrix, the matrix notation refers to the direct product $V_1\times V_2$, and the matrix elements act in V_3 ($V_i\equiv C^2$). The upper diagonal block corresponds to the one-dimensional antisymmetric subspace, the lower one to the three-dimensional symmetric one. $\tilde{R}(u)$ is regular when $u=-\eta$.

From (9), (10), and (12) we get relations for the transfer matrix $T(u)$:

$$T(u)T(u+\eta)=\operatorname{sh}u^N\operatorname{sh}(u+2\eta)^N+\operatorname{sh}(u+\eta)^N\tilde{T}(u), \tag{13}$$

$$T^t(u)=(-1)^NT(-u-\eta), \tag{14}$$

$$T(u+i\pi)=(-1)^NT(u). \tag{15}$$

We can easily evaluate immediately from (8) the main term in the asymptotic behavior of $T(u)$ as $u\to\infty$:

$$T(u)=\left(\frac{1}{2}\right)^Ne^{Nu}(e^{\eta(N-M)}+e^{\eta M})+O(e^{(N-1)u}), \tag{16}$$

$$M=\frac{1}{2}\left(N-\sum_{n=1}^{N}\sigma_n^z\right); \tag{17}$$

M is the magnon number operator which commutes with $T(u)$.

Operating with Eqs. (13) to (16) upon some eigenvector of $T(u)$ we get a set of functional equations for the eigenvalues of the transfer matrix:

$$\Lambda(u)\Lambda(u+\eta)=\operatorname{sh}u^N\operatorname{sh}(u+2\eta)^N+\operatorname{sh}(u+\eta)^N\tilde{\Lambda}(u), \tag{18}$$

$$\overline{\Lambda(u)}=(-1)^N\Lambda(-u-\eta), \tag{19}$$

$$\Lambda(u+i\pi)=(-1)^N\Lambda(u), \tag{20}$$

$$\Lambda(u)=\left(\frac{1}{2}\right)^Ne^{Nu}(e^{(N-m)\eta}+e^{m\eta})+O(e^{(N-1)u}),\quad u\to\infty. \tag{21}$$

$\Lambda(u)$ is the eigenvalue of $T(u)$ in the m-magnon state, $\tilde{\Lambda}(u)$ is the eigenvalue of $\tilde{T}(u)$ in the same state.

From the explicit form of $R(u)$ and from (16) and (17) it follows that

$$\Lambda(u)=\lambda^N\left(\frac{1}{2}\right)^N(e^{(N-m)\eta}+e^{m\eta})+\sum_{k=1}^{N}\lambda^{N-2k}c_k,\quad \lambda=e^u. \tag{22}$$

For the determination of $\Lambda(u)$ one must therefore find the N coefficients c_k, that is, $2N$ real unknowns. Equations (18) and (19) are a set of $2N+1$ equations for those unknowns. The cases $N=2,3$ show that there are necessarily amongst them dependent ones and the set (18), (19) has many solutions amongst which are all the eigenvalues of $T(u)$. It was suggested by Stroganov to use Eq. (18) for real eigenvalues.

One can easily find the solution of the set (18), (19) with $m=0$:

$$\Lambda^{(0)}(u)=\operatorname{sh}(u+\eta)^N+\operatorname{sh}u^N.$$

To construct other solutions we choose the following Ansatz:

$$\Lambda(u)=\operatorname{sh}(u+\eta)^NA(u)+\operatorname{sh}u^NB(u). \tag{23}$$

We shall call it the analytic Ansatz. It follows from (18) that A and B can only be rational functions of $\lambda^2=e^{2u}$ and each of them has the same number of zeros and poles. As $\Lambda(u)$ has no poles for finite u, the poles of A must be the same as the poles of B and the residues in them must cancel, i.e.,

$$A(u)=a\prod_{j=1}^{m}\frac{\lambda^2-\zeta_j}{\lambda^2-\xi_j},\quad B(u)=b\prod_{j=1}^{m}\frac{\lambda^2-\eta_j}{\lambda^2-\xi_j}. \tag{24}$$

The parameters η_j are determined by the sets ξ_j and ζ_j from the condition that the residue vanish for all $\lambda^2=\xi_j$. The asymptotic behavior as $\lambda^2\to\infty$ determines a and b and in (23) there remain $2m$ unknowns. Hence, when $2m\leqslant N$ the number of unknowns is not increased and Eq. (23) can be used as Ansatz for the solution of the set (14), (15).

