Superradiance of a multiatomic system with allowance for the Coulomb interaction

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Comparative estimates of the characteristic superradiance times and of the reciprocal atom–atom Coulomb interaction strength show that the interaction should have a significant effect on the superradiance of a system with a small Fresnel number. The effect of the Coulomb interaction on the superradiance of a linear chain of two-level atoms, which is the limiting case of a system with a small Fresnel number, is investigated on the basis of the semiclassical approach. The Coulomb interaction causes coherent transfer of excitation between the atoms, which leads to approximate spatial homogeneity of the inversion along the chain. This justifies the use of a spatially homogeneous model for systems with a small Fresnel number.

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1. INTRODUCTION

As a rule, in describing superradiance we consider the interaction of the atoms only via the transverse electromagnetic field, since it is precisely this field that phases the atomic radiators during the fluorescence, a phasing which leads to the well-known proportionality of the superradiance intensity to the square of the inversion density.[1–15] At the same time, for sufficiently dense systems (with atomic spacing smaller than the wavelength of the radiation), the Coulomb interaction can compete with the radiative interaction. It manifests itself, in particular, in the coherent transfer of excitation energy from one atom to another and, consequently, in the spatial variation of the population. The need for the consideration of the Coulomb part of the interaction can be judged by comparing the characteristic times τ_R and τ_C of the processes of collective spontaneous decay and excitation exchange between a pair of neighboring atoms. In the case of dipole emitters the time τ_C is of the order of the reciprocal dipole-dipole interaction strength, i.e., $\tau_C \sim \hbar a^3/\mu^2$, where a is the mean distance between the atoms and μ is the dipole moment of the transition. The superradiant-decay time τ_R depends on the relation between the wavelength λ of the radiation and the linear dimensions L of the active medium, as well as on the sample geometry.

The problem of collective spontaneous emission (without allowance for the Coulomb interaction) was formulated and solved[1] for a system of two-level atoms in a volume with linear dimensions $L \ll \lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-} = \lambda/2\pi$ (we shall call such a system the Dicke system). For it $\tau_R = \tau_0/N$, where N is the number of atoms in the system, $\tau_0 = 3\hbar\lambda\!\!\!^{-3}/4\mu^2$ being the radiative lifetime of one atom. For the ratio of τ_C to τ_R, we have $\tau_C/\tau_R \sim Na^3/\lambda\!\!\!^{-3} \sim (L/\lambda\!\!\!^{-})^3 \ll 1$. Thus, for the Dicke system the superradiant decay lags the variation of the population, and, consequently, the Coulomb interaction should be taken into account.

If the system extends greatly in one direction (i.e., if $L \gg \lambda\!\!\!^{-}$) and at the same time $D \ll \lambda\!\!\!^{-}$, where D is the transverse dimension, so that it is characterized by a Fresnel number $F = D^2/\lambda\!\!\!^{-}L \ll 1$ (a "pencil"), then $\tau_R \sim \tau_0 a/\lambda\!\!\!^{-}$ (Refs. 11 and 13). In this situation $\tau_C/\tau_R \sim (a/\lambda\!\!\!^{-})^2 \ll 1$, and, thus, the Coulomb interaction is, as in the case of the Dicke system, stronger than the radiative interaction.

Another limiting case of an extended system is the "disk" (i.e., the case in which $D^2/\lambda\!\!\!^{-}L \gg 1$). For the "disk"[11,13] $\tau_R \sim \tau_0/k_0 L$, where $k_0 = n_0\lambda\!\!\!^{-2}$ is the resonance absorption coefficient. Here $\tau_C/\tau_R \sim n_0 a^3 L/\lambda\!\!\!^{-}$. But $n_0 a^3 = 1$, and we obtain $\tau_C/\tau_R \sim L/\lambda\!\!\!^{-} \gg 1$. This inequality shows that, in a system with a large Fresnel number, the fluorescence can develop faster than the excitation transfer, so that the latter process can be neglected.

Attempts to take the effect of the Coulomb interaction on collective spontaneous emission into account have been made in a relatively small number of papers for the Dicke system ($L \ll \lambda\!\!\!^{-}$)[16–19] and for a small number of atoms.[20–22] In Ref. 16, using the semiclassical theory, Stroud *et al.* investigate the superradiance of the Dicke system with allowance for the Coulomb interaction under the assumption of spatial homogeneity of all the atomic characteristics over the sample volume. They show that in this approximation the electrostatic interaction has no effect on the superradiance dynamics, but leads to phase modulation. The effect of the Coulomb interaction on the superradiance of a short system is also discussed in Ref. 17, where it is concluded that the dephasing of the dipole moments of the atoms is a result of the inhomogeneity of the effective field over a period of time much shorter than the characteristic superradiance time. But the conclusion that the behavior of the atoms depends on their position is not quite consistent, since it is based on the assumption of spatial homogeneity of the polarization. In Ref. 20 the effect of the electrostatic interaction on the cooperative emission of two and three atoms is investigated on the basis of the quantum-mechanical equations for the reduced density matrix. It is noted that the Coulomb interaction has a dephasing effect if the surroundings of the various atoms are not equivalent. In Ref. 22 the Coulomb interaction is taken into consideration in a quantum-mechanical description of the initial stage of superradiant emission by a system consisting of a few atoms.

In the present paper we investigate the effect of the Coulomb interaction on the superradiant decay of a linear chain of equidistant equivalent two-level atoms that were in the

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excited state at the initial moment of time. The linear chain is the limiting case of a multiatomic system with a Fresnel number $F \ll 1$. It contains the Dicke model as a particular case.

