Dynamics of a nonequilibrium magnetized plasma in weakly stochastic fields

A. V. Gurevich, K. P. Zybin, and Ya. N. Istomin

P. N. Lebedev Physical Institute, Academy of Sciences of the USSR
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We consider the behavior of a plasma in electrical and magnetic fields which have random components. We derive kinetic equations for the electron and ion distribution functions averaged over the fluctuations. We determine the complete set of kinetic coefficients in the hydrodynamical plasma equations, taking fluctuations into account. Using these equations we consider the diffusion of the plasma and the damping of magnetohydrodynamic waves, and also the kinetic effect of the strong deformation of the tail of the distribution function in a fluctuating magnetic field.

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Magnetohydrodynamics meets with a number of difficulties when describing the fluxes of a nonequilibrium rarefied plasma; these are the consequences of the peculiarities of plasma behavior. Part of these peculiarities are connected with the fact that classical dissipative processes are weak. Due to this instabilities occur easily in a nonequilibrium rarefied plasma and they lead to the buildup of various kinds of waves and to the stochastization of the plasma state. As a result the dissipative processes are appreciably intensified–the transverse diffusion, the conductivity, the heat conductivity, and the viscosity are intensified. There arises a diffusion of the magnetic field and it is sharply enhanced.

Other important peculiarities are determined by the Coulomb nature of the collisions. Collisional dissipation due to Coulomb collisions is negligibly small for fast particles so that they easily gain energy in a nonequilibrium plasma–they are accelerated. As a result there occurs so to speak a special channel to dissipate the energy of the main plasma. If the fast particles are sufficiently well contained their number may become appreciable and then they contribute importantly both to the transfer processes and to the dynamics of the plasma as a whole.

The present paper is devoted to developing methods for describing the dynamics of a nonequilibrium rarefied plasma–so to speak a generalization of magnetohydrodynamics to the case when there are present in the plasma, due to its instability and turbulization, stochastic components of the electric and magnetic fields and there is an arbitrary number of fast particles. There has in recent years been an enhanced interest in the behavior of a plasma under such conditions in connection with problems of the Fermi acceleration of cosmic rays,[1–3] the diffusion of magnetic surfaces in toroidal traps,[4,5] anomalous heat conduction in tokamaks,[6,7] and the flow of the solar wind around the magnetospheres of the Earth and the other planets.

In §1 of our paper we derive the starting kinetic equation which describes the behavior of the electrons and the ions in the plasma when there are stochastic components of the electric and magnetic fields present. In §2 we consider a collisionless plasma and in §3 a plasma with collisions. We obtain additional terms in the hydrodynamic equations which describe the effect of the stochastic fields on the transport, heating, and acceleration of particles in the plasma.

In a collisional plasma one can then distinguish two components—the main plasma and the gas of the fast particles; we formulate a closed set of equations describing both components. In §§4 and 5 we consider as example solutions of these equations describing the diffusion in the plasma and the damping of magnetohydrodynamic waves, and also the kinetic effects of the deformation of the fast particle distribution function caused by the fluctuations in the magnetic field.

§1. STATEMENT OF THE PROBLEM. BASIC EQUATIONS

We consider a plasma in a magnetic field **B** and an electric field **E** which contain random fluctuating components $\mathbf{b}(\mathbf{r},t)$ and $\mathbf{e}(\mathbf{r},t)$ whose the amplitudes are small compared to the main components $\mathbf{B}_0(\mathbf{r},t)$ and $\mathbf{E}_0(\mathbf{r},t)$, viz.,

$$\begin{aligned} &\mathbf{B}=\mathbf{B}_0+\mathbf{b}, \quad |\mathbf{e}|\ll|\mathbf{E}_0|, \quad \langle\mathbf{e}\rangle=0,\\ &\mathbf{E}=\mathbf{E}_0+\mathbf{e}, \quad |\mathbf{b}|\ll|\mathbf{B}_0|, \quad \langle\mathbf{b}\rangle=0. \end{aligned} \tag{1}$$

We shall assume that we know the amplitudes of and spectra of the fluctuations and we study their effect on the averaged motion and the kinetics of the particles in the plasma.

The main quantities characterizing the fluctuations (the correlation length L_c and the correlation time τ_c) are assumed to be small compared to the characteristic scales of the motions–the spatial dimensions L of the inhomogeneity in the plasma and in the main fields and the times Δt over which they change, i.e.,

$$L_c\ll L, \quad \tau_c\ll\Delta t. \tag{2}$$

When we describe the plasma we shall start from the kinetic equations for the electron and ion distribution functions:

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{m}(\mathbf{E}_0+\mathbf{e})\frac{\partial f}{\partial \mathbf{v}}+\frac{e}{mc}[\mathbf{v}\times(\mathbf{B}_0+\mathbf{b})]\frac{\partial f}{\partial \mathbf{v}}=S(f). \tag{3}$$

Here $S(f)$ is the collision integral. Using (1) we write the distribution function f in the form

$$f=\bar{f}+\delta f, \quad |\delta f|\ll|\bar{f}|, \tag{4}$$

where $\bar{f}$ is the main part averaged over an ensemble of realizations of the random fields **b** and **e**, while δf are its fluctuations. We then have from (3) and (4)

$$\frac{\partial \bar f}{\partial t}+\mathbf{v}\frac{\partial \bar f}{\partial \mathbf{r}}+\frac{e}{m}\overline{\left(\mathbf{e}+\frac{1}{c}[\mathbf{v}\times\mathbf{b}]\right)\frac{\partial f}{\partial \mathbf{v}}}$$

$$+\frac{e}{m}\left(\mathbf{E}_0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}_0]\right)\frac{\partial \bar f}{\partial \mathbf{v}}=S(\bar f), \tag{5}$$

$$\frac{\partial}{\partial t}\delta f+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta f+\frac{e}{m}\left(\mathbf{E}_0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}_0]\right)\frac{\partial}{\partial \mathbf{v}}\delta f$$

$$=S_{\text{lin}}(\delta f)-\frac{e}{m}\left(\mathbf{e}+\frac{1}{c}[\mathbf{v}\times\mathbf{b}]\right)\frac{\partial}{\partial \mathbf{v}}\bar f. \tag{6}$$

Here S_{lin} is the collision integral linearized with respect to $\bar f$.