Indeed, (23) satisfies the set (18), (19) if

$$A(u)\overline{A(-u)}=1,\quad B(u)=\overline{A(-u-\eta)}. \tag{25}$$

Hence we get the explicit form of the solutions of the set (18), (19) in the m-magnon subspace:

$$\Lambda(u)=\operatorname{sh}(u+\eta)^N e^{-m\eta}\prod_{j=1}^{m}\frac{e^{2u}-e^{\eta}\mu_j}{e^{2u}-e^{-\eta}\mu_j}+\operatorname{sh}(u)^N e^{m\eta}\prod_{j=1}^{m}\frac{e^{2u}-e^{-3\eta}\mu_j}{e^{2u}-e^{-\eta}\mu_j}. \tag{26}$$

The numbers μ_j are solutions of the following set of equations:

$$e^{-N\eta}\left(\frac{\mu_j e^{\eta}-1}{\mu_j e^{-\eta}-1}\right)^N=\prod_{\substack{k\neq j\\ k=1}}^{m}\frac{\mu_k e^{-\eta}-\mu_j e^{\eta}}{\mu_k e^{\eta}-\mu_j e^{-\eta}}, \tag{27}$$

where $\{\bar{\mu}_j\}=\{\mu_j^{-1}\}$ and $m\leqslant N/2$.

Comparing expressions (26) and (27) with those known before[7] we check that the Ansatz (23) enables us to evaluate all eigenvalues of the transfer matrix of the 6-vertex model.

In the following sections we shall by means of the analytic Ansatz method obtain expressions for the eigenvalues of the transfer matrices in some unsolved models.

§3. THE IZERGIN-KOREPIN MODEL

The quantum-mechanical system considered in this section arose in the study of the quantum-field Zhiber-Shabat-Mikhaĭlov model.[17] The corresponding R-matrix was found in Ref. 15, it is a 9×9 matrix, and we shall not give all nonvanishing matrix elements because of their unwieldiness. The diagonal elements which we need have the form

$$\begin{aligned}
&R_{11}^{11}(\lambda)=R_{33}^{33}(\lambda)\equiv a(\lambda)=\lambda e^{5\eta}-\lambda^{-1}e^{-5\eta}+e^{-\eta}-e^{\eta},\quad \lambda=e^u,\\
&R_{12}^{12}(\lambda)=R_{21}^{21}(\lambda)=R_{23}^{23}(\lambda)\\
&\qquad=R_{32}^{32}(\lambda)\equiv b(\lambda)=\lambda e^{3\eta}-\lambda^{-1}e^{-3\eta}+e^{-3\eta}-e^{3\eta},\\
&R_{13}^{13}(\lambda)=R_{31}^{31}(\lambda)\equiv c(\lambda)=\lambda e^{\eta}-\lambda^{-1}e^{-\eta}+e^{-\eta}-e^{\eta}.
\end{aligned}$$

The Hamiltonian of the system is the logarithmic derivative of the transfer matrix for $u=0$. It describes an interacting chain of N sites with a three-dimensional space of states in the site:

$$H=\sum_{\alpha,\beta=1}^{8}\sum_{n=1}^{N}\lambda_n^{\alpha}\lambda_{n+1}^{\beta}J_{\alpha\beta},\quad \lambda_{N+1}^{\alpha}\equiv\lambda_1^{\alpha}. \tag{28}$$

Here the λ_n^{α} are Gell-Mann matrices acting in the nth site, $J_{\alpha\beta}=J_{\alpha\beta}(\eta)$ are constants characterizing the interaction; they are some functions of η (Ref. 15) the explicit form of which we shall not give, to save space. The Hamiltonian possesses mixed P-symmetry:

$$H(\eta)=\mathscr{P}H(-\eta)\mathscr{P},\quad \mathscr{P}\lambda_n^{\alpha}\mathscr{P}=\lambda_{N-n+1}^{\bar{\alpha}}.$$

We shall show below that the eigenvalues of H are independent of the sign of η. The Hamiltonian H can thus be interpreted as a system interacting with an electric field under very special relations between the interaction constants and the magnitude of the field.

