Nonlinear collective excitations in an easy plane magnet

A. M. Kosevich, V. P. Voronov, and I. V. Manzhos

Physicotechnical Institute of Low Temperatures, Ukrainian Academy of Sciences; Belgorod Technological Institute for Structural Materials

(Submitted 4 June 1982)
Zh. Eksp. Teor. Fiz. **84**, 148–159 (January 1983)

We describe soliton excitation of one-dimensional and two-dimensional ferromagnets with easy-plane anisotropy, placed in a magnetic field perpendicular to the easy plane. The excited states of such a ferromagnet were found to be similar to the excitations of a weakly nonideal Bose gas with repulsion between the particles. Thus, 1) soliton excitations analogous to Lieb states in a one-dimensional Bose gas exist in a one-dimensional ferromagnet, 2) vortical states similar to vortices in a nonideal Bose gas exist in a two-dimensional ferromagnet. A magnetic vortex is considered in a bounded ferromagnetic cylinder whose axis is perpendicular to the easy plane. It is shown that the state of uniform magnetization is unstable to formation of a magnetic vortex. The role of anisotropy in the basal plane is considered.

PACS numbers: 75.30.Gw, 75.10. – b

INTRODUCTION

A theoretical study of the excited states of low-dimensionality ferromagnets with easy-plane magnetization anisotropy has shown that the excitations of such magnets are similar in a certain sense to excitations of a weakly nonideal Bose gas with repulsion between the particles. The similarity manifests itself in the following: 1) The spectrum of the elementary excitations coincides with the Bogolyubov spectrum for a nonideal Bose gas. 2) In a ferromagnet there can exist vortical states similar to the nonideal Bose-gas vortices described by Pitaevskiĭ.[1] 3) A one-dimensional easy-plane ferromagnet contains, besides excitations of the Bogolyubov type, also the soliton excitations obtained in Ref. 2, which turn out to be equivalent to Lieb states in a one-dimensional Bose gas.[3]

In this paper we analyze in detail all the foregoing points of similarity between excitations of two different physical systems and discuss the consequences of their difference. The main difference is that the nonuniform magnetization of a ferromagnet is due, as a rule, to the appearance of a magnetic field in its volume (if div $\mathbf{M}\neq 0$, where $\mathbf{M}$ is the magnetization vector). Only a nonuniform state of special type (div$\mathbf{M}=0$) can exist in an unbounded ferromagnet without a magnetic field. If, however, a ferromagnetic sample has finite dimensions, a magnetic field appears in the space outside the sample even if the magnetization is uniform. We shall show that the presence of this field makes the state of a uniformly magnetized ferromagnetic cylinder unstable to formation of a magnetic vortex if the cylinder axis is perpendicular to the easy plane. To some degree, this instability is similar to the instability of rotating superfluid helium to formation of hydrodynamic vortices in it.

§1. MODEL OF FERROMAGNETIC AND SPECTRUM OF ITS ELEMENTARY EXCITATIONS

We are interested in a magnet subjected to macroscopic long-wave excitations such that the state of the ferromagnet is described by the field of the magnetization vector $\mathbf{M}$ ($\mathbf{M}^2=M_0^2$; $M_0=2\mu_0 s/a^3$, μ_0 is the Bohr magneton, s is the spin of the atom, and a^3 is the atomic volume. Since the vector $\mathbf{M}$ has only two independent components, it is convenient to introduce the angle variables θ and φ defined by

$$M_x=M_0\sin\theta\cos\varphi,\quad M_y=M_0\sin\theta\sin\varphi,\quad M_z=M_0\cos\theta. \tag{1}$$

Let the ferromagnet have a preferred anistropy axis and an easy-magnetization plane perpendicular to it. We choose the z axis along the preferred axis and direct along it the vector $\mathbf{H}$ of the constant and uniform external magnetic field. The crystal magnetic energy, neglecting the magnetic dipole interaction, then takes the form

$$w(\theta,\varphi)=\frac{\alpha M_0^2}{2}[(\nabla\theta)^2+\sin^2\theta(\nabla\varphi)^2]+\frac{\beta M_0^2}{2}\cos^2\theta-M_0H\cos\theta, \tag{2}$$

where α is the exchange constant and β the anisotropy constant. The latter is assumed positive ($\beta>0$), thereby specifying the easy plane $\theta=\pi/2$.

If the magnetic field $\mathbf{H}$ is weak enough ($H<\beta M_0$), the homogeneous-ferromagnet ground state that minimizes the quantity (2) corresponds to

$$\varphi=\varphi_0=\text{const},\quad \theta=\theta_0=\text{const},\quad \cos\theta_0=H/\beta M_0. \tag{3}$$

We see that the ground state of a uniaxial magnet in a uniform field perpendicular to the easy plane is degenerate, since φ_0 can be arbitrary.

The dynamic equations for the excited states of the magnetization field (the Landau-Lifshitz equations), neglecting magnetodipole interaction, are of the form

$$l_0^2\Delta\theta+[1-l_0^2(\nabla\varphi)^2]\sin\theta\cos\theta+\frac{1}{\omega_0}\left[\frac{\partial\varphi}{\partial t}-2\mu_0H\right]\sin\theta=0,$$
$$l_0^2\nabla(\sin^2\theta\nabla\varphi)-\frac{1}{\omega_0}\frac{\partial\theta}{\partial t}\sin\theta=0, \tag{4}$$

where $\Delta\equiv\nabla^2$ is the Laplace operator, l_0 is the magnetic length $l_0^2=\alpha/\beta$), and $\hbar\omega_0=2\beta\mu_0M_0$.