We can write the solution of Eq. (6) in the form

$$\delta f=-\frac{e}{m}\int_{-\infty}^{+\infty} d^3r'd^3v'\int_{-\infty}^{t} dt'\left(\mathbf{e}(r',t')+\frac{1}{c}[\mathbf{v}'\times\mathbf{b}(\mathbf{r}',t')]\right)\times\frac{\partial \bar f(r',v',t')}{\partial \mathbf{v}'}G. \tag{7}$$

Here G is the Green function of the linearized kinetic equation:

$$\frac{\partial G}{\partial t}+\mathbf{v}\frac{\partial G}{\partial \mathbf{r}}+\frac{e}{m}\left(\mathbf{E}_0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}_0]\right)\frac{\partial G}{\partial \mathbf{v}}$$

$$=S_{\text{lin}}(\bar f,G)+\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}'). \tag{8}$$

Substituting (6) and (7) into (5) and averaging we get the following equation for $\bar f$ (we shall drop here and henceforth the bar over f):

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{m}\left(\mathbf{E}_0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}_0]\right)\frac{\partial f}{\partial \mathbf{v}}=S(f)+I_{\text{fl}}(f),$$

$$I_{\text{fl}}=\frac{e^2}{m^2}\frac{\partial}{\partial v_\alpha}\int_{-\infty}^{+\infty} d^3r'd^3v'\int_{-\infty}^{t} dt' G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')$$

$$\times\Big\{\langle e_\alpha(\mathbf{r},t)e_\beta(\mathbf{r}',t')\rangle\frac{\partial f}{\partial v_\beta'}(\mathbf{r}',\mathbf{v}',t')$$

$$+\frac{1}{c}\langle e_\alpha(\mathbf{r},t)b_\beta(\mathbf{r}',t')\rangle\varepsilon_{\lambda\beta\mu}v_\mu'\frac{\partial f}{\partial v_\lambda'}(\mathbf{r}',\mathbf{v}',t')\Big\} \tag{9}$$

$$+\frac{e^2}{m^2c}\varepsilon_{\alpha\rho k}v_\rho\frac{\partial}{\partial v_\alpha}\int_{-\infty}^{+\infty} d^3r'd^3v'\int_{-\infty}^{t} dt' G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')$$

$$\times\Big\{\langle b_k(\mathbf{r},t)e_\mu(\mathbf{r}',t')\rangle\frac{\partial f}{\partial v_\mu'}+\frac{1}{c}\langle b_k(\mathbf{r},t)b_\mu(\mathbf{r}',t')\rangle$$

$$\times\varepsilon_{\nu\beta\mu}v_\beta'\frac{\partial f}{\partial v_\nu'}(\mathbf{r}',\mathbf{v}',t')\Big\}.$$

Here $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor. Summation occurs over repeated indexes.

The set of Eqs. (8) and (9) describes the averaged distribution function when there are collisions in the plasma and the relatively weak electromagnetic field fluctuations (1). Averaging was carried out over the fluctuations and their effect was taken into account by the correlators $\langle e_\alpha e_\beta\rangle$, $\langle e_\alpha b_\beta\rangle$, $\langle b_\alpha b_\beta\rangle$ which are assumed to be known. We note that these correlation functions are not independent but are connected through relations which follow from the Maxwell equations, for instance,

$$\text{rot}\langle \mathbf{e}(\mathbf{r},t)b_\beta(\mathbf{r}',t')\rangle=-\frac{1}{c}\frac{\partial}{\partial t}\langle \mathbf{b}(\mathbf{r},t)b_\beta(\mathbf{r}',t')\rangle.$$

The Green function G describes the dynamics of the perturbation which is localized in the point $\mathbf{r}'$, $\mathbf{v}'$ at the initial time $t=t'$.

The set (8), (9) is characterized, apart from by the correlation scales L_c and τ_c, by the mean free path time and mean free path $\tau_{e,i}$ and $l_{e,i}$ of the electrons and the ions. We shall in what follows consider both the case of rare collisions (a collisionless plasma in the limit)

$$\tau_{e,i}/\tau_c\to\infty,\quad l_{e,i}/L_c\to\infty, \tag{10}$$

and the inverse case of a plasma with a large number of collisions

$$\tau_c\gg\tau_{e,i};\quad L_c\gg l_{e,i}. \tag{11}$$

§2. COLLISIONLESS PLASMA

Under the conditions (10) of rare collisions Eq. (8) describes the free motion of particles in electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{B}_0$. It is clear from (9) that the solution of Eq. (8) is important only for spatial and temporal scales τ_c and L_c (we assume that the correlations decrease sufficiently rapidly when $L\gg L_c$ and $\tau\gg\tau_c$). Because of the condition (2) the fields change little on those scales and we can therefore take them to be quasi-uniform and quasi-stationary. Moreover, recognizing that quasi-stationary fields in a collisionless plasma are orthogonal $\mathbf{E}_0\perp\mathbf{B}_0$ and separating the slow $(\mathbf{r},\mathbf{v},t)$ and the fast $(\mathbf{r}-\mathbf{r}',\ \mathbf{v}-\mathbf{v}',\ t-t'=\tau)$ variables we find the Green function:

$$G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')=\theta(\tau)\delta(z-z'-v_z\tau)\delta\Big(x-x'-\frac{v_x}{\omega_H}\sin\omega_H\tau-\frac{v_y}{\omega_H}(\cos\omega_H\tau-1)\Big)\delta\Big(y-y'+\frac{v_x}{\omega_H}(\cos\omega_H\tau-1)-\frac{v_y}{\omega_H}\sin\omega_H\tau\Big)\delta(v_z-v_z')\cdot \tag{12}$$

$$\times\delta(v_x\cos\omega_H\tau-v_y\sin\omega_H\tau-v_x')\delta(v_x\sin\omega_H\tau+v_y\cos\omega_H\tau-v_y').$$