We proceed to the evalution of the eigenvalues of the transfer matrix of the model. The Izergin-Korepin R-matrix is skew-symmetric:

$$R_{12}^{t_2}(u)=Q_1^{-1}R_{12}(-u-6\eta-i\pi)Q_1,\quad Q=\begin{pmatrix}0&0&-e^{-\eta}\\0&1&0\\-e^{\eta}&0&0\end{pmatrix}. \tag{29}$$

The matrix $R_{12}(u)$ degenerates in the points $\lambda=-e^{6\eta}$, $-e^{-6\eta}$, $e^{4\eta}$, $e^{-4\eta}$ into the projectors on the subspaces of dimensionality 1, 8, 3, and 6, respectively. From the degeneracy of the R-matrix in those points one can obtain a block triangularity of the corresponding products[18] analogous to (12). Using the degeneracy points $\lambda=-e^{-6\eta}$, $\lambda=e^{4\eta}$ and the explicit form of $R(u)$ we get functional equations for the eigenvalues of $T(u)$:

$$\Lambda(\lambda)\Lambda(-\lambda e^{-6\eta})=a(\lambda)^N a(\lambda^{-1})^N+(\lambda-1)^N\tilde{\Lambda}(\lambda), \tag{30}$$

$$\Lambda(-\lambda e^{2\eta})\Lambda(-\lambda e^{-2\eta})=S(\lambda)^N\Lambda(\lambda)+(\lambda+e^{2\eta})^N\tilde{\tilde{\Lambda}}(\lambda). \tag{31}$$

Here $S(\lambda)=\lambda e^{3\eta}-\lambda^{-1}e^{-3\eta}+e^{5\eta}-e^{-5\eta}$, $\lambda=e^u$, $\tilde{\Lambda}(\lambda)$ and $\tilde{\tilde{\Lambda}}(\lambda)$ are the eigenvalues of other transfer matrices which have the same provenance as $\tilde{T}(u)$ in (13).

From the skew-symmetry and explicit form of the R-matrix we get

$$\overline{\Lambda(\lambda)}=\Lambda(-\lambda^{-1}e^{-6\eta}), \tag{32}$$

$$\Lambda(\lambda)=\lambda^N e^{3\eta N}(e^{2\eta(N-m)}+1+e^{-2\eta(N-m)})+O(\lambda^{N-1}),\quad \lambda\to\infty, \tag{33}$$

where m is an eigenvalue of the magnon number operator

$$M=\sum_{n=1}^{N}M_n,\quad M_n=\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix},$$

M commutes with $T(u)$.

We shall look for the eigenvalues of $T(u)$ in the form of the following Ansatz:

$$\Lambda(\lambda)=a(\lambda)^N A(\lambda)+b(\lambda)^N B(\lambda)+c(\lambda)^N C(\lambda). \tag{34}$$

Substitution of this expression into (30) to (32) shows that it is a solution of that set if the following relations hold

$$\begin{aligned}
&A(\lambda)\overline{A(\lambda^{-1})}=1,\\
&B(\lambda)=A(-\lambda e^{2\eta})A^{-1}(\lambda e^{4\eta}),\\
&C(\lambda)=A^{-1}(-\lambda e^{6\eta}).
\end{aligned} \tag{35}$$