Equations (4) have a number of integrals of motion[4,5] : total energy

$$E=\int [w(\theta,\varphi)-w(\theta_0,\varphi_0)]d^3x, \tag{5}$$

total number of spin deviations (number of real magnons)

$$N=\frac{M_0}{2\mu_0}\int (\cos\theta_0-\cos\theta)\,d^3x, \tag{6}$$

total momentum

$$\mathbf{P}=-\frac{\hbar M_0}{2\mu_0}\int (\cos\theta_0-\cos\theta)\,\nabla\varphi\,d^3x \tag{7}$$

and total angular momentum[6]

$$\mathbf{K}=-\frac{\hbar M_0}{2\mu_0}(\cos\theta_0-\cos\theta)\,[\mathbf{r}\nabla\varphi]d^3x. \tag{8}$$

The weakly excited states of a magnet with energy density (2) are described by a gas of elementary excitations (magnons or spin waves) that are the eigensolution of the linear equations obtained from (4) linearized near the ground state (3). It turns out that the frequency of a magnon with wave vector **k** is equal to

$$\omega(k)=\omega_0 k l_0[1-(H/\beta M_0)^2+(l_0k)^2]^{1/2},\quad H<\beta M_0. \tag{9}$$

The dispersion law (9), apart from the notation, gives the Bogolyubov spectrum of elementary excitations in a weakly nonideal Bose gas. At $l_0 k \ll 1-\cos^2\theta_0$ we obtain the acoustic dispersion law with sound velocity $S=\omega_0 l_0[1-(H/\beta M_0)^2]^{1/2}$, which is the minimum phase velocity of the excitations.

2. DYNAMICS OF ONE-DIMENSIONAL SOLITON STATES

In a one-dimensional easy-plane ferromagnet, besides elementary exitations such as spin waves or magnons with spectrum (9), there can exist also specific nonlinear soliton excitations.

In Ref. 2 were obtained two types of single-soliton one-dimensional solutions of Eqs. (4), which can be regarded as elementary excitations of an unusual type. These solutions describe waves of rotation of the magnetization vector **M**, in which the deviation of the angle θ from its equilibrium value θ_0 depends on the coordinate ξ and on the time t only via the difference $\xi - Vt$:

$$\cos\theta-\cos\theta_0=\frac{1-v^2-h^2}{(1-v^2)^{1/2}}\left[\frac{h}{(1-v^2)^{1/2}}\pm\operatorname{ch}\frac{(1-v^2-h^2)^{1/2}(\xi-Vt)}{l_0}\right]^{-1}, \tag{10}$$

where we put $v=V/V_0$, $V_0=\omega_0 l_0$, $h=H/\beta M_0$. The solution (10) with a $+$ $(-)$ sign in the square brackets will be called a soliton of the first (second) type. Since we must have $1-v^2-h^2>0$ or $h<(1-v^2)^{1/2}$, we get for the soliton of the first type $\theta<\theta_0$ and at the center of the soliton ($\xi=Vt$) we have $\cos\theta(0)=+(1-v^2)^{1/2}$, while for the second soliton $\theta-\theta_0$ and $\cos\theta(0)=-(1-v^2)^{1/2}$ at the center.

The gradient of the angle of rotation of the magnetization vector is given by the relation

$$\frac{d\varphi}{d\xi}=\frac{v}{l_0}\frac{\cos\theta-\cos\theta_0}{\sin^2\theta}. \tag{11}$$

The total angle of rotation of the magnetic moment in the soliton around the anisotropy axis is

$$\Delta\varphi=\int_{-\infty}^{\infty}d\xi\frac{d\varphi}{d\xi}$$

and depends on the wave propagation velocity. For the soliton of the first type it is equal to $\Delta\varphi_1=\Phi(v)-\pi$, and for the soliton of the second type to $\Delta\varphi_2=\Phi(v)+\pi$, where

$$\Phi(v)=2\operatorname{arctg}\frac{hv}{(1-v^2-h^2)^{1/2}}. \tag{12}$$

The considered soliton solutions exist at $0<v^2<1-h^2$. Thus, the maximum possible soliton velocity coincides with the minimum spinwave velocity S.

The solitons of the first and second type behave differently as $V\to S$. The amplitude of the soliton of the first type tends to zero, and the soliton becomes fully delocalized. The amplitude of the soliton of the second type remains constant at the limit $V=S$, and the excitation is transformed into an algebraic (power-law) soliton.[2]

Using the explicit form of the solution (10), we can easily find the dynamic integrals of motion E, P, and N for the solitons of both types and determine their dependence on the velocity V. The ranges of variation of E, P, and N are governed by the condition $0<v<1-h^2$.

The energy (5) of the soliton of the first type is found to be

$$E_1(v)=E_0\left\{(1-v^2-h^2)^{1/2}-h\operatorname{arctg}\frac{(1-v^2-h^2)^{1/2}}{h}\right\}, \tag{13}$$

where $E_0=2a^2(\alpha\beta)^{1/2}M_0^2$, and a is the interatomic distance [the factor a was introduced to preserve the dimensionalities of the physical quantities on going from integration over the volume in (5)–(7) to integration with respect to one coordinate ξ].