The z-axis is here taken along $\mathbf{B}_0(\mathbf{r})$, $\theta(\tau)$ is the unit step function, and ω_H is the cyclotron frequency of the particle considered. Using the actual form (12) of the Green function we can, by integrating by parts, transform the right-hand side of Eq. (9) for the collisionless case into

$$I_{\text{fl}}=\frac{e^2}{m^2}\frac{\partial}{\partial v_\alpha}\int_0^\infty d\tau\hat L_k\int_{-\infty}^{\infty} d^3r'd^3v'\Big\{\langle e_\alpha(\mathbf{r},t)e_k(\mathbf{r}',t')\rangle+\frac{v_m'}{c}\langle e_\alpha b_n\rangle\varepsilon_{kmn}\Big\}G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')f(\mathbf{r}',\mathbf{v}',t')$$

$$+\frac{e^2}{m^2c}\varepsilon_{\alpha\mu\nu}v_\rho\frac{\partial}{\partial v_\alpha}\int_0^\infty d\tau\hat L_k\int_{-\infty}^{+\infty} d^3r'd^3v'\Big\{\langle b_\nu e_k\rangle+\frac{1}{c}\langle b_\nu b_\mu\rangle\varepsilon_{k\lambda\mu}v_\lambda'\Big\}f(\mathbf{r}',\mathbf{v}',t')G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t'), \tag{13}$$

where the $\hat L_k$ are differential operators

$$\hat{L}_x=\left[\cos\omega_H\tau\frac{\partial}{\partial v_x}-\sin\omega_H\tau\frac{\partial}{\partial v_y}\right]+\frac{1}{\omega_H}\left[\sin\omega_H\tau\frac{\partial}{\partial x}-(1-\cos\omega_H\tau)\frac{\partial}{\partial y}\right],$$

$$\hat{L}_y=\left[\sin\omega_H\tau\frac{\partial}{\partial v_x}+\cos\omega_H\tau\frac{\partial}{\partial v_y}\right]+\frac{1}{\omega_H}\left[(1-\cos\omega_H\tau)\frac{\partial}{\partial x}+\sin\omega_H\tau\frac{\partial}{\partial y}\right],$$

$$\hat{L}_z=\frac{\partial}{\partial v_z}+\tau\frac{\partial}{\partial z}.$$

Using the form of the function G we now integrate over $d^3r'd^3v'$. There remains only a single integral over $d\tau$. The integrand in (13) then retains its form and, in agreement with (12), only the velocities v'_λ and the arguments of the function $f(\mathbf{v}',\mathbf{r}',t')$ are changed; one makes the following substitution:

$$v_x'\to v_x\cos\omega_H\tau-v_y\sin\omega_H\tau,$$

$$v_y'\to v_x\sin\omega_H\tau+v_y\cos\omega_H\tau \text{ etc.}$$

We consider further the particular case of strongly magnetized fluctuations:

$$\omega_H\tau_c\gg1. \quad (14)$$

Retaining terms of zeroth and first order in $(\omega_H\tau_c)^{-1}$ and using (12) and (14) we get a simple expression for I_{fl} in this limit:

$$I_{\text{fl}}(f)=\left(\frac{e}{m}\right)^2\left[\frac{\partial}{\partial v_\alpha}\left(\frac{\partial}{\partial v_\beta}R_{\alpha\beta}{}^ef+\frac{\partial}{\partial r_\beta}Q_{\alpha\beta}{}^ef\right)+\varepsilon_{\rho\mu\alpha}v_\mu\frac{\partial}{\partial v_\rho}\left(\frac{\partial}{\partial v_\beta}R_{\alpha\beta}{}^bf+\frac{\partial}{\partial r_\beta}Q_{\alpha\beta}{}^bf\right)\right],$$

$$R_{\alpha z}{}^\eta=\int_0^\infty\left\langle\eta_\alpha\left(e_z+\frac{1}{c}[\mathbf{u}_0\times\mathbf{b}]_z\right)\right\rangle d\tau,$$

$$R_{\alpha x}{}^\eta=\int_0\langle\eta_\alpha b_z\rangle\frac{1}{c}(v_y-u_{0y})\,d\tau,$$

$$R_{\alpha y}{}^\eta=-\int_0^\infty\langle\eta_\alpha b_z\rangle\frac{1}{c}(v_x-u_{0x})\,d\tau, \quad (15)$$

$$Q_{\alpha x}{}^\eta=\int_0^\infty\frac{1}{\omega_H}\left\langle\eta_\alpha\left(e_y+\frac{1}{c}[\mathbf{u}_0\mathbf{b}]_y+\frac{v_z}{c}b_x\right)\right\rangle d\tau,$$

$$Q_{\alpha y}{}^\eta=-\int_0^\infty\frac{1}{\omega_H}\left\langle\eta_\alpha\left(e_x+\frac{1}{c}[\mathbf{u}_0\mathbf{b}]_x-\frac{v_z}{c}b_y\right)\right\rangle d\tau,$$

$$Q_{\alpha z}{}^\eta=\int_0^\infty\tau\left\langle\eta_\alpha\left(e_z+\frac{1}{c}[\mathbf{u}_0\mathbf{b}]_z\right)\right\rangle d\tau.$$

Here $\mathbf{u}_0$ is the velocity of the plasma motion across the magnetic field $\mathbf{B}_0$ equal to

$$\mathbf{u}_0=c[\mathbf{E}_0\times\mathbf{B}_0]/B_0^2.$$

The quantity η_α takes on the values e_α and b_α. In the coefficients $R^{e,b}_{\alpha\beta}$ and $Q^{e,b}_{\alpha\beta}$ there occur integrals of the correlation functions $\langle e_\alpha e_\beta\rangle$, $\langle b_\alpha b_\beta\rangle$ in which the argument $\mathbf{x}=\mathbf{r}-\mathbf{r}'$ depends on the time according to (12):

$$x-x'=(v_x/\omega_H)\sin\omega_H\tau+(v_y/\omega_H)(\cos\omega_H\tau-1),$$

$$y-y'=(v_y/\omega_H)\sin\omega_H\tau-(v_x/\omega_H)(\cos\omega_H\tau-1),$$

$$z-z'=v_z\tau.$$

Hence it is clear [see (15)] that the interaction with the fluctuations leads not only to diffusion in energy space (the terms $R^{e,b}_{\alpha\beta}$) but also to a drift in coordinate space (the terms $Q^{e,b}_{\alpha\beta}$). This distinguishes $I_{\text{fl}}{}^{(f)}$ from the usual collision integral.