Using the analytical properties of $R(u)$ this leads to the following answer for $A(\lambda)$ in an m-magnon state:

$$\Lambda(\lambda)=a(\lambda)^N\prod_{j=1}^{m}\frac{\lambda e^{-\eta}-\mu_j e^{\eta}}{\lambda e^{\eta}-\mu_j e^{-\eta}}+c(\lambda)^N\prod_{j=1}^{m}\frac{\lambda e^{4\eta}+\mu_j e^{-4\eta}}{\lambda e^{2\eta}+\mu_j e^{-2\eta}}+b(\lambda)^N\prod_{j=1}^{m}\frac{\lambda e^{3\eta}-\mu_j e^{-3\eta}}{\lambda e^{\eta}-\mu_j e^{-\eta}}\,\frac{\lambda+\mu_j}{\lambda e^{2\eta}+\mu_j e^{-2\eta}}. \tag{36}$$

The condition for the analyticity of $\Lambda(\lambda)$ for finite λ gives an equation for the numbers μ_j:

$$\left(\frac{\mu_k e^{\eta}-e^{-\eta}}{\mu_k e^{-\eta}-e^{\eta}}\right)^N=\prod_{\substack{j=1\\ j\neq k}}^{m}\frac{\mu_k e^{2\eta}-\mu_j e^{-2\eta}}{\mu_k e^{-2\eta}-\mu_j e^{2\eta}}\frac{\mu_k e^{-\eta}+\mu_j e^{\eta}}{\mu_k e^{\eta}+\mu_j e^{-\eta}}. \tag{37}$$

The set of numbers is such that $\{\mu_j^{-1}\}=\{\bar{\mu}_j\}$ and $m\leqslant N$.

When $u=\eta\theta$, $\eta\to 0$ and at fixed θ the Izergin-Korepin R-matrix becomes an SU(3) invariant R-matrix.[19] The corresponding limit leads in Eqs. (36) and (37) to results obtained in Ref. 19 using the algebraic Bethe Ansatz method. The results of the present section were obtained with V. I. Vichirko.

§4. *O(n)*-INVARIANT MAGNET

First of all we give a short description of the $O(n)$-invariant solutions of Eq. (1). The R-matrix is called $O(n)$ invariant if the spaces in which it acts are spaces of the representation of the $O(n)$ group and

$$R_{12}(u)=T_1(g)T_2(g)R_{12}(u)T_1(g)^{-1}T_2(g)^{-1}.$$

Here g is an element of the $O(n)$ group; T_1 and T_2 are representations of $O(n)$ in the spaces V_1 and V_2, respectively. Up to the present, three such R-matrices are known: $R^{(v,v)}(u)$ acting in the direct product of a vector representation by a vector one,[10] $R^{(v,s)}(u)$ in the product of a vector by a spinor representation,[20] and $R^{(s,s)}(u)$ in a product of a spinor by a spinor representation.[20] These R-matrices satisfy the Yang-Baxter relation:

$$R_{12}^{(a,b)}(u)R_{13}^{(a,c)}(u+v)R_{23}^{(b,c)}(v)=R_{23}^{(b,c)}(v)R_{13}^{(a,c)}(u+v)R_{12}^{(a,b)}(u), \tag{38}$$

a, b, c take on the values v or s independently, and the lower indexes indicate the number of the spaces.

The following R-matrices have the simples form:

$$R^{(v,v)}(u)_{j_1j_2}^{i_1i_2}=u(u+2-n)\delta_{j_1}^{i_1}\delta_{j_2}^{i_2}-2(u-n+2)\delta_{j_2}^{i_1}\delta_{j_1}^{i_2}+2u\delta^{i_1i_2}\delta_{j_1j_2}, \tag{39}$$

$$R^{(v,s)}(u)_{j\beta}^{i\alpha}=\left(u-\frac{n-2}{2}\right)\delta_j^{i}\delta_\beta^{\alpha}+(\sigma_{ij})_\beta^{\alpha}. \tag{40}$$

The explicit form of $R^{(s,s)}(u)$ is rather unwieldy and we do not need it. The dimensionality of the vector representation space equals n. The spinor representation is irreducible for $n=2k+1$ and its dimensionality equals 2^k, for $n=2k$ it splits into two irreducible representations with fixed chirality: $V^{(+)}$ with positive and $V^{(-)}$ with negative; the dimensionality of each equals 2^{k-1}. σ_{ij} are the $O(n)$ generators in the spinor representation:

$$\sigma_{ij}=\gamma_i\gamma_j-\delta_{ij},\quad \gamma_i\gamma_j+\gamma_j\gamma_i=2\delta_{ij}. \tag{41}$$

The chirality operator is defined for even n:

$$\Gamma=(-i)^{n/2}\gamma_1\gamma_2\ldots\gamma_n,\quad \Gamma^2=1,\quad [\Gamma,\gamma_i]=0. \tag{42}$$

For even n, solution (40) splits into two: $R^{(v,+)}$ and $R^{(v,-)}$ with fixed chiralities.