The energy $E_2(v)$ of the soliton of the second type differs from (13) by the velocity-independent quantity

$$E_2(v)=E_1(v)+\pi hE_0. \tag{14}$$

The momenta (7) of solitons of the two types also differ by a certain constant quantity

$$P_1(v)=\frac{P_0}{\pi}\left\{\operatorname{arctg}\frac{(1-v^2-h^2)^{1/2}}{v}-h\operatorname{arctg}\frac{(1-v^2-h^2)^{1/2}}{hv}\right\}, \tag{15}$$

$$P_2(v)=P_1(v)+hP_0\operatorname{sign}v, \tag{16}$$

where sign (v) is the sign function (by definition sign$x=+1$, $x>0$; sign$x=-1$, $x<0$), and $P_0=\pi\hbar a^2M_0/\mu_0$ is the limiting momentum of the magnetization field, and arises also in the description of solitons in an easy-axis ferromagnet.[4] We note that $\pi E_0=\omega_0 l_0 P_0$.

The energy and momentum of a soliton are respectively even and odd functions of the velocity V. A characteristic feature is that the energy of a soliton of the second type cannot be less than $\pi h E_0$ and its momentum cannot be less than hP_0 in value.

The numbers of the spin deviations (6), which correspond to the magnetization distributions in solitons of the first and second type, are

$$N_1(v) = -N_0 \operatorname{arctg} \frac{(1-v^2-h^2)^{1/2}}{h}, \qquad (17)$$

$$N_2(v) = N_1(v) + N_0\pi, \qquad (18)$$

where $N_0(a^2M_0/\mu_0)(\alpha/\beta)^{1/2}$. We note again that a characteristic number of magnons on the order of N_0 appears also in nonlinear dynamics of the magnetization of an easy-axis ferromagnet.[4] We note that $E_0 = \hbar\omega_0 N_0$ and it is assumed that $N_0 \gg 1$. It can be seen from (19) that upon deviation from the equilibrium state the angle θ is smaller than θ_0 in the rotation wave of the first type and larger than θ_0 in the rotation wave of the second type. In accordance with the definition (6), the number of spin deviations is $N_1 < 0$ in a wave of the first type and $N_2 > 0$ in a wave of the second type. A negative number of spin deviations N_1 means that the projection of the vector **M** on the z axis is larger in the rotation wave of the first type than the value of this projection in the equilibrium state.

Comparing (13), (14) and (17), (18) we can note that for the solitons of both types

$$E = E_0(1-v^2-h^2)^{1/2} + 2\mu_0 NH, \qquad (19)$$

where the first term can be taken to mean the kinetic energy of the soliton and the second its energy in an external magnetic field.

It must be borne in mind, however, that a solution of the type (10) and (11) is a single-parameter solution, so that N and v are uniquely related, as follows directly from (17) and (18). Therefore, in contrast to the situation with two-parameter solitons,[4] the kinetic energy of a soliton cannot vary independently of its potential energy in an external field, and the division of the energy into two terms (19) is in a certain sense formal.

On the basis of the obtained dependences of the dynamic integrals of motion of the solitons on the velocity V it is easy to analyze the Hamiltonians of the solitons, i.e., the dependences of their energies on the momenta. Since we have verified that at a fixed velocity V the energies and momenta of the solitons of the two types differ by constant quantities, it suffices to study the $E = E(P)$ dependence for one of them, say for the soliton of the first type.

We note first that the momentum of the soliton of the first type cannot exceed $P_1^{\max} = \frac{1}{2} P_0(1-h)$. At any rate, independent excited states are classified in accord with the momentum values in the interval $-P_{\max} < P < P_{\max}$. Following the reasoning of Ref. 4, we can formally introduce values of P outside this interval, assuming the energy to be a periodic function of P. In our case, however, this formal generalization does not lead to any physical conclusions. The appearance of a soliton momentum limit in the long-wave approximation is the consequence of the Galilean invariance of the initial field equations.

A plot of $E_1 = E_1(P_1)$ is shown in Fig. 1 (curve 1). Since the relation $V = dE/dP$ holds for the soliton, curve 1 of Fig.

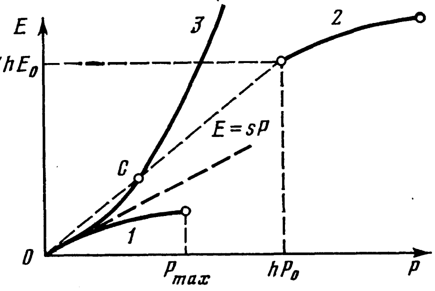

FIG. 1. Soliton dispersion law: 1—for a soliton of the first type, 2—for a soliton of the second type, 3—dispersion law of free magnons.

1 is monotonic in the interval $0 < P < \frac{1}{2} P_0(1-h)$ with a constant derivative

$$\frac{dV}{dP} = \frac{S}{\partial P/\partial v} = -\frac{\pi S}{P_0}\left(\frac{(1-v^2-h^2)^{1/2}}{1-v^2}\right)^{-1} < 0. \qquad (20)$$

Since it turns out that $V \to S$ as $P \to 0$, the entire curve 1 lies below the straight line $E = SP$, i.e., below the plot of the magnon dispersion slaw (curve 3). This means that solitons of the first type can be regarded as ferromagnet collective excitations having a momentum P and an energy lower than that of a magnon with the same momentum. In other words, they are excitations similar to the Lieb excitations in a nonideal Bose gas with repulsion described in detail in Refs. 3 and 7.