2. THE SEMICLASSICAL APPROXIMATION

To solve the formulated problem, we use the semiclassical approach in which the two-level atoms are described quantum-mechanically with the aid of the single-particle density matrix $\rho_{ab}^{(k)}$ (the indices a and b number the ground and excited states of the atom and k numbers the sites along the chain) and the electromagnetic field is described classically. For the density matrix $\rho_{ab}^{(k)}$ we have the system of equations

$$\begin{aligned}
&\dot{\rho}_{11}^{(k)} = -i\hbar^{-1}\mu_k \mathbf{E}_k[\rho_{12}^{(k)} - \rho_{21}^{(k)}],\\
&\dot{\rho}_{22}^{(k)} = i\hbar^{-1}\mu_k \mathbf{E}_k[\rho_{12}^{(k)} - \rho_{21}^{(k)}],\\
&\dot{\rho}_{21}^{(k)} = -i\omega_0\rho_{21}^{(k)} + i\hbar^{-1}\mu_k \mathbf{E}_k[\rho_{11}^{(k)} - \rho_{22}^{(k)}], \quad \rho_{12}^{(k)} = [\rho_{21}^{(k)}]^*,
\end{aligned} \tag{1}$$

where μ_k is the transition dipole moment of the k-th atom, ω_0 is the resonance frequency of the atom, and $\mathbf{E}_k$ is the electric field acting on the atom at the site k. This field is represented in the form of a superposition of the fields $\mathbf{E}_{lk}$ produced at the site k by all the remaining atoms l, plus the self-action field $\mathbf{E}_{kk}$:

$$\mathbf{E}_k = \sum_l \mathbf{E}_{lk}. \tag{2}$$

The electromagnetic field emitted by an atom is treated as the radiation emitted by a classical dipole with electric moment d equal to the mean quantum-mechanical dipole-moment operator. The intensity of the electric field produced by the atoms l at the site $k \neq l$ is equal to

$$\begin{aligned}
\mathbf{E}_{lk}(t) = &\left[\frac{3d_l(t')}{r_{lk}^3} + \frac{3\dot{d}_l(t')}{cr_{lk}^2} + \frac{\ddot{d}_l(t')}{c^2 r_{lk}}\right](\mathbf{m}_l\mathbf{n}_{lk})\mathbf{n}_{lk}\\
&-\left[\frac{d_l(t')}{r_{lk}^3} + \frac{\dot{d}_l(t')}{cr_{lk}^2} + \frac{\ddot{d}_l(t')}{c^2 r_{lk}}\right]\mathbf{m}_l,
\end{aligned} \tag{3}$$

where

$$d_l = \mu_l[\rho_{12}^{(l)} + \rho_{21}^{(l)}], \quad t' = t - \frac{r_{lk}}{c}, \quad \mathbf{m}_l = \frac{\mathbf{d}_l}{d_l}, \quad \mathbf{n}_{lk} = \frac{\mathbf{r}_{lk}}{r_{lk}}.$$

The allowance for the self-action field $\mathbf{E}_{kk}$ in (2) guarantees the preservation of the energy balance[1]:

$$dU_a/dt = -I, \quad I = \oint \mathbf{S}(t + r_0/c)\, d\sigma. \tag{4}$$

Here U_a is the energy stored in the atomic subsystem, I is the total radiation intensity, and $\mathbf{S}$ is the Poynting vector averaged over the period of the atomic vibrations. The integration in (4) is performed over a sphere of radius $r_0 \gg L, \lambda$.

3. THE SLOWLY-VARYING-AMPLITUDE APPROXIMATION

Let us assume that the characteristic time of the variation of the amplitudes $R_k^{\pm}(t)$ is significantly longer than ω_0^{-1}, and let us separate the fast dependences in the atomic and field characteristics:

$$\rho_{21}^{(k)} = R_k^- e^{-i\omega_0 t}, \quad \rho_{12}^{(k)} = R_k^+ e^{i\omega_0 t}. \tag{5}$$

Substituting (5) into (2) and (3), we obtain for $\mathbf{E}_{lk}$ and $\mathbf{E}_k$ expressions similar to (5):

$$\mathbf{E}_{lk} = \mathbf{E}_{lk}^- e^{-i\omega_0 t} + \mathbf{E}_{lk}^+ e^{i\omega_0 t}, \quad \mathbf{E}_k = \mathbf{E}_k^- e^{-i\omega_0 t} + \mathbf{E}_k^+ e^{i\omega_0 t}, \tag{6}$$

where the amplitudes $\mathbf{E}_{lk}^{\pm}$ and $\mathbf{E}_k^{\pm}$ are given by the formulas

$$\begin{aligned}
\mathbf{E}_{lk}^{\pm}(t) = &\left\{\left[\frac{3}{r_{lk}^3} \pm \frac{3ik_0}{r_{lk}^2} - \frac{k_0^2}{r_{lk}}\right](\mu_l\mathbf{n}_{lk})\mathbf{n}_{lk}\right.\\
&\left.-\left[\frac{1}{r_{lk}^3} \pm \frac{ik_0}{r_{lk}^2} - \frac{k_0^2}{r_{lk}}\right]\mu_l\right\} R_l^{\pm}(t')\exp(\mp ik_0 r_{lk}), \quad k_0 = \frac{\omega_0}{c},
\end{aligned} \tag{7}$$

$$\mathbf{E}_k^{\pm} = \sum_{l \neq k} \mathbf{E}_{lk}^{\pm}. \tag{8}$$

As follows from (7), the retardation of the interaction manifests itself in the form of an oscillating exponential factor and in the argument of the amplitude of the off-diagonal element of the density matrix. We shall neglect the retardation in the amplitudes, assuming that the time L/c of propagation of light through the system is shorter than the characteristic superradiance time τ_R. Notice that this condition imposes a limitation on the number of particles in the linear chain: $N < \omega_0\tau_0$.