In this section we shall obtain by means of the analytic Bethe Ansatz method explicit expressions for the eigenvalues of the transfer matrices which we construct using (39) and (40):

$$T_v(u)=\mathrm{tr}_a R_{1a}^{(v,v)}(u)\ldots R_{Na}^{(v,v)}(u), \tag{43}$$

$$T_s(u)=\mathrm{tr}_a R_{1a}^{(v,s)}(u)\ldots R_{Na}^{(v,s)}(u). \tag{44}$$

For odd n the index s indicates the spinor representation, for even n the index $s=\pm$ which corresponds to R-matrices with fixed chirality.

The simplest $O(n)$ invariant Hamiltonian is the logarithmic derivative of $T_v(u)$ for $u=0$:

$$H=T_v'(0)T_v(0)^{-1}$$

$$=\sum_{l=1}^{N}\frac{1}{4}M_{ij}^{(l)}M_{ij}^{(l+1)}-\frac{n}{2(n-2)}e_{ij}^{(l)}e_{ij}^{(l+1)}\qquad M_{ij}^{(l)}=e_{ij}^{(l)}-e_{ji}^{(l)},$$

where the $e_{ij}^{(l)}$ are the basis matrices in the l th site of the chain $(e_{ij})_{ab}=\delta_{ia}\delta_{jb}$, $M_{ij}^{(l)}$ are the $O(n)$ generators in the l th site.

From (38) there follows the commutability of the sets T_v and T_s:

$$[T_v(u),T_v(w)]=[T_v(u),T_s(w)]=[T_s(u),T_s(w)]=0. \tag{45}$$

The R-matrices (39) and (40) are skew-symmetric:

$$R_{12}^{(v,v)}(u)^{t_1}=R_{12}^{(v,v)}(-u+n-2), \tag{46}$$

$$R_{12}^{(v,s)}(u)^{t_1}=R_{12}^{(v,s_c)}(-u+n-2); \tag{47}$$

$s_c=s$ for $n=2k+1$, $n=4k$, $s_c=-s$, $n=2k+2$. Hence we get for the eigenvalues of T_v and T_s

$$\overline{\Lambda_v(u)}=\Lambda_v(-u+n-2), \tag{48}$$

$$\overline{\Lambda_s(u)}=(-1)^N\Lambda_{s_c}(-u+n-2). \tag{49}$$

Using the degeneracy points of the matrices (39), (40), and $R^{(s,s)}(u)$ and the block triangularity of the corresponding products of the R-matrices we can obtain for Λ_v and Λ_s the following set of functional equations:

$$\Lambda_v(u)\overline{\Lambda_v(-u)}=(2^2-u^2)^N((n-2)^2-u^2)^N+u^N\Lambda_1(u), \tag{50}$$

$$\Lambda_v(u)\Lambda_s\left(u-\frac{n}{2}\right)=u^N(u-n)^N\Lambda_{s_b}\left(u-\frac{n-4}{2}\right)+(u-n+2)^N\Lambda_2(u), \tag{51}$$

$$\Lambda_s(u)\overline{\Lambda_s(-u)}=((n/2)^2-u^2)^N+\Lambda_3(u), \tag{52}$$

$$\Lambda_s\left(u+\frac{n-4}{2}\right)\Lambda_s\left(u-\frac{n-4}{2}\right)=\Lambda_v(u)+\Lambda_4(u). \tag{53}$$

Here $s_b=s$ for $n=2k+1, 4k+2$; $s_b=-s$ for $n=4k$; Λ_1, Λ_2, Λ_3, and Λ_4 have the same meaning as $\tilde{\Lambda}$ in (14).

