The Hall effect in a unipolar inductor: a possible dynamo or antidynamo

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The Hall effect in a magnetized disk is considered. If the disk is used as a unipolar inductor the original magnetic field decreases, whereas if a current of appropriate direction is driven through a rotating disk the magnetic field increases (without limit in our simple analysis at constant current I). The effects should be testable for laboratory sized objects and may be of crucial importance in pulsars. The relative change of the magnetic field is of order βN for pulsars, where $\beta = \Omega R / c$ is the equatorial velocity of the pulsar in units of the velocity of light and, $N = eB\tau/mc$ is the dimensionless Hall constant. Under favorable conditions $\delta B / B \approx 10^{-5}$ for laboratory sized objects and $\beta N \approx 1$ for pulsars.

PACS numbers: 72.20.My, 85.70. - w

§1. INTRODUCTION

It is well known that the magnetic field in an ordinary disk dynamo is amplified if the inequality

 $M\Omega/R > 2\pi$

is satisfied, where M is the mutual inductance of the solenoid and the disk, Ω is its angular velocity, and R is the Ohmic resistance of the circuit. If the inequality is not satisfied, the current will decay as soon as the initially given magnetic field has disappeared (a diagram of such a dynamo is shown in Fig. 1).

The disk dynamo depends on the special arrangement of wires and it has remained doubtful for a long time whether a dynamo process can be realized in a singly connected domain of nearly uniform electrical conductivity, such as that represented by the Earth's core. These doubts have been dispelled by the work of different authors who demonstrated in a mathematically rigorous way that the so-called "homogeneous dynamo" is indeed possible. Since then, a large number of solutions have been obtained, and it is generally believed nowadays that almost all velocity fields are capable of generating magnetic fields in a singly connected body of fluid if the magnetic Reynolds number R_m is sufficiently high. R_m is defined in analogy to the ordinary Reynolds number except that the kinematic viscosity is replaced by the magnetic diffusivity $\lambda = (\mu \sigma)^{-1}$, where σ is the electric conductivity and μ is the magnetic permeability.

Here we shall pursue the alternative idea that in the case of pulsars i.e., neutron stars, the Hall effect may play a crucial part in determining the external magnetic field.

§2. THE HALL EFFECT IN A MAGNETIZED NONROTATING DISK

Consider a magnetized disk with inner radius r_i and outer radius r_e . Let σ be the conductivity of the disk, I the total current driven through the disk (from r_e to r_i). We introduce the dimensionless Hall constant

 $N = e\tau B/mc$,

where the collision time τ is related to σ by the well known relation¹ $\sigma = ne^2 \tau/m$. Ohm's law^{1,2} reads

$$\mathbf{j} + [\mathbf{N} \times \mathbf{j}] = \sigma \mathbf{E}, \tag{1a}$$

or in tensor notation

$$\sigma_{ab}^{-4} j_b = E_a, \quad \sigma_{ab} = \frac{1}{1 + N^2} (\delta_{ab} + N_a N_b + e_{abc} N_c)$$
(1b)

with the inverse

$$j_a = \sigma_{ab} E_b \tag{1c}$$

(we use the summation convention). For future use Eq. (1c) also in vector notation

$$\mathbf{j} = \sigma_{\perp} \mathbf{E} + (\sigma_{\parallel} - \sigma_{\perp}) (\mathbf{b} \mathbf{E}) \mathbf{b} + \sigma_{\perp} [\mathbf{E} \times \mathbf{N}], \qquad (\mathbf{1d})$$

where $\sigma_{+} = \sigma/(1 + N^2)$, $\sigma_{\parallel} = \sigma$, and **b** is a unit vector in the direction of the field **B**.