Recall, however, that there are also solitons of the second type. In a nonideal Bose gas, whose dynamics is described in the self-consistent-field approximation by a nonlinear Schrödinger equation, there are no such solitons. They are restricted only to an easy-plane ferromagnet.

A plot of the dispersion laws for solitons of the second type is obtained by a parallel shift of curve 1 by an amount hP_0 along the P axis and by an amount $\pi h E_0$ along the E axis, i.e., by a shift along a straight line having a slope

$$\pi E_0/P_0 = \omega_0 l_0 = V_0 > S = V_0(1-h^2)^{1/2}.$$

This plot corresponds to curve 2 of Fig. 1. It also lies below the dispersion curve of an individual magnon (curve 3). The energy of a soliton of the second type is consequently also lower than the energy of a single magnon having the same momentum.

It is useful to ascertain what determines the position of the point $(hP_0, \pi h E_0)$, i.e., of the left edge of the $E(P)$ plot for solitons of the second type. We note that $N_2 = \pi N_0$ magnons correspond to an algebraic soliton with momentum hP_0 (and velocity $V = s$). Following Refs. 4 and 5, we regard the soliton of the second type as a bound state of magnons. It is found then that at $P = hP_0$ the energy per bound magnon is

$$\varepsilon = hE_0/N_0 = \hbar\omega_0 h. \qquad (21)$$

Comparing (21) with the magnon dispersion law (21) we conclude that ε coincides with the energy of a free magnon having a momentum

$$\hbar k = \hbar h/l_0 = hP_0/\pi N_0,$$

i.e., to the total soliton momentum divided by the number of bound magnons. We see that the point $(\hbar k,\varepsilon)$ is found to lie on the dispersion curve of the free magnon (this is point C in Fig. 1). We can thus conclude that when the velocity limit $V=S$ is reached the interaction between the magnons that make up the soliton of the second type vanishes completely, and the soliton "crumbles" into free spin waves.

3. MAGNETIZATION VORTEX (MAGNETIC DISCLINATION)

We proceed now to analyze the localized two-dimensional solutions of Eqs. (4). We introduce in a plane perpendicular to the vector **H** the polar coordinates r and χ: $x=r\cos\chi, y=r\sin\chi$ and seek a localized static solution of Eqs. (4) in the form

$$\theta=\theta(r),\quad \varphi=\varphi_0+\nu\chi,\quad \varphi_0=\text{const},\tag{22}$$

where $\nu=0,\ \pm1,\ \pm2,\ \dots$. We substitute (22) in (4) and obtain the following equation for the function $\theta(r)$ in dimensionless variables:

$$\frac{d^2\theta}{d\rho^2}+\frac{1}{\rho}\frac{d\theta}{d\rho}+\left(1-\frac{\nu^2}{\rho^2}\right)\sin\theta\cos\theta-\Omega\sin\theta=0,\tag{23}$$

where $\Omega=H/\beta M_0$ and $\rho=r/l_0$.

We stipulate that the magnetization at infinity correspond to the equilibrium state

$$\theta=\theta_0,\quad \cos\theta_0=\Omega \text{ for } \rho=\infty,\tag{24}$$

and that on the axis $\rho=0$ all the physical quantities are bounded, in particular

$$\theta=0 \quad\text{for}\quad \rho=0.\tag{25}$$

Satisfying the boundary conditions (24) and (25) is a solution having the following limiting properties at infinity ($\rho=\infty$):

$$\theta=\theta_0-\frac{\nu^2}{\rho^2}\frac{\Omega}{(1-\Omega^2)^{1/2}},\quad \Omega\neq0,\quad \Omega\neq1,\tag{26}$$

$$\theta=\frac{\pi}{2}-c\frac{e^{-\rho}}{\rho^{1/2}},\quad c=\text{const},\quad \Omega=0.\tag{27}$$

Near the axis ($\rho\to0$) we have

$$\theta=(\rho/\rho_0)^{|\nu|},\quad \rho_0=\text{const}.\tag{28}$$

The constant coefficients c in (17) and ρ_0 in (28) cannot be obtained from the asymptotic equations and must be found from the condition that one and the same solution of Eq. (4) satisfy (28) near the axis and fall off like (26) or (27) at large distances. We note that at $\Omega=1$ we have $\theta_0=0$ and there are no localized solutions of the type considered.

A solution of the type (22), which vanishes at infinity in accord with (26) or (27), has a nonzero z-component of the total angular momentum (8):

$$K_z=-\frac{\hbar M_0}{2\mu_0}\int(\cos\theta_0-\cos\theta)\,[r\nabla\varphi]_z\,dV=-\hbar\nu N,\tag{29}$$

where N is the number of spin states (6). The relation (29) between K_z and N was obtained earlier in Refs. 6 and 8. The fact that the solution has the property $K_z\neq0$ at $\nu\neq0$ makes it possible to call such a state of the magnetization field a magnetic vortex characterized by a topological parameter ν. According to another classification of topologically singular solutions in two-dimensional systems, such a solution should be called a magnetic disclination.