We now consider the hydrodynamic macroscopic plasma motion. We recognize that even rare electron-electron and ion-ion collisions succeed in Maxwellizing the distribution function provided the average particle energy ε in the plasma increases not too rapidly, i.e.,

$$d\varepsilon_e/dt<\varepsilon_e/\tau_e,\quad d\varepsilon_i/dt<\varepsilon_i/\tau_i, \quad (16)$$

where τ_e and τ_i are the mean free path times of the electrons and the ions. Assuming that conditions (16) are satisfied, i.e., assuming the electron and ion distribution functions to be Maxwellian, we change from the kinetic Eq. (15) to the hydrodynamic equations

$$\partial N/\partial t+\operatorname{div}(N\mathbf{u})+\operatorname{div}\mathbf{j}_D=0,$$

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}=\frac{1}{Nm_ic}[\mathbf{j}\times\mathbf{B}_0]-\frac{1}{Nm_i}\nabla p+\mathbf{R}, \quad (17)$$

$$\frac{3}{2}N\left[\frac{\partial T}{\partial t}+(\mathbf{u}\nabla)T\right]+p\operatorname{div}\mathbf{u}=\frac{1}{2}\frac{e^2}{m_e}R_{zz}{}^eN-\operatorname{div}\mathbf{q}+\mathbf{jE}_0.$$

As usual we have assumed that the non-uniformity scales are much larger than the Debye radius so that the plasma is quasi-neutral, $N_e=N_i=N$. The hydrodynamic velocity is then $\mathbf{u}=(m_e\mathbf{u}_e+m_i\mathbf{u}_i)/(m_e+m_i)$. We assume also for the sake of simplicity that the electron and ion temperatures are equal: $T_e=T_i=T$. The quantity $p=N_eT_e+N_iT_i$ is the total plasma pressure, $p_e=N_eT_e$ the electron pressure; $\mathbf{j}_D$ is the diffusion flux and $\mathbf{R}$ the frictional force:

$$j_D{}^\mu=\frac{c^2}{B_0^2}\frac{\partial}{\partial r_{k_\perp}}\left[\varepsilon_{\mu z\nu}D^e_{\nu k}N+\frac{1}{c}\varepsilon_{\mu z\nu}O^e_{\nu k}u_zN+\frac{1}{c}D^b_{zk}u_\perp{}^\mu N-\frac{1}{c}D^b_{\perp\perp}u_zN\right],$$

$$R_\mu=\frac{e^2}{m_im_ec}\varepsilon_{\mu z\lambda}R^b_{\lambda z}+\frac{1}{N}\frac{\partial}{\partial r_k}\left\{-\frac{e^2}{m_im_e}\delta_{kz}D_{\mu k}{}^eN-\frac{1}{B_0m_i}C_{\mu k}{}^ej_z-\varepsilon_{\mu\rho\lambda}\left[\delta_{kz}\frac{e^2}{m_im_ec}D_{\lambda k}{}^bN\left(u_\rho-\frac{1}{eN}j_\rho\right)+\frac{1}{m_iB_0}\delta_{kk_\perp}D^b_{\lambda k}j_\rho-\delta_{\rho z}\frac{e}{m_im_ecB_0}C^b_{\lambda k}NT\right]\right\}; \quad (18)$$

$$D_{nk}^{\eta}=\begin{cases}\int\limits_0^\infty \tau\left\langle \eta_n\left(e_z+\frac{1}{c}[\mathbf{u}_0\times\mathbf{b}]_z\right)\right\rangle d\tau, & k=z;\quad k_\perp=(x_x,y_y,0)\\ \int\limits_0^\infty \left\langle \eta_n\left(e_y+\frac{1}{c}[\mathbf{u}_0\times\mathbf{b}]_y\right)\right\rangle d\tau, & k=x;\quad r_{k_\perp}=(x,y,0),\\ -\int\limits_0^\infty \left\langle \eta_n\left(e_x+\frac{1}{c}[\mathbf{u}_0\times\mathbf{b}]_x\right)\right\rangle d\tau, & k=y;\quad b_\perp=(b_x,b_y,0)\end{cases}$$

$$C_{nk}^{\eta}=\begin{cases}0, & k=z\\ \int\limits_0^\infty\langle\eta_n b_x\rangle d\tau, & k=x.\\ \int\limits_0^\infty\langle\eta_n b_y\rangle d\tau, & k=y\end{cases}\tag{19}$$

As before, η_n has here two values, e_n and b_n; $q_\alpha=(e/m_eB_0)NTC^{\,e}_{z\alpha}$ is the heat flux caused by the fluctuations to which only the correlation function $\langle e_z b_\perp\rangle$ contributes. Furthermore, $\frac{1}{2}(e^2/m_e)R^{\,e}_{zz}N$ is the work done by the stochastic electric field on the plasma and causes its heating. Ohm's law for the longitudinal current takes the form

$$\frac{dj_z}{dt}+j_z\,\mathrm{div}\,\mathbf{u}=\frac{e^2N}{m_e}E_{0z}+\frac{e}{m_e}\nabla_z p_e$$
$$-\frac{ce^2}{B_\beta m_e}\frac{\partial}{\partial r_k}\left\{D_{zk}^eN+\frac{1}{c}C_{zk}^eN\left(u_z-\frac{j_z}{eN}\right)+\varepsilon_{\tau\mu\nu}\frac{1}{c}\cdot\right.$$
$$\left.\times D_{\nu k}^b\left(u_\mu-\frac{j_\mu}{eN}\right)N\right\}.\tag{20}$$