If we apply a potential difference $\Delta \Phi$ between inner and outer rims, the electric field in the disk will be given by $\mathbf{E} = -\nabla \Phi$, div $\mathbf{E} = 0$. So that we have

$$\mathbf{E} = \frac{\Delta \Phi}{\rho \ln (r_e/r_i)} \, \mathbf{e}_{\rho}$$

We use coordinates (ρ, φ, z) in the disk and \mathbf{e}_{ρ} , \mathbf{e}_{φ} , \mathbf{e}_{z} are unit vectors in the (ρ, φ, z) directions, respectively.

In the following it is convenient to consider a disk of finite height $h \ll r_i$. To determine the energy dissipated in the disk we evaluate $W = \int \mathbf{j} \cdot \mathbf{E} \cdot dV$ and obtain the following three equivalent expressions

$$W = I \Delta \Phi = \sigma \frac{2\pi h}{1 + N^2} \frac{(\Delta \Phi)^2}{\ln(r_e/r_i)} = \frac{(1 + N^2)I^2}{2\pi\sigma h} \ln \frac{r_e}{r_i}.$$
 (2)



FIG. 1. Sketch of a disk dynamo.

whence we deduce by means of $\Delta \Phi = RI$ that the resistance R is

$$R = \frac{1+N^2}{2\pi\sigma h} \ln \frac{r_e}{r_i} \,. \tag{3}$$

Note that

$$I=2\pi h j_{\rho}(r_c) r_e=2\pi h j_{\rho}(r_i) r_{f_i}$$

Owing to the Hall effect the dissipation increases by a factor $1 + N^2$ for a given current I and decreases by the same factor for a given potential difference $\Delta \Phi$. The point is that owing to the magnetic field the current I spirals from r_e to r_i in N turns. The current density j_{φ} (which equals $-Nj_{\rho}$, according to Eq. (1b), will however generate in the disk an extra magnetic field δBz of order $\delta B_z = -NI/c r_e$ (see below). Depending on the direction of the current I, the existing magnetic field will either increase or decrease. If we keep I fixed (which implies that we increase the potential difference by $1 + N^2$ times) we can arrive at very large magnetic fields without using any coils.

In our simple analysis, we neglect the dependence of τ on B^2 and the mechanical magnetic stresses exerted on the disk. Infinitely strong magnetic fields are therefore possible at a finite current *I*. The additional magnetic field can be calculated by standard techniques by using the Green's function in polar coordinates (r, θ, φ) . We put $\delta \mathbf{B} = \operatorname{curl} \delta \mathbf{A}$ and div $\delta \mathbf{A} = 0$. We obtain

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$$\delta \mathbf{A} = -\Delta \delta \mathbf{A} = \frac{4\pi}{c} \mathbf{j}_{\varphi} = -\frac{4\pi}{c} IN \frac{1}{2\pi r^2} \delta \left(\theta - \frac{\pi}{2} \right) \mathbf{e}_{\varphi}.$$
(4a)

The solution of this equation for $r \ge r_e$ is

$$\delta A_{\varphi} = \frac{NI}{cr_e} \sum_{n=1}^{\infty} \alpha_n \left(\frac{r}{r_e}\right)^{-2n-1} P_{2n}(\cos \theta), \qquad (4b)$$

where the α_n can be found e.g., in Ref. 3. For the dipole part we find

$$\delta B_r = -\frac{NI}{cr_e} \left(\frac{r_e}{r}\right)^2 \cos\theta \qquad (4c)$$

as stated above.

§3. THE UNIPOLAR INDUCTOR

We apply now the foregoing considerations to the rotating disk (unipolar inductor). For an observer at rest, Ohm's law reads now

$$\mathbf{j} + [\mathbf{N} \times \mathbf{j}] = \sigma(\mathbf{E} + [\boldsymbol{\beta} \times \mathbf{B}]), \quad \boldsymbol{\beta} = \frac{\rho}{c} [\boldsymbol{\Omega} \times \mathbf{e}_{\rho}], \quad (5)$$

where we have dropped terms of order β^2 . The electric field at the rim of the disk (index e) is therefore

$$\mathbf{E}_{e} = -[\boldsymbol{\beta}_{e} \times \mathbf{B}] + \frac{1}{\sigma} (\mathbf{j}_{e} + [\mathbf{N} \times \mathbf{j}_{e}]).$$
(6)