Substituting the relation (6) into the system (1) and neglecting the rapidly oscillating terms we arrive at the following system of equations for the slow amplitudes:

$$\begin{aligned}
&\dot{R}_k^{\pm} = \pm 2i\frac{\mu_k \mathbf{E}}{\hbar} Z_k, \quad \dot{Z}_k = i\frac{\mu}{\hbar}(\mathbf{E}_k^- R_k^+ - \mathbf{E}_k^+ R_k^-),\\
&Z_k = \frac{\rho_{22}^{(k)} - \rho_{11}^{(k)}}{2}.
\end{aligned} \tag{9}$$

It is not difficult to verify that the square $Z_k^2 + R_k^+ R_k^-$ of the length of the Bloch vector for each atom is conserved in the course of the evolution described by Eqs. (9). Thus, the system (9) has N integrals of motion.

The initial conditions for the system of equations in question are prescribed in the standard—for the semiclassical approach—form: all the atoms are excited at the initial moment of time, and a small initial polarization simulating the spontaneous decay is prescribed:

$$Z_k(0) = 1/2, \quad R_k^{\pm}(0) = R_0^{\pm}. \tag{10}$$

We did not, in deriving the truncated system, separate out the spatial factor $\exp(i\mathbf{k}_0\cdot\mathbf{r})$, as is often done in the semiclassical approximation.[3,8,10-14] Therefore, the truncated system of equations (9) allows us to follow the changes that occur in the character of the superradiance as we go from the Dicke model ($L \ll \lambda$) to the extended system ($L \gg \lambda$).

The system (9) can be rewritten in the form

$$\begin{aligned}
&\dot{R}_k^{\pm} = \mp i\sum_{l \neq k} \Omega_{lk} R_l^{\pm} Z_k + \sum_{l \neq k} \gamma_{lk} R_l^{\pm} Z_k,\\
&\dot{Z}_k = -1/2\, i\sum_{l \neq k} \Omega_{lk}(R_l^- R_k^+ - R_l^+ R_k^-)\\
&\qquad -1/2\sum_{l \neq k} \gamma_{lk}(R_l^- R_k^+ + R_l^+ R_k^-),
\end{aligned} \tag{11}$$

where the matrices Ω_{lk} and γ_{lk} are given by the formulas

$$\Omega_{lk}=\frac{2}{\hbar}\left[\left(\frac{\chi_{lk}}{r_{lk}^3}-k_0^2\frac{\varkappa_{lk}}{r_{lk}}\right)\cos k_0r_{lk}+k_0\frac{\chi_{lk}}{r_{lk}^2}\sin k_0r_{lk}\right], \quad (12)$$

$$\gamma_{lk}=\frac{2}{\hbar}\left[\left(k_0^2\frac{\varkappa_{lk}}{r_{lk}}-\frac{\chi_{lk}}{r_{lk}^3}\right)\sin k_0r_{lk}+k_0\frac{\chi_{lk}}{r_{lk}^2}\cos k_0r_{lk}\right], \quad (13)$$

$$\chi_{lk}=(\mu_l\mu_k)-3(\mu_l\mathbf{n}_{lk})(\mu_k\mathbf{n}_{lk}), \qquad \varkappa_{lk}=(\mu_l\mu_k)-(\mu_l\mathbf{n}_{lk})(\mu_k\mathbf{n}_{lk}). \quad (14)$$

The matrix γ_{lk} coincides with the relaxation matrix computed in the quantum theory using the Hamiltonian for the interaction of the atoms with only the transverse field, and describes the decrease, due to the emission, of the energy stored in the atomic subsystem. This can easily be verified by summing the second equation in (11) over the atoms:

$$\sum_k \dot{Z}_k=-\sum_{l,k}{}' \gamma_{lk}R_l^+R_k^-. \quad (15)$$

The matrix Ω_{lk} describes the frequency shifts caused by both the static dipole–dipole and the radiative interactions. It also determines the redistribution of the excitations among the atoms of the chain.

The matrices γ_{lk} and Ω_{lk} have their simplest forms in the case of the Dicke system. For $k_0r_{lk}\ll 1$, it follows from (12) and (13) that

$$\Omega_{lk}=(2/\hbar)\,[1/r_{lk}^3-k_0^2/2r_{lk}]\chi_{lk}, \quad (16)$$

$$\gamma_{lk}=4\mu^2k_0^3/3\hbar=\gamma_0. \quad (17)$$

The dominant contribution to the frequency-shift matrix Ω_{lk} is made by the static dipole–dipole interaction, and the matrix element γ_{lk} coincides with the radiative constant of the atom.