The transverse current is defined as follows:

$$j_\perp^\alpha=-\frac{c}{B_0^2}[\nabla p\times\mathbf{B}_0]_\alpha+\frac{e^2c}{m_eB_0}\left\{\frac{1}{c}R_{\alpha z}^bN+\frac{\partial}{\partial z}\right.$$
$$\left.\times\left[\varepsilon_{\alpha zm}D_{mz}^eN+\frac{1}{c}D_{zz}^bu_\perp^\alpha N-\frac{1}{c}D_{\alpha\perp z}^bu_zN\right]\right\}.\tag{21}$$

The hydrodynamic equations are closed by the Maxwell equations for the main fields $\mathbf{E}_0$ and $\mathbf{B}_0$, i.e.,

$$\mathrm{div}\,\mathbf{B}_0=0,\quad \mathrm{rot}\,\mathbf{H}_0=4\pi\mathbf{j}/c,$$
$$\mathrm{rot}\,\mathbf{E}_0=-\frac{1}{c}\frac{\partial\mathbf{B}_0}{\partial t},\quad \mathbf{H}_0=\mathbf{B}_0\left(1+\frac{4\pi p}{B_0^2}\right).\tag{22}$$

The longitudinal electric field (the polarization field) is determined by the quasi-neutrality equation:

$$\mathrm{div}\,\mathbf{j}=0.$$

The set of Eqs. (17) to (22) describes the magnetohydrodynamics of a Maxwell plasma with rare collisions in stochastic fields. It is clear that in that case the fluctuations determine both the forces acting upon the plasma and the transport processes leading to diffusion, heating, and particle acceleration. For instance, the magnitude of the transverse transport coefficient $D_\perp$, determined by (19), is

$$D_\perp\approx v_T^2\tau_c\langle b_\perp^2/B_0^2\rangle,$$

where $v_\perp$ is the ion or electron thermal velocity.

We give below in §§4 and 5 particular examples of applications of the hydrodynamic equations obtained here.

§3. PLASMA WITH COLLISIONS

We now consider the dynamics of a plasma under conditions (11) when not only the fluctuations of the electromagnetic field but also the electron and ion collisions are important. We shall assume the plasma to be magnetized

$$\omega_H\tau_{e,i}\gg1,\tag{23}$$

where $\tau_{e,i}$ are the mean free path times of the thermal particles. By virtue of (11) the conditions $\rho_H/L_c\ll1$ and $\omega_H\tau_c\gg1$ are then satisfied, where ρ_H is the Larmor radius of a particle in the field $\mathbf{B}_0$. We can then average Eqs. (8) and (9) over the gyrorotation angle which is equivalent to changing to drift variables. We get

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}+\frac{c}{B_0^2}[\mathbf{E}_0\times\mathbf{B}_0]\frac{\partial f}{\partial\mathbf{r}_\perp}+\left\{\frac{eE_{0z}}{m}-\frac{v_\perp^2}{2B_0v_z}\cdot\right.$$
$$\left.\times\left(v_z\frac{\partial B_0}{\partial z}+\frac{c}{B_0^2}[\mathbf{E}_0\times\mathbf{B}_0]\nabla B_0\right)\right\}\frac{\partial f}{\partial v_z}+\frac{v_\perp}{2B_0}\left\{\frac{\partial B_0}{\partial t}\right.$$
$$\left.+v_z\frac{\partial B_0}{\partial z}+\frac{c}{B_0^2}[\mathbf{E}_0\times\mathbf{B}_0]\nabla B_0\right\}\frac{\partial f}{\partial v_\perp}$$
$$=\frac{1}{B_0}\left(v_z\frac{\partial}{\partial r_\alpha}+\frac{eE_{0\alpha}}{m}\frac{\partial}{\partial v_z}\right)\int\limits_{-\infty}^{+\infty}d^3r'\,dv_z'\,dv_\perp'$$
$$\times\int\limits_{-\infty}^{t}dt'\left\{\langle b_\alpha b_\beta\rangle\frac{1}{B_0}\left(v_z'\frac{\partial f}{\partial r_\beta}+\frac{eE_{0\beta}}{m}\frac{\partial f}{\partial v_z'}\right)\right.$$
$$\left.+\langle b_\alpha e_z\rangle\frac{e}{m}\frac{\partial f}{\partial v_z'}\right\}G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')$$
$$+\frac{e}{m}\frac{\partial}{\partial v_z}\int\limits_{-\infty}^{+\infty}d^3r'\,dv_z'\,dv_\perp'\int\limits_{-\infty}^{t}dt'\left\{\langle e_zb_\beta\rangle\right.$$
$$\left.\times\frac{1}{B_0}\left(v_z'\frac{\partial f}{\partial r_\beta}+\frac{eE_{0\beta}}{m}\frac{\partial f}{\partial v_z'}\right)+\frac{e}{m}\langle e_ze_z\rangle\frac{\partial f}{\partial v_z'}\right\}$$
$$\times G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')+S(f).\tag{24}$$

Here v_z is the velocity along and $v_\perp$ the velocity across the magnetic field $\mathbf{B}_0+\mathbf{b}$. We denote the left-hand side of Eq. (24) by Df; the equation for the Green function G then takes the form

$$DG=S_{\mathrm{lin}}(f,G)+\delta(t-t')\delta(v_z-v_z')\delta(v_\perp-v_\perp')\delta(\mathbf{r}-\mathbf{r}').\tag{25}$$