It is curious to note that the number N of the spin deviations linked with the magnetic vortex is different at $\Omega=0$ and $\Omega\neq0$. In the absence of a magnetic field ($\Omega=0$) the behavior (27) of the function $\theta(\rho)$ at infinity ensures convergence of the integral (29) and N is therefore finite. If, however, $\Omega\neq0$, it follows from (26) that the integral (29) diverges logarithmically and we can state that with logarithmic accuracy.

$$N(R)=\frac{\pi M_0}{\mu_0}l_0^2\int_0^{R/l_0}(\cos\theta_0-\cos\theta)\rho\,d\rho=-\frac{\pi M_0 l_0^2\nu\Omega}{\mu_0}\ln\left(\frac{R}{l_0}\right),\tag{30}$$

where R is the transverse dimension of the magnet. The number of spin deviations N is given per unit length of the vortex.

The magnetic-vortex energy behaves differently: it is logarithmically large at $\Omega\neq1$. The energy density (2) contains a term of the type

$$(\nabla\varphi)^2\sin^2\theta=\frac{\nu^2}{\rho^2}\sin^2\theta,\tag{31}$$

which leads to a logarithmic divergence of the integral (5) at any law governing the approach of $\theta(\rho)$ to the equilibrium value $\theta_0\neq0$ ($\Omega\neq1$).

We assume that $\Omega\neq0$ and $\Omega\neq1$ and write down expressions for the integrals of motion E and N in the form ($R\gg l_0$)

$$E=E_0^1+\pi\alpha M_0^2\nu^2(1-\Omega^2)\ln(R/l_0),\tag{32}$$

$$N=N_0-\frac{\pi M_0 l_0^2}{\mu_0}\nu^2\Omega\ln\frac{R}{l_0},\tag{33}$$

where E_0^1 and N_0 are constants that depend little on R (they have at any rate no singularities as $R\to\infty$).

We eliminate from (32) the Zeeman energy of the vortex in the external field. We then obtain the internal magnetic energy of the vortex

$$E_m=E-2\mu_0NH=E-\hbar\omega_0\Omega N.\tag{34}$$

From (32) and (33) follows

$$E_m=E_m^0+\pi\alpha M_0^2\nu^2(1+\Omega^2)\ln(R/l_0),\tag{35}$$

where

$$E_m^0=E_0^1-2\mu_0N_0H=E_0-\hbar\omega_0\Omega N_0.$$

We proceed now to discuss the solutions of Eq. (23) that satisfies the conditions (24) and (25). We obtained these solutions for $\nu=\pm1$ with a computer. The method of finding such solutions is described in Ref. 9. It is a certain realization of the "random shot" method[10] and reduces to the following. A value of the parameter Ω is specified and the value of ρ_0 in (28) is chosen such that the solution approaches monotonically the limiting value (24). An idea of the character of the solution is given by the plots in Fig. 2. At Ω close to unity the vortical perturbation is weakly localized, and the function $\theta(\rho)$ approaches slowly, with increasing ρ, the limiting value $\theta_0=[2(1-\Omega)]^{1/2}$. When Ω is changed the slope of the plots increases and in the limit as $\Omega\to0$ a function is obtained

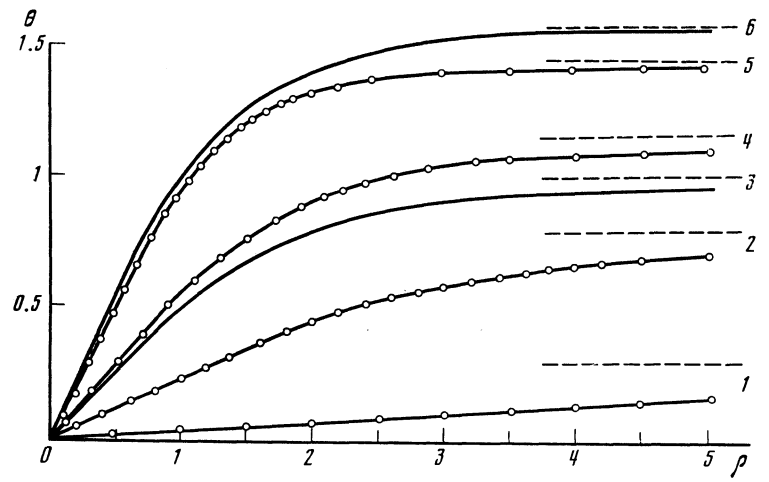

FIG. 2. Dependence of θ on ρ of a magnetization vortex in a two-dimensional ferromagnetic: $1 - \Omega = 0.95$, $2 - \Omega = 0.7$, 3—result of Ref. 1, $4 - \Omega = 0.4$, $5 - \Omega = 0.1$, $6 - \Omega = 0.001$.

(curve 6 of Fig. 2) that differs at $\rho = 3$ by only 2.5% from its limiting value $\theta_0 = \pi/2$.