For good conductivity the second term in Eq. (6) can usually be neglected but it is clear that it is this term which limits the maximum producible current I. If we connect the inner and outer rims by nonrotating wires and a resistor R, a current Iwill flow and energy is thus extracted from the rotation of the disc. Neglecting the Hall effect for a moment the analysis goes like this. The electric field E is given by Eq. (6)

$$\mathbf{E} = -\frac{\mathbf{p}}{c} (\mathbf{\Omega} \mathbf{B}) \mathbf{e}_{\boldsymbol{\nu}},\tag{7a}$$

The potential inside the disk is

$$\Phi = \frac{1}{2c} (\Omega \mathbf{B}) \rho^2. \tag{7b}$$

The potential difference is therefore

$$\Delta \Phi = \Phi(r_e) - \Phi(r_i) = \frac{1}{2c} (\Omega B) (r_e^2 - r_i^2).$$

It produces a current

$$I = \frac{\Delta \Phi}{R} = \frac{1}{2cR} (\Omega \mathbf{B}) (r_e^2 - r_i^2).$$
(8)

The dissipated energy is $W = I\Delta\Phi$. It is drawn from the rottional energy of the disk and therefore angular momentum must be dissipated at a rate J such that $W = \Omega \cdot J$. Externally this is achieved in the immobile wires and resistor, and inside the disk the angular momentum is extraced by the electromagnetic torque T

$$\mathbf{T} = \mathbf{j} = \frac{1}{c} \int \rho[\mathbf{e}_{\rho} \times [\mathbf{j} \times \mathbf{B}]] dV.$$
⁽⁹⁾

Multiplying Eq. (9) by $\mathbf{\Omega}$ we have

$$\Omega \mathbf{T} = \frac{1}{c} \Omega \int \rho j_{\rho} B dV = \frac{1}{2c} (\Omega B) \left(r_{e}^{2} - r_{i}^{2} \right) I = W, \quad (10)$$

as it must.

Let us now take into account the Hall effect. Under stationary conditions and for axisymmetric flow the derivatives with respect to t and φ vanish. We then have from curl $\mathbf{E} = -\mathbf{B}/\mathbf{c} = 0$ that $E_{\varphi} = 0$ everywhere since E_{φ} vanishes at the boundaries r_a and r_i . With $E_{\varphi} = 0$ we have, neglecting the second term of Ohm's law (6) $j_{\varphi} + N j_{\rho} = 0$ so that as in the nonrotating disk an extra current will flow which will give rise to an additional magnetic field δB_z in the disk and the subsequent analysis can be directly carried over from the preceeding section. According to Lenz' rule the induced magnetic field will oppose the original field if we introduce a resistance. However, the reverse is obviously also possible: if we drive a current through the unipolar inductor so that the disk speeds up, the magnetic field will then increase and the energy must be supplied by the external e.m.f. which produces the current.

§4. APPLICATION TO LABORATORY AND ASTROPHYSICS OBJECTS

For a pure copper crystal at 4K a time $\tau \approx 2 \cdot 10^{-9}$ sec is possible, whereas at room temperature $\tau \approx 2 \cdot 10^{-14}$ sec for ordinary copper. We find therefore for these two cases, respectively

$$N=100(B\cdot 10^{-4}) (\tau \cdot 10^{9}) \text{ and } N=10^{-3}(B\cdot 10^{-4}) (\tau \cdot 10^{44}).$$
(11)

For a copper disk in a magnetic field of 10^4 G, having a radius of 10 cm and rotating at $\Omega = 10^3 \text{ sec}^{-1}$ (≈ 160 Hz), the potential difference will be 5 V. At a load resistance 10^{-4} Ohm, a current of 1 A will flow and induce a magnetic field (cf. (4c))

$$B[G] = 2 \frac{N}{200} I[A].$$
 (12)

compared to the original 10^4 G, which should be measurable in the laboratory.