In deriving Eqs. (24) and (25) we have used conditions (1) and (23), and also the additional condition

$$v_T\tau_c/L_c\ll(\omega_H\tau_c)^{1/2}.\tag{26}$$

Because the number (11) of collisions is large, the electron and ion distribution functions are nearly Maxwellian so that we can linearize S_{ii} and S_{ee} relative to the Maxwell distribution function. The integrals for S_{ei} and S_{ie} have the form[8]

$$S_{ei}(f_e)=\nu_1\left(\frac{2T_e}{m_e}\right)^{3/2}\frac{\partial}{\partial v_\alpha}\left(V_{\alpha\beta}\frac{\partial f}{\partial v_\beta}\right)+\frac{m_e}{m_i}\nu_1\left(\frac{2T_e}{m_e}\right)^{3/2}\times\frac{\partial}{\partial v_\alpha}\left\{\frac{2v_\alpha}{v^3}f_e+\frac{T_e}{m_i}\tilde{V}_{\alpha\beta}\frac{\partial f_e}{\partial v_\beta}\right\},$$
$$S_{ie}(f_i)=\nu_2\frac{\partial}{\partial v_\alpha}\left[v_\alpha f_i+\frac{T_e}{m_i}\frac{\partial f_i}{\partial v_\alpha}\right]+\frac{1}{m_iN_i}R_\alpha\frac{\partial f_i}{\partial v_\alpha},$$
$$V_{\alpha\beta}=(v^2\delta_{\alpha\beta}-v_\alpha v_\beta)/v^3,\quad \tilde{V}_{\alpha\beta}=(3v_\alpha v_\beta-v^2\delta_{\alpha\beta})/v^5,$$
$$\nu_1=\frac{3\sqrt{\pi}}{8}\frac{1}{\tau_e};\quad \tau_e=\frac{3m_e^{1/2}}{4(2\pi)^{1/2}\Lambda_e}\frac{T_e^{3/2}}{N_ie^4},\tag{27}$$
$$\nu_2=\frac{m_eN_e}{m_iN_i}\frac{1}{\tau_e};\quad R_\alpha=-\frac{m_eN_e}{\tau_e}(u_{e\alpha}-u_{i\alpha}),$$

where $u_{i,e}$ are the ion and electron macroscopic velocities, and Λ_e is the Coulomb logarithm. We have written the electron distribution function in the coordinate system moving with the ion velocity.

It is important that both the electron and the ion collision frequency decreases when their energy increases: $\nu\propto\varepsilon^{-3/2}$. Condition (11) is therefore satisfied only, if

$$\varepsilon<\varepsilon_k\approx T(\tau_c/\tau_{e,i}(T))^{2/3}.$$

Since $\tau_c\gg\tau_{e,i}(T)$ and $\varepsilon_k\gg T$, for particles with energies $\varepsilon>\varepsilon_k$ the condition which is the opposite of condition (11) is satisfied. The plasma is for them collisionless. This distinguishes the energetic particles; they form, so to speak, a gas in the plasma. In what follows we get the equations which describe separately the main, thermal plasma and the energetic component.

A. Main plasma

We consider the main plasma. The particle velocity in it is of the order of the thermal velocity v_T. We take it that the drift velocity $u_{dr}\ll v_T$. The equation for the Green function G can then be written in the form

$$\frac{\partial G}{\partial t}+v_z\frac{\partial G}{\partial z}=S(G)+\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')\delta(v_z-v_z')\delta(v_\perp-v_\perp').\tag{28}$$

We give in the Appendix the detailed solution of Eq. (28). It can be written in the form

$$G(\mathbf{r},\mathbf{v},t,\mathbf{r}',\mathbf{v}',t')=\sum_{\alpha=1}^{4}\Theta_\alpha(\mathbf{r},\mathbf{r}',t,t')f_M(\mathbf{v})\chi_\alpha(\mathbf{v})\chi_\alpha(\mathbf{v}')+\sum_{\beta=5}^{\infty}G_\beta.\tag{29}$$

Here $f_M(\mathbf{v})$ is the Maxwell distribution function normalized to unity and

$$\Theta_1=(4\pi\eta_1\tau)^{-1/2}(4\pi\eta_2\tau)^{-1}\exp\{-(R_z-u_z\tau)^2/4\eta_1\tau-(\mathbf{R}_\perp-\mathbf{u}_\perp\tau)^2/4\eta_2\tau\},$$
$$\Theta=({}^8/{}_3\pi\varkappa_1\tau)^{-1/2}({}^8/{}_3\pi\varkappa_2\tau)^{-1}\exp\{-(R_z-u_z\tau)^2/{}^8/{}_3\varkappa_1\tau-(\mathbf{R}_\perp-\mathbf{u}_\perp\tau)^2/{}^8/{}_3\varkappa_2\tau\},$$
$$\Theta_2=\Theta_1\exp\{-\nu^{e,i}\tau\},\quad \Theta_3=\Theta_1|_{\eta\to{}^1/{}_6\eta}\exp\{-\nu^{e,i}\tau\},\quad \Theta_4=\Theta\exp\{-2\nu_2\tau\},$$
$$\nu^{e,i}=\begin{cases}\nu_2 \text{ — for ions,}\\ 2/\tau_e \text{ — for electrons}\end{cases}\qquad \mathbf{R}=(x-x',\ y-y',\ z-z'),\quad \tau=t-t'$$
$$\chi_1=1,\quad \chi_2=(m/T)^{1/2}v_\perp,\quad \chi_3=(m/T)^{1/2}v_z,\quad \chi_4=({}^2/{}_3)^{1/2}[-{}^3/{}_2+mv^2/2T].\tag{30}$$

As usual we have taken the z-axis along $\mathbf{B}_0$ and η_1, η_2, $\varkappa_1$, $\varkappa_2$ are the longitudinal and transverse kinematic viscosities and thermal conductivities (see §1). In (30) we have selected the four main terms of the Green function which describe the relaxation of the first moments of the initial perturbation. The remaining part of G describes higher moments, it relaxes more rapidly.