AT $\Omega \sim 1$ the plots in Fig. 2 do not differ qualitatively from the Bose-gas density distribution in the vortex described by Pitaevskiĭ.[1] This is not at all surprising, since Eq. (23) at $\theta < \pi/2$ is similar to the Gross-Pitaevskiĭ equation. Indeed, putting $\nu = 1$, we expand in (23) the trigonometric functions accurate to terms of third order in θ and leave out the term θ^3/x^2:

$$\frac{d^2\Omega}{dx^2}+\frac{1}{x}\frac{d\theta}{dx}+\left(1-\frac{1}{x^2}\right)\theta-\frac{4-\Omega}{6(1-\Omega)}\theta^3=0, \qquad (36)$$

where $x = (1 - \Omega)^{1/2}\rho$. At $\Omega = 0.4$ Eq. (36) coincides with the equation considered in Ref. 1. A plot of the solution discussed in Ref. 1 is curve 3 of Fig. 2. It is curious to note that this solution differs by about 15% from the solution of Eq. (23) at $\Omega = 0.4$. This means that the term θ^3/x^2 plays no significant role: it is small as $x \to 0$ (when $\theta \sim x$) and as $x \to \infty$.

Having the explicit form of the solutions of (23) at $\nu = 1$ and at various values of Ω, we can determine how well the vortex energy is represented by the sum (32). We write down the energy of the excited state of the magnet in the form

$$E=\pi\alpha M_0^2(1-\Omega^2)\ln(RA(\Omega)/l_0), \qquad (37)$$

where the function $A(\Omega)$ can be easily obtained by numerical methods.

We use the limiting (25) of the function θ at large distances and assume that at $\rho \geqslant \rho_1$ Eq. (26) describes correctly the sought solution $\theta(\rho)$. We choose the point $\rho = \rho_1$ to satisfy the condition that the numerically constructed functions $\theta(\rho)$ and their derivatives $d\theta/d\rho$ agree at the specified accuracy with the function (26) and its derivative. After finding such a point we calculate the magnetic vortex energy as a sum of two integrals (on the intervals $0 < \rho < \rho_1$ and $\rho_1 < \rho < R/l_0$), using Eq. (26) on the second interval. We represent the resultant expression in the form (27) and plot $A(\Omega)$. This plot is shown in Fig. 3. As $\Omega \to 1$ the plot describes the linear function Ω, extrapolation of which to $\Omega = 1$ yields $A \approx 0.2$. As $\Omega \to 0$ the plot approaches the point $A(0) \approx 4.78$.

The analysis of the vortical solutions of Eq. (23) leads to an important conclusion. In a cylindrical easy-plane ferromagnet of radius R there can exist in the absence fo an external magnetic field a static vortex (disclination) with energy $E = \pi\alpha M_0^2 \times \ln(4{,}78R/l_0)$.

4. MAGNETIC VORTEX IN A CYLINDRICAL SAMPLE OF FINITE RADIUS

We have shown that in a cylindrical sample of an easy-plane ferromagnet there can arise a static magnetic disclination for which $\nu = \pm 1$. But it must be borne in mind that Eq. (23) was written without allowance for the magnetic field produced by the inhomogeneous magnetization distribution itself. Therefore the question of the existence of a static magnetic vortex in a real magnet and its energy should have been analyzed by taking the last circumstance into consideration. However, if we restrict ourselves to the case $\nu = 1$, there is no need for additional investigations. Indeed, using (22) at $\nu = 1$ and choosing $\varphi_0 = \pi/2$ we obtain in cylindrical coordinates the following expression for the components of the magnetization produced by the magnetic vortex:

$$\begin{aligned} M_r &= M_0 \sin\theta\cos(\varphi-\chi)=0,\\ M_\chi &= M_0 \sin\theta\sin(\varphi-\chi)=M_0\sin\theta(r), \qquad (38)\\ M_z &= M_0\cos\theta(r). \end{aligned}$$

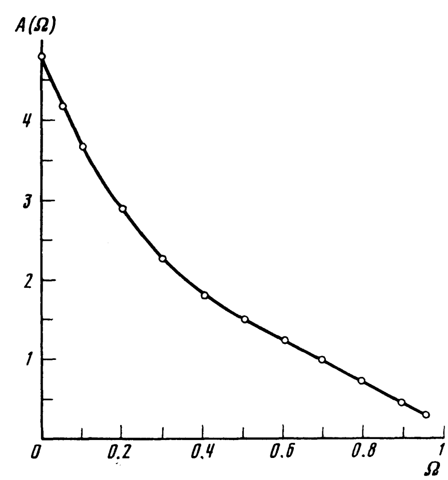

FIG. 3. Functional dependence of the parameter $A(\Omega)$. The points mark the results of a numerical calculation.

It follows from (38) that div$\mathbf{M}=0$, i.e., the inhomogeneity of the magnetization does not contain volume sources of magnetic-field intensity. On the other hand, on the outer surface of the cylinder, $r=R$, there is no normal component of the vector $\mathbf{M}$, therefore an infinitely long magnetized cylinder produces no magnetic field in the surrounding space. Thus, the onset of a magnetic vortex in a cylindrical easy-plane ferromagnet is equivalent in a certain sense to breakdown of the easy-plane ferromagnetic sample into domains.

A state competing in energy may be one of an easy-plane magnet with the so-called helical magnetization. We study this state in the case when the energy of the magnetostatic field is low compared with the anisotropy energy, i.e., at $\beta \gg 4\pi$. We can then start as before from the solutions of Eqs. (4), assuming a ground state corresponding to $\theta=\theta_0=\pi/2$.