We shall determine the intensity of the radiation in a given direction as the Poynting vector in the wave zone $r_0\gg L,\lambda$ averaged over the period of the atomic vibrations:

$$\mathbf{S}(\mathbf{r}_0,\ t)=(c/4\pi)\,[\overline{\mathbf{EH}}], \quad (18)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields produced by the system at the point $\mathbf{r}_0$ (the middle of the chain is taken as the origin). When $r_0\gg L,\lambda$, the vectors $\mathbf{E}$ and $\mathbf{H}$ are equal in magnitude, and for $\mathbf{E}$ we have from (7) and (8) the expression

$$\mathbf{E}(\mathbf{r}_0,t)=2k_0^2\sum_l\frac{\mu_l-(\mu_l\mathbf{n}_l)\mathbf{n}_l}{r_l}\,|R_l^{\pm}(t')|\cos[\omega_0t'+\varphi_l(t')]. \quad (19)$$

Here r_l is the distance, which can be assumed to be approximately equal to r_0, of the l-th dipole to the point of observation, $\mathbf{n}_l=\mathbf{r}_l/r_l$, $t'=t-r_l/c$, and φ_l is the phase of the amplitude $R_l^{\pm}$. The time scale of the variation of the φ_l, like that of the $|R_l^{\pm}|$, is significantly greater than ω_0^{-1}; therefore, we shall consider them to be constants in the averaging over the period of the atomic vibrations in (18). Thus,

$$\mathbf{S}(\mathbf{r}_0,t)=\mathbf{n}_0\frac{ck_0^4}{2\pi r_0^2}\sum_{l,l'}[(\mu_l\mu_{l'})-(\mu_l\mathbf{n}_0)(\mu_{l'}\mathbf{n}_0)]$$

$$\times R_l^+(t')R_{l'}^-(t')\exp\{ik_0r_{ll'}\cos\theta_0\}, \quad (20)$$

where $\mathbf{n}_0$ is the unit vector along $\mathbf{r}_0$ and θ_0 is the angle between the axis of the system and the direction $\mathbf{n}_0$ of observation. The expression for the Poynting vector assumes an especially simple form when all the transition dipole moments are equal in magnitude and oriented perpendicularly to the axis of the chain. In this case

$$\mathbf{S}(r_0,t)=\mathbf{n}_0\frac{\hbar\omega_0}{4\pi r_0^2}\left[1-\frac{(\mu\mathbf{n}_0)^2}{\mu^2}\right]S_c(t), \quad (21)$$

$$S_c(t)=\frac{3}{2}\gamma_0\sum_{l,l'}R_l^+(t')R_{l'}^-(t')\exp\{ik_0r_{ll'}\cos\theta_0\}. \quad (22)$$

The factor in the square brackets in (21) is the directivity pattern of a single dipole and $S_c(t)$ is a collective factor due to the interference of the fields of the atomic radiators. Below we shall assume that the transition dipole moments of all the atoms are oriented identically in a direction perpendicular to the axis of the chain.

4. THE SPATIALLY HOMOGENEOUS APPROXIMATION

The simplest model that allows us to solve analytically the problem of the effect of the Coulomb interaction on superradiance is the model in which the density matrices of all the atoms are assumed to be identical. Strictly speaking, such an approximation is justified only for an infinite chain in the case of homogeneous initial conditions. The merit in considering this approximation in the general case will be seen below. So, let $R_k^{\pm}=R^{\pm}$ and $Z_k=Z$. Then the system of equations (11) becomes significantly simpler.

$$\dot{R}^{\pm}=\mp i\Omega R^{\pm}Z-\frac{1}{\tau_R}R^{\pm}Z, \quad \dot{Z}=-\frac{1}{\tau_R}R^+R^-. \quad (23)$$

The quantities Ω and τ_R are given by the expressions

$$\Omega=\frac{3\gamma_0}{(k_0a)^3}\sum_{l=1}^{N/2}\left\{\left[\frac{1}{l^3}-\frac{(k_0a)^2}{l}\right]\cos k_0al+\frac{k_0a}{l}\sin k_0al\right\}, \quad (24)$$

$$\frac{1}{\tau_R}=\frac{3\gamma_0}{(k_0a)^3}\sum_{l=1}^{N/2}\left\{\left[\frac{(k_0a)^2}{l}-\frac{1}{l^3}\right]\sin k_0al+\frac{k_0a}{l}\cos k_0al\right\}, \quad (25)$$

which follow from (12), (13), and (17) with allowance for the fact that for the conditions stipulated at the beginning of the section $\chi_{lk}=\varkappa_{lk}=\mu^2$ and, furthermore, for the fact that $r_{lk}=a|l-k|$ for an equidistant set of points. Summing the series in (24) and (25) in the limit $k_0a\ll 1$ of interest to us, we obtain

$$\Omega=3\zeta(3)\gamma_0\left(\frac{\lambda}{a}\right)^3=3\zeta(3)\frac{\gamma_0}{(k_0a)^3}, \quad (26)$$

$$\frac{1}{\tau_R}=\gamma_0N, \qquad L\ll\lambda. \quad (27)$$

$$\frac{1}{\tau_R}=\frac{3\pi}{4}\gamma_0\frac{\lambda}{a}=\frac{3\pi}{4}\frac{\gamma_0}{k_0a}, \qquad L\gg\lambda, \quad (28)$$

where $\zeta(x)$ is the Riemann zeta function. As can be seen from (28), for an extended system ($L\gg\lambda$) the superradiant constant τ_R^{-1} is determined not by the total number of atoms in the system, but by the number of atoms located over a

distance equal to the wavelength. Notice that $\Omega\tau_R \gg 1$ irrespective of the relation between the length L of the system and the wavelength λ of the radiation, i.e., the fluorescence process significantly lags behind the population movement, a fact pointed out in the Introduction.