Substituting (30) into (24) we get the required kinetic equation for the plasma distribution functions. Under conditions (11) the distribution function must be close to Maxwellian. Indeed, to first approximation in the small parameters $l_{e,i}/L_c$ and $\tau_{e,i}/\tau_c$ the collision integral $S(f)$ plays the main role in Eq. (24). Hence, a solution of Eq. (24) in this approximation is the local Maxwellian distribution.

We obtain equations for the electron and ion densities N_e and N_i, their average macroscopice velocities $\mathbf{u}_e$ and $\mathbf{u}_i$, and their temperatures T_e and T_i, as usual, by selecting the first moments of Eq. (24). Using the quasi-neutrality of the plasma, $N_e=N_i=N$, and introducing the hydrodynamic velocity $\mathbf{u}$ and current $\mathbf{j}=eN(\mathbf{u}_i-\mathbf{u}_e)$, and also putting $T_e=T_i=T$, we obtain hydrodynamic equations which have the same form as (17), but the quantities $\mathbf{j}_D$, R_{zz}, $\mathbf{R}$ are different. We cite them for the case of not too high velocities when $u^2\ll v_T^2$. We have

$$j_D{}^\alpha=j_{\rm dif}^\alpha+j_{\langle eb\rangle}+q_T{}^\alpha/({}^3/{}_2NT)+j_{\rm fast}^\alpha,$$
$$Nm_iR_\alpha=\partial\Pi_{z\alpha}/\partial r_{\perp\alpha},\quad R_{zz}=2D,$$
$$\Pi_{z\alpha}=\pi_{z\alpha}+eE_{0\beta}Q_{\alpha\beta}^iNu_z+\frac{e}{B_0}(D_\alpha{}^i+Q_\alpha{}^i)Nu_z-2u_zj_D{}^\alpha m_i,$$
$$q^\alpha=q_T{}^\alpha+q_{\rm fast}^\alpha,\quad q_T{}^\alpha=-\frac{3}{2}N^2T\frac{D_{\alpha\beta}}{m_iB_0}\frac{\partial T}{\partial r_\beta},\tag{31}$$
$$j_{\rm dif}^\alpha=-\frac{D_{\alpha\beta}^i}{m_iB_0}\left(T\frac{\partial N}{\partial r_\beta}-eE_{0\beta}N\right),\quad D=\int_{-\infty}^{+\infty}\int_0^\infty\Theta_3{}^e\langle e_ze_z\rangle d^3r'\,d\tau,$$
$$\pi_{z\alpha}=-T\frac{\sigma_{\alpha\beta}^i}{B_0}\frac{\partial}{\partial r_\beta}(Nu_z),\quad j_{\langle eb\rangle}^\alpha=eND_\alpha{}^i/m_iB_0$$
$$D_{\alpha\beta}^{e,i}=\int_{-\infty}^{+\infty}\int_0^\infty\Theta_3^{e,i}\left\langle\frac{b_\alpha b_\beta}{B_0}\right\rangle d^3r'\,d\tau,\quad D_\alpha^{e,i}=\int_{-\infty}^{+\infty}\int_0^\infty\Theta_3^{e,i}\langle b_\alpha e_z\rangle d^3r'\,d\tau,$$
$$Q_{\alpha\beta}^{e,i}=\frac{1}{3}\int_{-\infty}^{+\infty}\int_0^\infty\Theta_4^{e,i}\left\langle\frac{b_\alpha b_\beta}{B_0^2}\right\rangle d^3r'\,d\tau,\quad Q_\alpha^{e,i}=\frac{1}{3}\int_{-\infty}^{+\infty}\int_0^\infty\Theta_4^{e,i}\langle b_\alpha e_z\rangle d^3r'\,d\tau,$$
$$\sigma_{\alpha\beta}^{e,i}=\int_{-\infty}^{+\infty}\int_0^\infty\left(\Theta_1^{e,i}+\frac{1}{3}\Theta_4^{e,i}\right)\left\langle\frac{b_\alpha b_\beta}{B_0}\right\rangle d^3r'\,d\tau.$$

Ohm's law for the longitudinal current takes the following form:

$$\frac{dj_z}{dt}+j_z\operatorname{div}\mathbf{u}=\frac{e^2N}{m_e}E_{0z}+\frac{e}{m_e}\nabla_zp_e+\frac{1}{B_0}\frac{\partial}{\partial r_\alpha}\times\left\{\frac{T_e}{m_e}\sigma_{\alpha\beta}{}^e\frac{\partial}{\partial r_\beta}j_z+\frac{e}{m_e}Q_\alpha{}^ej_z+\frac{2j_z}{N}\left[D_{\alpha\beta}^e\left(\frac{\partial}{\partial r_\beta}\frac{NT_e}{m_e}+\frac{eE_{0\beta}}{m_e}N\right)+\frac{e}{m_e}D_\alpha{}^eN\right]+\frac{e}{m_e}E_{0\beta}Q_{\alpha\beta}^eB_0j_z\right\}-D_\alpha{}^e\frac{e}{m_eB_0}\frac{\partial j_z}{\partial r_\alpha},\tag{32}$$

and the equation of quasi-neutrality can be written in the form

$$\partial j^{\alpha}_{D_i}/\partial r_\alpha=\partial j^{\alpha}_{D_e}/\partial r_\alpha. \qquad (33)$$

The current $j^{\alpha}_{\rm diff}$ describes the plasma diffusion, $j^{\alpha}_{\langle eb\rangle}$ is the fluctuation drift which exists when there is the correlator $\langle e_z b_\perp\rangle$; further, q^{α}_T is the heat flux, $\pi_{z\alpha}$ is the momentum flux across the magnetic field caused by the scattering of the longitudinal momentum $m_i N u_z$ by the magnetic field fluctuations.

The terms j_f and q_f describe the contributions of the fast particles to the diffusion and heat transfer. To determine them we must consider also the energetic collisionless component. The next subsection of the paper is devoted to it.