In pulsars, which are believed to work essentially like unipolar inductors, the effect may be much larger for low enough temperatures, i.e., for high enough conductivity at the surface. If the surface is anything like a crystal lattice we expect the conductivity due to electron-phonon scattering to depend on temperature like $(T/\Theta)^{-5}$, where Θ is the Debye temperature of the lattice. For one of the models, calculations⁴ yield for the conductivity directly below the pulsar surface $(\rho \sim 10^4 \text{ g} \cdot \text{cm}^{-3}, B \sim 10^{12} \text{ G})$ a value $\sigma = 10^{20}(T \cdot 10^{-6})^{-5} \text{ sec}^{-1}$. Together with the value of the electron density $n_e \sim 10^{20} \text{ cm}^{-3}$ and the time $\sim 10^{-14} \text{ sec}$ this gives

$$N=10^{5} (B\cdot 10^{-12}) (\tau\cdot 10^{14}) (T\cdot 10^{-6})^{-5}.$$
(13)

For the induced magnetic field due to the Hall effect we obtain, knowing the current I (which can be determined from observation of the deceleration of the pulsar rotation⁵⁾

$$\frac{\delta B}{B} \sim \frac{\Delta F}{F} \beta N \sim 1. \tag{14}$$

We have used here the relation

$$I = en_e c \Delta F = -\frac{\Omega B}{2\pi} \Delta F,$$

where F is the pulsar surface area and ΔF is the surface area of its polar cap.

5. EXTENDED BODIES AND THE BUILDUP PROBLEM

So far we have considered only thin disks, for which the calculations can be carried out in analytic form. From the viewpoint of astrophysics the most interesting case is that of magnetized spheres. What happens when current passes through such a sphere? (Expressions for the resistance of a nonmagnetized sphere can be found in Ref. 2). The buildup problem is determined by Maxwell's equations

$$\operatorname{rot} \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \dot{\mathbf{E}}, \qquad \mathbf{j} = \hat{\sigma} \mathbf{E},$$
$$\operatorname{rot} \mathbf{E} = -\frac{1}{c} \mathbf{B},$$

where $\hat{\sigma}$ is the conductivity tensor given by Eq. (1b). For our case we can neglect with good approximation the term \dot{E}/c , and we arrive at the equation

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = -\frac{4\pi}{c}\,\partial\mathbf{E}.\tag{15}$$

With the Hall effect neglected, the solution of the problem of the damping of a magnetic field in a sphere is well known.² Separating the time dependence in the form $B = B_0 e^{-\gamma t}$, we obtain for the eigenvalues of (15)

$$\gamma_n = \frac{c^2}{4\pi\sigma} k_n^2,$$

where the values of k_n are determined by the roots of the equation $J_0(k_n r_e) = 0$ in the case of a sphere of radius r_e or of the equations $J_1(k_n r_e) = 0$ for a cylinder (of infinite length in the z direction); J_0 and J_1 are Bessel functions. The lowest decay mode is $k_0 = \pi/r_0$ for a sphere and $k_0 = 1.22\pi/r_e$ for

an infinite cylinder; in other words, the magnetic field in a cylinder attenuates $(1.22)^2 = 1.5$ times more slowly than in a sphere of the same radius.

For a cylinder it possible to obtain an analytic solution of the problem also with the Hall effect taken into account. The complete solution is given by the equations

$$\mathbf{B} = B_0 J_0(k_n \rho) e^{-\tau_n t} \mathbf{e}_z, \qquad (16)$$

$$\mathbf{j} = -\frac{cB_0}{4\pi} J_0'(k_n \rho) e^{-\tau_n t} \mathbf{e}_o,$$

$$\mathbf{E} = \left[-\frac{B_0}{\sigma} J_0'(k_n \rho) \mathbf{e}_{\varphi} + N J_0'(k_n \rho) \mathbf{e}_{\rho} \right] e^{-\tau_n t}$$

with the same eigenvalues (this can be seen from (16) by recalling that $J'_0 = -J_1$). Thus, the influence of the magnetic field reduces to production of an additional electric field $E_{\rho} = -NE_{\varphi}$.