1. Fluorescence dynamics.

Using the constancy of the square of the length of the Bloch vector, i.e., the fact that $Z^2 + |R^{\pm}|^2 = 1/4 + |R_0^{\pm}|^2 \approx 1/4$, we integrate the equation (23) for the inversion:

$$Z(t) = -{}^1/_2 \operatorname{th}[(t-t_D)/2\tau_R]. \tag{29}$$

Here t_D is the superradiance delay time, which is given by

$$t_D = -2\tau_R \ln |R_0^{\pm}|. \tag{30}$$

The integration of the equation for $R^{\pm}$ with allowance for (29) yields

$$R^{\pm}(t) = \frac{R_0^{\pm}}{2|R_0^{\pm}|} \operatorname{sech}\frac{t-t_D}{2\tau_R} \exp\left\{\pm i\Omega\tau_R \ln\left[2|R_0^{\pm}| \operatorname{ch}\frac{t-t_D}{2\tau_R}\right]\right\}. \tag{31}$$

The radiation intensity is equal to:

$$I(t) = -N\hbar\omega_0 \dot{Z} = \frac{N\hbar\omega_0}{4\tau_R} \operatorname{sech}^2\frac{t-t_D}{2\tau_R}. \tag{32}$$

Notice that the parameter Ω, which has the meaning of the inversion oscillation frequency for neighboring atoms, does not enter into the expressions for Z and I. Thus, in the case of a spatially homogeneous system the Coulomb interaction has no effect on the superradiance dynamics. The expression (31) for $R^{\pm}$ differs from the corresponding expression in the case when there is no Coulomb interaction (i.e., when $\Omega = 0$) by the rapidly oscillating factor, the oscillation frequency being determined, according to (24), by the strength of the interaction between two neighboring dipoles.

At the beginning of the process, when $t < t_D$,

$$R^{\pm}(t) = \frac{R_0^{\pm}}{2|R_0^{\pm}|} \operatorname{sech}\frac{t-t_D}{2\tau_R} e^{\mp i\Omega t}. \tag{33}$$

This behavior of $R^{\pm}$ is valid up to a time $t \sim t_D$. At $t > t_D$ (at the end of the fluorescence process)

$$R^{\pm}(t) = \frac{R_0^{\pm}}{2|R_0^{\pm}|} \operatorname{sech}\frac{t-t_D}{2\tau_R} \exp\{\pm i\Omega(t-2t_D)\}. \tag{34}$$

A comparison of (34) with (33) shows that, in the course of the fluorescence process, the phase of the amplitude of the off-diagonal density-matrix element $R^{\pm}$ changes sign, a fact reflected in the increase in time of the superradiance frequency from the value $\omega_0 - \Omega/2$ to $\omega_0 + \Omega/2$. The superradiance spectrum becomes broadened by an amount Ω. Since $\Omega \gg \tau_R^{-1}$, it is this broadening that will determine the width of the superradiance spectrum.

2. The directivity pattern.

The spatially homogeneous approximation allows us to compute exactly not only the dynamics, but also the directivity pattern of the superradiance of a finite chain. We can sum the series in (22) in this approximation, obtaining for the collective part S_c of the Poynting vector the expression

$$S_c(t) = \frac{3}{2}\gamma_0 \operatorname{sech}^2\frac{t-t_D}{2\tau_R} \frac{\sin^2(k_0 aN\cos\theta_0/2)}{\sin^2(k_0 a\cos\theta_0/2)}. \tag{35}$$

If the system is short (i.e., if $k_0 aN \gg 1$), the collective factor S_c does not depend on the angle θ_0:

$$S_c(t) = {}^3/_2\gamma_0 \operatorname{sech}^2[(t-t_D)/2\tau_R]N^2, \quad L \ll \lambda, \tag{36}$$

and the directivity pattern of the superradiance coincides, according to (21), with the directivity pattern of the radiation emitted by a single dipole. In the case of an extended system (i.e., for $k_0 aN \gg 1$) the collective factor has a principal maximum, (36), of width of the order of $(k_0 aN)^{-1}$ at $\theta_0 = \pi/2$ and subordinate maxima when

$$\cos\theta_0 = (n+{}^1/_2)\lambda/L, \quad n = 0, 1, 2, \ldots. \tag{37}$$

From this it follows that the number of subordinate maxima is determined by the number of wavelengths that fit into the length of the system. The physical meaning of the condition (37) is fairly simple: the quantity $L\cos\theta_0$ is the difference between the paths of the two waves emitted by the extreme dipoles of the chain in the direction θ_0. According to (37), this path difference is a half-integer multiple of the wavelength. Consequently, on the average, the waves from all the dipoles arrive at the observation point in phase, and give energy-flux maxima in the direction in question. The height of the n-th subordinate maximum is equal to

$$S_c^{(n)}(t) = \frac{3}{2\pi^2}\gamma_0 \operatorname{sech}^2\frac{t-t_D}{2\tau_R}\left(\frac{N}{n+{}^1/_2}\right)^2 \tag{38}$$

and decreases rapidly with increasing number n.

Thus, in the spatially homogeneous approximation the directivity pattern of the superradiance of a linear bounded extended chain of atoms is characterized by a sharp directivity "to the side" in the direction perpendicular to the axis of the chain and to the direction of the dipole moments (similarly to the directivity pattern of a linear chain of classical dipole radiators oscillating in phase.