In contrast to the analogous equations in the XXZ model, Eqs. (48) to (53) do not enable us to evaluate the expansion coefficients Λ_v and Λ_s in powers of the spectral parameter. The number of equations in them is less by N than the number of unknowns. This is connected with the fact that the set of transfer matrices T_v and T_s is not complete. To construct a complete set of transfer matrices we must find R-matrices acting in the product of a vector representation by an antisymmetric tensor representation of arbitrary rank. However, this problem is outside the framework of the present paper.

Nonetheless, the analytic Ansatz for the set (48) to (53) has a unique solution. Namely, we shall look for $\Lambda_v(u)$ and $\Lambda_s(u)$ in the following form:

$$\Lambda_v(u)=u^N(u+4-n)^NF(u)+u^N(u+2-n)^N\sum_{i=1}^{n-2}G_i(u)+(u-2)^N(u+2-n)^NH(u), \tag{54}$$

$$\Lambda_s(u)=\left(u-\frac{n-2}{2}+1\right)^N\sum_{i=1}^{\left[\frac{n-1}{2}\right]/2}A_i(u)$$

$$+\left(u-\frac{n-2}{2}-1\right)^N\sum_{i=1}^{\left[\frac{n-1}{2}\right]/2}B_i(u). \qquad (55)$$

The eigenvalues of T_v and T_s above the "ghost vacuum" have such a form with $F=G_i=H=A_i=B_i=1$. By analogy with models solved by the Bethe Ansatz[7,19] we may assume that arbitrary eigenvalues of T_v and T_s have the form (54) and (55) with some rational functions F, G_i, H, A_i, and B_i.

From the explicit form of the dependence of $R^{(v,v)}$ and $R^{(v,s)}$ on the spectral parameter it follows that the unknown functions in (54) and (55) can only be rational and that for each of them the number of zeros must equal the number of poles. The condition of analyticity of Λ_v and Λ_s for finite u imposes, as in previous models, a restriction upon the structure of the poles of these functions; it is necessary that the residues in the poles cancel in pairs. Substituting the expressions (54) and (55) into the set (48) to (53) we get for F, G_i, H, A_i, B_i functional equations which have a unique solution satisfying the above enumerated analyticity properties.

In order to write the solution compactly we introduce a set of functions:

$$\tau_2(u|\{u_1\}\{w\})=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\prod_{w}\frac{u-iw-2}{u-iw}+\prod_{w}\frac{u-iw+2}{u-iw}, \qquad (56)$$

$$\tau_3(u|\{u_1\}\{u_2\}\{w_+\}\{w_-\})$$
$$=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\overline{\tau_2(-u+1|\{u_2\}\{w_+\})}+\tau_2(u+1|\{u_2\}\{w_-\}),$$

$$\cdots$$

$$\tau_{2k}(u|\{u_1\}\ldots\{u_{2k-1}\}\{w_+\}\{w_-\})$$
$$=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\overline{\tau_{2k-1}(-u+1|\{u_2\}\ldots}$$
$$\overline{\ldots\{u_{2k-1}\}\{w_+\}\{w_-\})}+\tau_{2k-1}(u+1|\{u_2\}\ldots\{u_{2k-1}\}\{w_+\}\{w_-\}),$$

$$\tau_{2k+1}(u|\{u_1\}\ldots\{u_{2k}\}\{w_+\}\{w_-\})$$
$$=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\overline{\tau_{2k}(-u+1|\{u_2\}\ldots}$$
$$\overline{\ldots\{u_{2k}\}\{w_+\}\{w_-\})}+\tau_{2k}(u+1|\{u_2\}\ldots\{u_{2k}\}\{w_-\}\{w_+\}),$$

$$\cdots$$

$$\sigma_1(u|\{u_1\}\{w\})=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\prod_{w}\frac{u-iw-1}{u-iw}+\prod_{w}\frac{u-iw+1}{u-iw}, \qquad (57)$$

$$\sigma_k(u|\{u_1\}\ldots\{u_k\}\{w\})$$
$$=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}\overline{\sigma_{k-1}(-u+1|\{u_2\}\ldots\{u_k\}\{w\})}$$
$$+\sigma_{k-1}(u+1|\{u_2\}\ldots\{u_k\}\{w\}).$$