The equilibrium equation (4) admits of the solution

$$\theta=\theta_0=\pi/2, \qquad \varphi=kz, \tag{39}$$

which describes the "curling" of the magnetization around the axis along a helix with pitch $2\pi/k$. We assume that $kR\gg 1$ and that the helix pitch is much less than the cylinder radius.

Choosing the solution (39) as the zeroth approximation, we obtain the magnetic field produced by such a magnetization distribution. We introduce, in accordance with the usual rules ($\mathbf{H}=-\nabla\Phi$) the magnetostatic potential Φ, which is by virtue of the condition div $\mathbf{M}=0$ a harmonic function in all of space. It is easy to verify that

$$\Phi=\Phi_0(r)\cos(kz-\chi),$$

$$\Phi_0(r)=AI_1(kr), \qquad 0<r<R, \tag{40}$$

$$\Phi_0(r)=BK_1(kr), \qquad r>R, \tag{41}$$

where $I_1(z)$ and $K_1(z)$ are Bessel functions of imaginary argument z of the first and second kind, respectively.

If $kR\gg 1$, the constants A and B are equal to

$$A=4\pi M_0(\pi R/2k)^{1/2}e^{-kR},$$
$$B=4\pi M_0(R/2\pi k)^{1/2}e^{kR}. \tag{42}$$

The maximum magnetic-field intensity in the cylinder is reached on its outer surface ($r=R$) where, in order of magnitude,

$$H_r\sim 4\pi M_0, \quad H_\chi\sim 4\pi M_0/kR, \quad H_z\sim 4\pi M_0. \tag{43}$$

We see that at $\beta\gg 4\pi$ the magnetostatic field (43) is small compared with the "anisotropy field." This justifies our successive-approximation method.

With the aid of (40) and (41) it is easy to calculate the magnetostatic-field energy per unit cylinder length:

$$E_H=\int\frac{H^2}{8\pi}d\chi\rho\,d\rho=\frac{\pi^2M_0^2R}{k}+o\left[\frac{1}{(kR)^2}\right]. \tag{44}$$

The corresponding inhomogeneous-exchange energy is

$$E_{\text{ex}}=\frac{\alpha M_0^2}{2}\int(\nabla\chi)^2\rho\,d\rho\,d\chi=\frac{\pi}{2}\alpha k^2M_0^2R^2. \tag{45}$$

The equilibrium value of the pitch of the helix should ensure a minimum of the energy $E=E_H+E_{\text{ex}}$. From the condition $dE/dk=0$ we obtain $k\approx(\pi/\alpha R)^{1/3}$. We note that the condition $kR\gg 1$ means that the radius of the cylinder should greatly exceed the "exchange length": $R\gg\alpha^{1/2}$. Only in this case are all our calculations self-consistent.

We return now to the total equilibrium energy of a cylinder with helical magnetization:

$$E={}^3/_2\pi^{5/3}\alpha^{1/3}M_0^2R^{4/3}. \tag{46}$$

Comparing (46) with the energy of the static vortex we conclude that for a cylinder having a radius

$$R>\alpha^{1/2}[\ln(R/l_0)]^{3/4} \tag{47}$$

the energy of the state with helical magnetization exceeds the energy of a cylinder with a magnetic vortex along its axis.

The last conclusion was undoubtedy obtained under the assumption $\beta\gg 4\pi$ (precisely because the parameter β is contained in the inequality (47) as the argument of a logarithm). This assumption has enabled us to start from the solution (39), although it follows from (43) that the presence of the component $H_z\sim 4\pi M_0$ "takes out" the equilibrium magnetization vector from the plane $\theta=\pi/2$. It is easy to verify, however, that the resultant correction to the cylinder energy is of the order of

$$\Delta E\sim(4\pi/\beta)\pi^{5/3}\alpha^{1/3}M_0^2R^{4/3},$$

i.e., it is small compared with (46).

Thus, the equilibrium state of a cylindrical easy-plane ferromagnet with a cylinder axis coinciding with the anisotropy axis is unstable to formation of a magnetic vortex.

5. INFLUENCE OF ANISOTROPY IN THE BASAL PLANE ON THE DISTRIBUTION OF THE MAGNETIZATION AROUND THE VORTEX

In the preceding section we discussed the role of the restriction on the size of a cylindrical easy-plane ferromagnet, disregarding a very important circumstance connected with the anisotropy in the easy plane. In the description of inhomogeneous states of a ferromagnet with a vector $\mathbf{M}$ in the easy plane, the anisotropy in this plane may turn out to be important. In the presence of even a weak anisotropy there are always distances large enough such that the inhomogeneity of the magnetization over them cannot be described without taking this anisotropy into account.

Assume that the z axis has fourfold symmetry. The ferromagnet anisotropy energy can then be written in the form

$$W_a=\int dx\left\{\frac{\beta M_0^2}{2}\cos^2\theta+\frac{\beta' M_0^2}{4}\sin^4\theta\cos^2 2\varphi\right\}, \tag{48}$$

where β' is the anisotropy constant in the easy plane ($\beta'\ll\beta$) and φ is the angle between the magnetization vector and some direction in the xy plane.