To understand this result better, we consider the dissipated energy with the aid of the phenomenological equation (for a thin conductor)

$$c^{-2}LI + RI + C^{-1}Q = U_{\text{ext}},$$
 (17)

where L is the self-inductance, R the resistance, and C the capacitance of the system, and U_{ext} is the external applied emf.² In the damping problem, the total charge is equal to zero, and there is no external emf. We have therefore $c^{-2}L\dot{I} + RI = 0$, whence

$$\gamma = 1/\tau = c^2 R/L. \tag{18}$$

Since both R and L have the same dependence on N, we see that the decay time, determined by the ratio of these two quantities, is independent of N. For a sphere we should use the telegraphy equation

$$c^{-2}LI + RI + \frac{\partial V}{\partial r} = 0, \tag{19}$$

$$-\frac{1}{r^2}\frac{\partial}{\partial r}r^2I + C\frac{\partial V}{\partial t} + RV = 0, \qquad (20)$$

whence

$$c^{-2}RLI + R^{2}I - \frac{\partial}{\partial r} \left(\frac{1}{r^{2}} \frac{\partial}{\partial r} r^{2}I \right) = 0.$$
(21)

We see therefore that in a sphere, too, the Hall effect does not change qualitatively the decay (or buildup) of the magnetic field.

We turn now to a detailed application to pulsars.

§6. EVOLUTION OF PULSAR MAGNETISM VIA THE FARADAY DYNAMO MECHANISM

We shall first give some background information on pulsars and some motivation why we think the preceding considerations could be of importance for pulsars.

Soon after the identification of pulsars with neutron stars it was pointed out that the observation of absence of long-period pulsars could be understood if one assumed that the magnetic field decayed on a time scale of some 10^6 years. As shown in §5, the time-scale τ_d for a magnetized body to lose its magnetic field via Ohmic dissipation is

$$\tau_d = 4\sigma r^2 / \pi c^2. \tag{22}$$

It amounts to $\sim 10^6$ years if the average conductivity is

 $\sigma \sim 10^{23}$ sec⁻¹, a rather large value for nondegenerate matter. However, unlike in ordinary stars, the matter of a neutron star is extremely degenerate and due to Pauli principle the conductivity is very large. In fact the protons may in some part of the neutron star actually form a type-II superconductor. Consequently only in the crust of a neutron star can the magnetic field decay and this would not lead to any appreciable reduction of the star's dipole moment. Impressed by such theoretical considerations, observers continue to this day to discuss their observational results in terms of magnetic-field decay.⁶

How can pulsars turn-off then if magnetic-field decay is not possible? Three viewpoints have been offered. The first is a clever variant of the magnetic-field decay hypothesis.⁷ It is clear that a magnetic field cannot be anchored stably in a liquid body. So either a crust must be present, or a toroidal field which confines the poloidal field. If this toroidal field is mainly anchored in the crust it can decay by Ohmic dissipation (see §5), the poloidal field may subsequently crack the crust and reorient itself, lowering the magnetic energy thereby and form a quadrupole field.

The second viewpoint is based on the fact that external or internal torques may lead to considerable alignment of the pulsar spin axis with the axis of its dipole moment.⁸ There is little observational evidence that alignment alone is the mechanism that turns off a pulsar, although the angle between dipole and spin axes is probably important for pulsar evolution.⁹ Therefore some other mechanism must be at work. In line with earlier work by Sturrock,¹⁰⁾ Ruderman and his group¹¹ have developed the idea that sparking in gaps in the magnetosphere is responsible for the coherent radio emission of radio pulsars and that this process is sensitive to the surface temperature and to the rotation period.