Here $\{u_k\}$ indicates the set of numbers $\{u_k^{(i)}\}_{i=1}^{m_k}$, and similarly $\{w_\pm\}$, $\{w\}$. The products in (56) and (57) are taken over all numbers from the appropriate set. For the eigenvalues of $T_s(u)$ we have

$n=2k+1$,

$$\Lambda_s(u)=\left(u-\frac{n-2}{2}+1\right)^N\overline{\sigma_{(n-3)/2}\left(-u+\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-3)/2}\}\{w\}\right)}$$
$$+\left(u-\frac{n-2}{2}-1\right)^N\sigma_{(n-3)/2}\left(u-\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-3)/2}\}\{w\}\right), \qquad (58)$$

$n=4k$,

$$\Lambda_\pm(u)=\left(u-\frac{n-2}{2}+1\right)^N\overline{\tau_{(n-2)/2}\left(-u+\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-4)/2}\}\{w_\pm\}\{w_\mp\}\right)}$$
$$+\left(u-\frac{n-2}{2}-1\right)^N\tau_{(n-2)/2}\left(u-\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-4)/2}\}\{w_\pm\}\{w_\mp\}\right), \qquad (59)$$

$n=4k+2$,

$$\Lambda_\pm(u)=\left(u-\frac{n-2}{2}+1\right)^N\overline{\tau_{(n-2)/2}\left(-u+\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-4)/2}\}\{w_\pm\}\{w_\mp\}\right)}$$
$$+\left(u-\frac{n-2}{2}-1\right)^N\tau_{(n-2)/2}\left(u-\frac{n-2}{2}+1|\{u_1\}\ldots\{u_{(n-4)/2}\}\{w_\mp\}\{w_\pm\}\right). \qquad (60)$$

For the functions F, G_l, H from (55) we get

$$n=2k+1, \quad H(u)=\prod_{u_1}\frac{u-iu_1+1}{u-iu_1-1}=\overline{F(-u+n-2)},$$

$$\ldots$$

$$G_l(u)=\prod_{u_l}\frac{u-iu_l-2-l}{u-iu_l-l}\prod_{u_{l+1}}\frac{u-iu_{l+1}-l+1}{u-iu_{l+1}-l-1},$$

$$\ldots \tag{61}$$

$$G_{k-1}(u)=\prod_{u_{k-1}}\frac{u-iu_{k-1}-1-k}{u-iu_{k-1}+1-k}\prod_{w}\frac{u-iw-k+2}{u-iw-k},$$

$$G_k(u)=\prod_{w}\frac{u-iw-2-k}{u-iw-k}\frac{u-iw-k-1}{u-iw-k+1},$$

$$G_l(u)=\overline{G_{n-l-1}(-u+n-2)}.$$

For $n=2k$ the functions $F, H, G_1 \ldots G_{k-3}$ retain their form, while

$$G_{k-2}(u)$$
$$=\prod_{u_{k-2}}\frac{u-iu_{k-2}-k}{u-iu_{k-2}-k+2}\prod_{w_+}\frac{u-iw_+-k+3}{u-iw_+-k+1}\prod_{w_-}\frac{u-iw_--k+3}{u-iw_--k+1},$$

$$G_{k-1}(u)=\prod_{w_+}\frac{u-iw_+-k+2}{u-iw_+-k+1}\prod_{w_-}\frac{u-iw_--k-1}{u-iw_--k+1}. \tag{62}$$