5. SUPERRADIANCE OF A DICKE SYSTEM ($L \ll \lambda$) WITH ALLOWANCE FOR THE COULOMB INTERACTION

The neglect of the Coulomb interaction (i.e., the setting of $\Omega_{lk} = 0$) in the description of a short system ($L \ll \lambda$) together with the imposition of homogeneous initial conditions automatically leads to the spatially homogeneous model considered in the preceding section. Indeed, according to (26), in the limit $L \ll \lambda$ the relaxation matrix γ_{lk} is equal to γ_0 and does not depend on the position of the atom in the chain. Consequently there are no inhomogeneity sources of any kind and we have a superradiant pulse in the shape of the square of the hyperbolic secant (32).

The Coulomb interaction destroys the spatial homogeneity even in the case of a short system ($L \ll \lambda$). This is a consequence of the fact that the square of the total Bloch vector ceases to be a conserved quantity (in the quantum-mechanical description the Coulomb-interaction operator does not commute with the operator for the square of the total energy spin of the system).

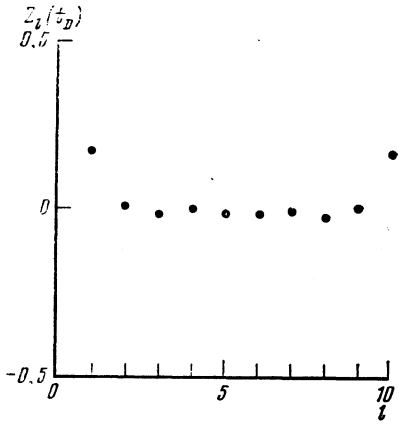

FIG. 1. Inversion profile for the Dicke system at the moment of time t_D, computed with allowance for the Coulomb interaction ($N = 10$).

To estimate the effect of the Coulomb interaction on the spatial inversion distribution, the polarizations, and the superradiance dynamics of the Dicke system, we solved the system (11). Figure 1 shows the distribution of the inversion along the chain ($L \ll \lambda$) at the moment when the superradiance intensity is maximal. The initial value of the polarizaton was chosen to be equal to $R_k^{\pm}(0) = 0.01$. A significant change in the inversion occurs only for the boundary atoms. For the inner atoms Z_k depends weakly on the site number. In consequence, the superradiant pulse with allowance made for the Coulomb interaction (the curve 1 in Fig. 2) differs from the pulse obtained in the spatially homogeneous model or in the absence of the Coulomb interaction (the curve 2 in Fig. 2) only by an increase in the time lag. The shapes and the amplitudes of the pulses differ only slightly. The increase in the time lag when allowance is made for the Coulomb interaction is due, in our opinion, to the fact that the Coulomb interaction gives rise to a spatially inhomogeneous phase modulation of the polarization at the initial stage of the superradiant emission. This leads to the slowing down of the growth of the polarization, i.e., to the increase of the time lag. The super-radiance of a system with dimensions $L \lesssim \lambda$ is similar to the superradiance described in the present section.

6. SUPERRADIANCE OF AN EXTENDED SYSTEM ($L \gg \lambda$) WITHOUT ALLOWANCE FOR THE COULOMB INTERACTION

The neglect of the Coulomb interaction in the description of the dynamics of the superradiance of an extended system does not lead to a spatially homogeneous model, as was the case for the Dicke system. The collective relaxation constant

$$\tilde{\gamma}_l = \sum_{l \neq k} \gamma_{lk}$$

for $L \gtrsim \lambda$ is a function of the site number, and, consequently, all the atomic characteristics will also depend on the position of the atom in the chain.

The fluorescence-dynamics calculations without allowance for the Coulomb interaction revealed the existence of two emission regimes—single- and two-pulse (Fig. 3)—which alternate with each other as the length of the chain is increased. We can see the reason for this behavior by studying the spatial dependence of the field acting at the initial moment of time, and determined by the collective constant $\tilde{\gamma}_l$. In the continuum limit, which can be used on account of the condition $k_0 a \ll 1$,

$$\tilde{\gamma}(x) = \frac{1}{k_0 a}\left[\int_0^{k_0(L/2+x)} dx'\,\gamma(x') + \int_0^{k_0(L/2-x)} dx'\,\gamma(x')\right], \quad (39)$$

$$\gamma(x) = \frac{3}{2}\gamma_0\left(\frac{\sin x}{x} + \frac{\cos x}{x^2} - \frac{\sin x}{x^3}\right). \quad (40)$$

The plots of the functions $\gamma(x)$ and $\tilde{\gamma}(x)$ are shown in Figs. 4 and 5. For $N = 100$ we have the single-pulse fluorescence regime (curve 2 in Fig. 3); for $N = 120$, the two-pulse regime (curve 5 in Fig. 3).

Figure 5 demonstrates the existence in the sample of regions differing from each other in the de-excitation rates $\tilde{\gamma}(x)$. The centers of these regions can be found from the condition $d\tilde{\gamma}/dx = 0$ for extrema, which yields an equation for the determination of the extremum points:

$$\gamma[k_0(L/2+x)] - \gamma[k_0(L/2-x)] = 0. \quad (41)$$

The region with a higher de-excitation rate $\tilde{\gamma}(x)$ begins to develop faster than the region with a lower de-excitation rate. The subsequent evolution significantly depends on the

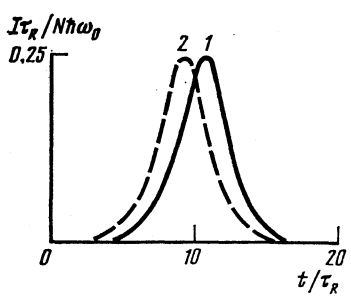

FIG. 2. Superradiant pulse emitted by the Dicke system ($\Omega\tau_R = 360$), as computed: 1) with allowance for the Coulomb interaction; 2) in the spatially homogeneous model.