If the inhomogeneity of the magnetization distribution is characterized by a parameter L with dimension of length, then at $L\ll(\alpha/\beta')^{1/2}=l_0(\beta/\beta')^{1/2}$ the magnetic anisotropy in the easy plane can be neglected. In particular, the distribution of the magnetization near the magnetic-vortex axis can be calculated neglecting the anisotropy in the easy plane, and is given by the formulas obtained for the uniaxial model. At a distance $\rho\gg l_0(\beta/\beta')^{1/2}\gg l_0$ from the vortex axis, however, account must be taken of the anisotropy in the basal plane.

Since the distance from the vortex axis is much larger in this case than the vortex characteristic dimensions, it can be assumed that the vector **M** lies in the easy plane ($\theta = \pi/2$).

The equilibrium state of a ferromagnet with $\theta = \pi/2$ corresponds to its breakdown into domains in which the orientation of the vector **M** ensures a minimum of the energy (48). The need for breaking up the sample into domains and the requirement that the total phase advance φ be preserved on going around the vortex axis can be reconciled only if a radial system with a fully defined number of domain walls is produced. For a vortex with $\nu = 1$ the most symmetric is breakdown of the sample into four domains divided by 90-degree domain walls.

We consider now the distribution of the magnetization in one of the domain walls far from the vortex axis. If the x axis from which the angle φ is measured is directed perpendicular to the domain-wall plane, the magnetization in the wall will depend only on x. The Landau–Lifshitz equation for the variable $\varphi(x)$ takes the form

$$\left(\frac{\beta}{\beta'}\right) l_0^2 \frac{d^2\varphi}{dx^2} + \sin 2\varphi \cos 2\varphi - \frac{h_x}{\beta'} \sin\varphi = 0, \qquad (49)$$

where h_x is the magnetic-field component produced by the magnetization **M** and determined by the static Maxwell's equations. The solution of Maxwell's equations yields the for h_x

$$h_x = 4\pi(2^{-1/2} - \cos\varphi). \qquad (50)$$

Substituting (50) in (49) and integrating (49) we get

$$\frac{x}{l_0}\left(\frac{\beta'}{\beta}\right)^{1/2} = \int_0^{\varphi} d\varphi \left[\frac{\cos^2 2\varphi}{2} + \frac{4\pi}{\beta'}\left(\cos\varphi - \frac{1}{\sqrt{2}}\right)^2\right]^{-1/2}. \qquad (51)$$

This expression is an implicit form of the sought solution $\varphi = \varphi(x)$ that describes the domain wall. At $x = \pm\infty$, i.e., far from the domain wall, we have $\varphi = \pm\pi/4$, and $d\varphi/dx$ at $x = 0$ has a maximum equal to

$$\left(\frac{d\varphi}{dx}\right)_{max} = \frac{1}{2l_0}\left(\frac{\beta'}{\beta}\right)^{1/2}\left[1 + \frac{4\pi}{\beta'}(3 - 2^{3/2})\right].$$

The domain-wall energy is proportional to its length, whereas the magnetic energy of a cylindrical easy-plane ferromagnet with a magnetic vortex along its axis, is proportional to $\ln R$ (R is the cylinder radius). Consequently the energy of a cylinder containing a magnetic vortex and an associated radial system of domain walls increases in direct proportion to the cylinder radius.

If $\beta' < 4\pi$, a continuous deformation of the magnetization distribution takes place near the cylinder surface and causes the vector **M** to become parallel to the outer surface. As a result, the magnetic field outside the cylindrical sample is practically zero, and the entire magnetic energy is proportional to R.

A competing structure might be one with a sequence of uniformly magnetized domains alternating along the cylinder axis and rotated 90 degrees relative to one another. If L is the period of such a structure, the energy of its magnetostatic field per unit cylinder length is proportional to $M_0^2 RL$. It is easy to verify that $L \propto R^{1/2}$, so that the total magnetic energy is proportional to $M_0^2 R^{3/2}$.

Thus, even if account is taken of the anisotropy energy in the basal plane at a sufficiently large radius R an energywise favored state is that of a cylindrical sample containing a vortex with its associated radial system of domain walls.

[1]L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].

[2]B. A. Ivanov, A. M. Kosevich, and I. V. Manzhos, Sol. St. Commun. **34**, 417 (1980).

[3]E. Lieb and W. Linger, Phys. Rev. **130**, 1605 (1963).

[4]A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, in: Nelineĭnye volny (Nonlinear Waves), Nauka, 1979, p. 45.

[5]A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Fiz. Nizk. Temp. **3**, 906 [Sov. J. Low Temp. Phys. **3**, 440 (1977)].

[6]A. S. Kovalev, A. M. Kosevich, and K. B. Maslov, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 321 (1979) [JETP Lett. **30**, 296 (1979)].

[7]M. Ishikawa and H. Takayama, J. Phys. Soc. Japan **49**, 1242 (1980).

[8]I. E. Dzyaloshinskii and B. A. Ivanov, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 592 (1979) [JETP Lett. **29**, 540 (1979)].

[9]A. M. Kosevich and V. P. Voronov, Fiz. Nisk. Temp. **7**, 908 (1979) [Sov. J. Low Temp. Phys. **7**, 442 (1981)].

[10]A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov, Differentsial'nye uravneniya (Differential Equations), Nauka, 1980, p. 231

Translated by J. G. Adashko