From the condition that the residues in Λ_v and Λ_s must vanish for finite u we get a set of equations for the numbers $\{u_l\}$, $\{w_\pm\}$, $\{w\}$:

$$n=2k+1, \quad \left(\frac{iu_1+1}{iu_1-1}\right)^N=\prod_{u_1'\neq u_1}\frac{iu_1-iu_1'+2}{iu_1-iu_1'-2}\prod_{u_2}\frac{iu_1-iu_2-1}{iu_1-iu_2+1},$$

$$\ldots$$

$$\prod_{u_{l-1}}\frac{iu_l-iu_{l-1}+1}{iu_l-iu_{l-1}-1}=\prod_{u_l'\neq u_l}\frac{iu_l-iu_l'+2}{iu_l-iu_l'-2}\prod_{u_{l+1}}\frac{iu_l-iu_{l+1}-1}{iu_l-iu_{l+1}+1},$$

$$\ldots \tag{63}$$

$$\prod_{u_{k-2}}\frac{iu_{k-1}-iu_{k-2}+1}{iu_{k-1}-iu_{k-2}-1}=\prod_{u_{k-1}'\neq u_{k-1}}\frac{iu_{k-1}-iu_{k-1}'+2}{iu_{k-1}-iu_{k-1}'-2}\prod_{w}\frac{iu_{k-1}-iw-1}{iu_{k-1}-iw+1},$$

$$\prod_{u_{k-1}}\frac{iw-iu_{k-1}+1}{iw-iu_{k-1}-1}=\prod_{w'\neq w}\frac{iw-iw'+1}{iw-iw'-1}.$$

For $n=2k$ the first $k-3$ equations are the same, while the last ones have the form

$$\prod_{u_{k-3}}\frac{iu_{k-2}-iu_{k-3}+1}{iu_{k-2}-iu_{k-3}-1}$$

$$=\prod_{u_{k-2}'\neq u_{k-2}}\frac{iu_{k-2}-iu_{k-2}'+2}{iu_{k-2}-iu_{k-2}'-2}\prod_{w_+}\frac{iu_{k-2}-iw_+-1}{iu_{k-2}-iw_++1}\prod_{w_-}\frac{iu_{k-2}-iw_--1}{iu_{k-2}-iw_-+1}$$

$$\tag{64}$$

$$\prod_{u_{k-2}}\frac{iw_\pm-iu_{k-2}+1}{iw_\pm-iu_{k-2}-1}=\prod_{w_\pm'\neq w_\pm}\frac{iw_\pm-iw_\pm'+2}{iw_\pm-iw_\pm'-2}.$$

One can easily find the isotopic structure of the eigenvectors from the asymptotic behavior of $T_s(u)$ as $u\to\infty$:

$$T_s\left(u+\frac{n-2}{n}\right)=Du^N+Du^{N-2}\left(C_2-\frac{N(n-1)}{2}\right)+O(u^{N-4}),$$
$$C_2=M_{ij}M_{ji}, \quad M_{ij}=\sum_{l=1}^{N}M_{ij}^{(l)}. \tag{65}$$

Here D is the dimensionality of the spinor representation. Comparing the asymptotic behavior of (58) to (60) as $u\to\infty$ we find that the eigenvectors corresponding to (58) to (62) are the leading vectors in the irreducible representation with signature $(N-m_1, m_1-m_2, \ldots, m_{k-1}-m)$ for $n=2k+1$ and $(N-m_1, \quad m_1-m_2, \ldots, m_{k-2}-m_+m_-, m_+-m_-)$ for $n=2k$.

In conclusion we note that as $N\to\infty$ Eqs. (58) to (62) duplicate the answers found in Refs. 10, 20 by the "inverse transfer matrix" method.

§5. CONCLUSION

The application of the analytic Ansatz model to other models[8,16,21] is of interest. No less important is the problem of the construction of the eigenvectors of the transfer matrix. To solve it we must, apparently, use the requirement of reasonable analyticity and symmetry of the eigenfunctions.

The example of the chiral Gross-Neveu model[5] has shown that the solution of the model of a G-invariant magnet turns out to be useful to find the spectrum of the corresponding G-invariant relativistic quantum-field model. The results of §4 may thus turn out to be useful for solving the $O(n)$ invariant Gross-Neveu model and the **n**-field model. The latter are of interest as they contain asymptotic freedom.

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