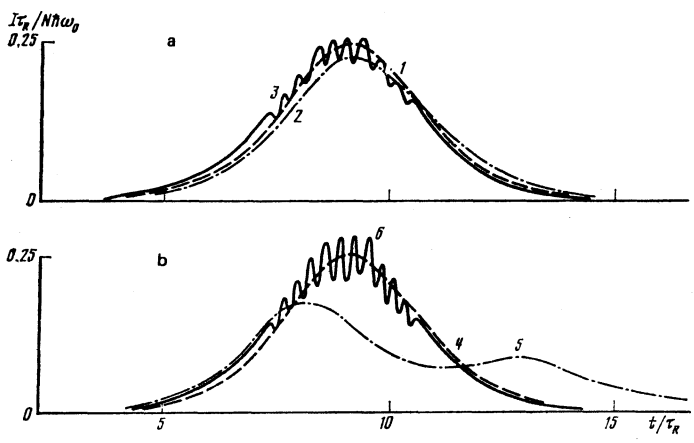

FIG. 3. Superradiance dynamics for the linear chain for the cases: a) $N = 100$, $a/\lambda = 0.1$; b) $N = 120$, $a/\lambda = 0.1$; 1) and 4) the spatially homogeneous model; 2) and 5) without allowance for the Coulomb interaction; 3) and 6) with allowance for the Coulomb interaction.

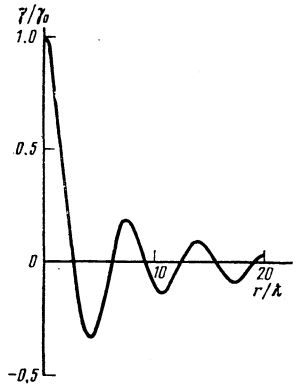

FIG. 4. Dependence of the off-diagonal element of the relaxation matrix γ on the atomic spacing.

character of the coupling existing between these regions, i.e., on whether this coupling is positive or negative. The coupling will be positive (negative) if $\gamma(\Delta x) > 0 (< 0)$, where Δx is the distance between the centers of the neighboring regions. In systems with positive coupling the regions with a lower de-excitation rate $\tilde{\gamma}(x)$ follow in their development the regions with a higher rate (some of them lagging behind (Fig. 6, curves 1–4)). This is ensured by the fact that, because $\gamma(\Delta x)$ is positive, the electric fields of the indicated regions are in phase. As a result, there is formed a superradiant pulse that is close in shape to the pulse obtained in the spatially homogeneous model (Fig. 3, curve 2).

In systems with negative coupling the electric field of a rapidly developing region cancels out the field of a region that develops more slowly, thereby virtually completely blocking its evolution until the inversion of the first region is completely depleted (Fig. 6, curves 5–8). After the first region has completely disposed of its inversion and has formed one peak in the superradiant pulse, the second region begins to develop which development eventually yields a second peak (Fig. 3, curve 5).

7. SUPERRADIANCE OF AN EXTENDED SYSTEM ($L \gg \lambda$) WITH ALLOWANCE FOR THE COULOMB INTERACTION

The distinctive features of the dynamics of collective spontaneous emission by a linear extended system without allowance for the Coulomb interaction of the atoms is closely tied with the possibility of the appearance and existence

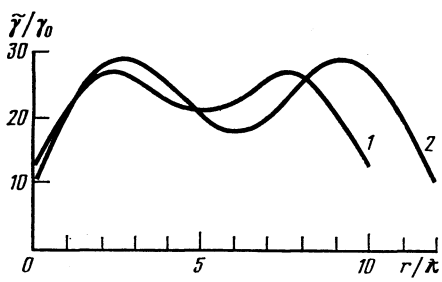

FIG. 5. Dependence of the collective relaxation constant $\tilde{\gamma}$ of an atom on the position of the atom: 1) $N = 100$; 2) $N = 120$.

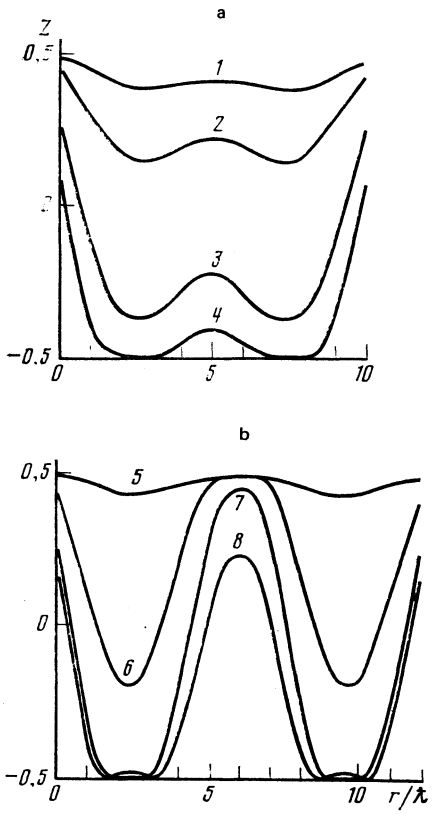

FIG. 6. Invesion profile in a linear chain at different moments of time, as computed without allowance for the Coulomb interaction for the cases: a) $N = 100$, $a/\lambda = 0.1$: 1) $t = 7.1\,\tau_R$, 2) 8.5, 3) 10.6, 4) 12.7; b) $N = 120$, a/λ 0.1: 5) $t = 5.7\,\tau_R$, 6) 8.5, 7) 11.3, 8) 12.7.