The following discussion is also in line with these considerations and stresses, as will be seen, the importance of the surface temperature. In Refs. 5 and 9 was described in detail a valid model for slowing-down or speeding-up a neutron star, based on the mechanism of a disk dynamo in which one replaces the wires by those conducting field lines of the



FIG. 2. Scheme of the current distribution in the pulsar polar cap. The forward current \mathbf{j}_d flows away from the polar cap of area ΔF along the magnetic field lines **B**. The return current \mathbf{j}_r flows axially symmetrically about the forward current further away from the rotation axis Ω , which coincides in the present case with the magnetic dipole axis. θ is the angle subtended by the polar cap. Only a few turns of the toroidal component of the current within the polar cap are shown.

magnetic dipole which cannot rotate with the pulsar, i.e., the so called open field lines. In Ref. 5 are described the details of the current distribution in a magnetosphere which does not rotate with the pulsar and it is demonstrated that the anomalous braking index of the Crab nebula pulsar can be explained in a quite natural manner. This is a major success of the model, since the explanation of the anomalous braking index has presented so far a major difficulty for any theory. The neutron star is slowed down or accelerated by a magnetic torque produced by a current which flows along the magnetic field lines away from the surface of the polar caps. In the simplest case the return current (j_r in Fig. 2) will flow axially symmetrically around the forward current (j_d in Fig. 2) and further away from the center of the polar cap.

According to the considerations presented in §4 no Hall field can be established for geometrical reasons so the current which flows across the magnetic field lines inside the neutron star must follow a spiral and satisfy divj = 0. The total current *I* can be inferred from the observed slow-down or speed-up and we can write instead of Eq. (14)

$$\delta B_p / B_p = \sigma I \Omega / e n_e c r_e^3 \theta^3 \tag{23}$$

(r_e is the pulsar radius; the angle θ is shown in Fig. 2). If the conductivity σ is mainly due to electron-phonon scattering, it will depend on temperature like T^{-5} , i.e., the conductivity will be extremely temperature dependent. In this case it is convenient to turn around the meaning of Eq. 23 and solve for T instead. For those pulsars^{12,13} which are believed to become extinct, we have typically $\dot{\Omega} = -10^{-16} \sec^{-2}$ $\theta \approx 10^{-1}$, $B_p = \delta B_p \approx 10^{12}$ G, which implies a temperature $T \approx 10^{4.7}$ K. For x-ray pulsars we may on the other hand put $T \approx 10^{7}$ % for the surface temperature. Putting also $\theta \approx 10^{-1.5}$, we obtain $\delta B_p \approx 10^{12}$ G, in good agreement with the observations if we use $\dot{\Omega} \approx 10^{-12} \sec^{-2}$ as inferred from their speed-up.¹⁵)

§7. CONCLUDING REMARKS

The evidence that radio-pulsars are slowed down and xray pulsars accelerated predominantly by magnetic torques is now very strong. Angular momentum is transferred away from the neutron star in the case of radio-pulsars or fed to the neutron star in the case of x-ray pulsars by means of a magnetic spring, the physical origin of which is an appropriate current flowing along the magnetic field lines. Since this current must be closed on the neutron-star surface and no Hall field can be built up, a Faraday dynamo mechanism sets in. It is pointed out that this mechanism could turn on a radiopulsar or turn off an x-ray pulsar. Many contradictory pulsar observations could thus be explained,¹⁵ if it is assumed that radio-pulsars can be reactivated in the galactic plane by means of accretion in dense clouds and that x-ray pulsars must first create a sufficiently strong magnetic field to function as a regularly pulsed emitter.

This paper was finished when I was a guest at the Institute for theoretical problems in Moscow. I thank Prof. E. M. Lifshitz for his hospitality and Drs. A. V. Gurevich, Ya. N. Istomin, and V. S. Beskin for enlighting discussions. I would further like to thank Prof. I. M. Khalatnikov for the invitation and the Academy of Sciences of the USSR and the Deutsche Forschungsgemeinschaft for financial support.

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Translation provided by the author