during the fluorescence of large-scale—in comparison with λ—inversion gradients (see Fig. 6). As has already been noted, the Coulomb interaction between the atoms leads to the transfer of excitation from one atom to another, i.e., to inversion equalization, this process occurring in the system under consideration significantly faster than the fluorescence process. It is therefore natural to expect that allowance for the Coulomb interaction can have a strong effect on the superradiance dynamics of an extended system. Figure 3 (curves 3 and 6) shows superradiant pulses obtained by numerical in-

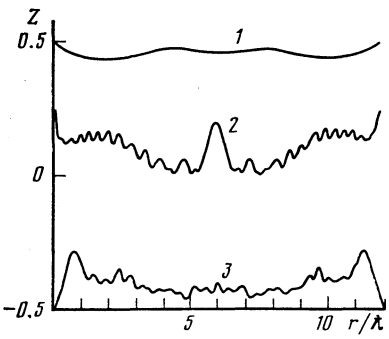

FIG. 7. Inversion profile in a linear chain at different moments of time, as computed with allowance for the Coulomb interaction for $N = 120$, $a/\lambda = 0.1$: 1) $t = 5.7\,\tau_R$, 2) 8.5, 3) 11.3.

tegration of the system (11) with allowance for the Coulomb interaction. The occurrence of a structure in the pulse and the proximity "on the average" of this pulse to the pulse obtained in the spatially homogeneous model are noteworthy. The later circumstance is explained by the mixing effect of the Coulomb interaction, an effect which inhibits the appearance of large-scale inversion gradients during the fluorescence. The inversion profiles at different moments of time show (Fig. 7) that there is indeed spatial homogeneity "on the average."

The cause of the structure in the superradiant pulse is, apparently, the exchange of excitation between macroscopic regions with different de-excitation rates $\tilde{\gamma}(x)$ (see the preceding section). The modulation of the de-excitation rate with frequency equal to the rate of redistribution of the inversion among these regions is the cause of the structure in the fluorescence pulse. The time scale of the structure is thus determined by the transfer of energy from one region into another. We can estimate the characteristic time τ_R of this process by recognizing that the velocity of the coherent motion of excitation along a linear chain is of the order of Ωa (Ref. 23). The distance between the indicated regions is $\Delta x \sim \lambda$ (see Fig. 5). Therefore, $\tau_{tr} \sim \Delta x/\Omega a \sim k_0 a \tau_R$. This value of τ_{tr} corresponds in order of magnitude to the structural scale of the superradiant pulse (see Fig. 3, curves 3 and 6). Corroborating the proposed explanation of the structure in the superradiant pulse is the fact that this structure disappears as the length of the system is decreased. This is due to the equalization of the de-excitation rates for the various regions as L is decreased. Let us recall that, for $L \ll \lambda$, the rate $\tilde{\gamma}(x) = N\gamma_0$.

8. CONCLUSION

The atoms acquire nonzero mean dipole moments during the superradiant emission. This suggests that it is necessary to take the dipole–dipole interaction into account. It follows from comparative estimates of the characteristics superradiance times and the reciprocal Coulomb-interaction strength that the Coulomb interaction should have a significant effect on the superradiance of a system with a small Fresnel number (i.e., a system for which $D^2/\lambda L \ll 1$) and the Dicke system.

For the Dicke system ($L \ll \lambda$) the Coulomb interaction modifies the superradiance dynamics only slightly, but has a strong effect on the shape and width of the spectrum. If without allowance for the Coulomb interaction the width of the spectrum is determined by the reciprocal time $\tau_R^{-1} = N/\tau_0$, with allowance for the interaction the spectrum has a width $\mu^2/\hbar a^3 \gg \tau_R^{-1}$. The manifestation of the Coulomb interaction in the fluorescence dynamics amounts, as calculations show, to an increase in the delay time of the superradiant pulse as compared to the case in which the interaction is neglected.

In the case of an extended linear chain of atoms ($L \gg \lambda$), treated as the limiting case of a system with a small Fresnel number, allowance for the Coulomb interaction has the same effect on the spectrum as in the case of the Dicke system. There is in this case a radical change in the superradiance dynamics. The Coulomb interaction causes coherent excitation transfer, which leads to approximate spatial homogeneity of the inversion along the length of the system. Therefore, the superradiant pulse turns out to be close to the pulse found in the spatially homogeneous model. This justifies to some extent the spatially homogeneous model as applied to a system with a small Fresnel number, and possibly explains the qualitative agreement of the theory[7,10] based on this model with experiment[24] for the case $F \approx 1$. But the dipole–dipole interaction induced excitation exchange between regions with different de-excitation rates leads to the appearance of structure in the superradiant pulse. The scale of this structure is determined by the time characterizing the transfer of excitation from one region to another, and is of the order of $k_0 a \tau_R \sim (k_0 a)^2 \tau_0$.

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[1)]For systems containing a large number of particles, the effect of the self-action field on the development of superradiance turns out to be unimportant, and we shall neglect it in what follows.

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