The transport coefficients evaluated in Eqs. (31) and (32) are determined solely by the effect of the electromagnetic field fluctuations. Electron and ion collisions lead to the appearance of additional fluxes which are described by well known classical expressions.[8] The total fluxes are given as the sum of the collisional and fluctuation fluxes. The longitudinal transfer coefficients are then determined mainly by the collisions. On the other hand, the transverse coefficients can, if the level of fluctuations is sufficiently high, when

$$\left\langle\frac{b_\perp^2}{B_0^2}\right\rangle\int_0^{\tau_c}\int_{|\mathbf{r}'|<L_c}\Theta_3^{e,i}(\mathbf{r}',\tau)\,d^3r'\,d\tau>\frac{1}{\omega_H^2\tau_{e,i}},$$

be larger than the classical ones.

B. Kinetic equation for the fast particles

For the fast particles we have the following relations

$$\tau_{e,i}(\mathrm{f})>\tau_c,\quad l_{e,i}(\mathrm{f})>L_c,\quad v_{\mathrm{f}}\gg v_T>u_{\mathrm{dr}},$$

where $\tau_{e,i}$(fast), $l_{e,i}$(fast) are the mean free flight time and the mean free path of the fast particles. Using also the fact that the scale of the main fields $L\gg L_c$, (2), and retaining in (25) the main terms we get the following equation for G:

$$\frac{\partial G}{\partial t}+v_z\frac{\partial G}{\partial z}=\delta(t-t')\delta(v_z-v_z')\delta(v_\perp-v_\perp')\delta(\mathbf{r}-\mathbf{r}'). \qquad (34)$$

Its solution is obvious. Substituting (34) into (24), using (2), we get

$$Df=S(f)+\frac{1}{B_0}\left(v_z\frac{\partial}{\partial r_\alpha}+\frac{eE_{0\alpha}}{m}\frac{\partial}{\partial v_\alpha}\right)\left\{\frac{1}{|v_z|}\frac{q_{\alpha\beta}}{B_0}\times\left(v_z\frac{\partial}{\partial r_\beta}+\frac{e}{m}E_{0\beta}\frac{\partial}{\partial v_z}\right)f+\frac{1}{|v_z|}\frac{e}{m}q_{\alpha z}\frac{\partial f}{\partial v_z}\right\}$$
$$+\frac{e}{m}\frac{\partial}{\partial v_z}\left\{\frac{q_{z\beta}}{B_0}\frac{1}{|v_z|}\left(v_z\frac{\partial}{\partial r_\beta}+\frac{e}{m}E_{0\beta}\frac{\partial}{\partial v_z}\right)f+q_{zz}\frac{1}{|v_z|}\frac{e}{m}\frac{\partial f}{\partial v_z}\right\}, \qquad (35)$$

$$q_{\alpha\beta}=|v_z|\int_0^\infty\langle b_\alpha b_\beta\rangle_{\tau,v_z\tau}\,d\tau,\quad q_{\alpha z}=|v_z|\int_0^\infty\langle b_\alpha e_z\rangle_{\tau,v_z\tau}\,d\tau,$$

$$q_{zz}=|v_z|\int_0^\infty\langle e_z e_z\rangle_{\tau,v_z\tau}\,d\tau,\quad (\alpha,\beta)=\{x,y\},$$

where the subscripts of the correlators of the fluctuating fields indicate their dependence on the time $t-t'=\tau$ and on the coordinate $z-z'=v_z\tau\lesssim L_{\parallel c}$, $L_{\parallel c}$ is the longitudinal correlation length. Equation (35) is the required equation for the fast particles and as $v_{\rm fast}\gg v_T\gtrsim L_c/\tau_c$ the coefficients q are independent of v_z and are determined solely by the amplitude of the electromagnetic field fluctuations.

The fast particles make additional contributions to the heat flux, the particle flux, and to the pressure, namely,

$$\mathbf{j}_{\mathrm f}=\int_{-\infty}^{+\infty}\mathbf{v}f\,d^3v,\quad \mathbf{q}_{\mathrm f}=\int_{-\infty}^{+\infty}\mathbf{v}\frac{mv^2}{2}f\,d^3v,\quad P_{\mathrm f}=\frac{m}{3}\int_{-\infty}^{+\infty}v^2 f\,d^3v. \qquad (36)$$

The distribution function of the main plasma determines the boundary conditions for Eq. (35), being expressed in terms of the flux of the run-away particles.[9] Hence, Eq. (35) forms with the currents (36) a closed set of hydrodynamic equations and at the same time is itself defined by the currents through the boundary condition.

We have thus obtained the complete set of equations describing both the main plasma and the fast component. Below, in §§ 4 and 5 we shall give some examples of a solution of problems described by these equations.

§4. DIFFUSION OF THE PLASMA. DAMPING OF MAGNETOHYDRODYNAMIC WAVES

As an example we consider the simplest problem of plasma diffusion. We put $\mathbf{B}_0=\mathrm{const}$, $\mathbf{u}_0=0$, $\mathbf{j}=0$, $T=\mathrm{const}$, and determine the diffusion of the plasma across the magnetic field caused by the fluctuations in the field $\mathbf{b}$. In a one-dimensional geometry we get from (17) and (31)

$$\frac{\partial N}{\partial t}=\frac{\partial}{\partial x}\left\{D_i\left(\frac{\partial}{\partial x}\frac{NT}{m_i}-\frac{e}{m_i}NE_{0x}\right)-D_x^i\frac{e}{m_i}N\right\}, \qquad (37)$$

where $D^{e,i}_{\alpha\beta}=\delta_{\alpha\beta}D_{e,i}B_0$ (we assume that the fluctuations are isotropic) and the $D^{e,i}_{\alpha\beta}$ are determined by Eqs. (31).

Using the quasi-neutrality Eq. (33) we get from (37) with the given accuracy

$$\frac{\partial N}{\partial t}=\frac{D_eD_i}{D_em_i+D_im_e}\frac{\partial}{\partial x}\left(2T\frac{\partial N}{\partial x}\right).$$

When the correlator $\langle b^2/B_0^2\rangle_{\mathbf{r}',t'}$ changes slowly as compared to $\Theta_3^{e,i}(\mathbf{r}',t')$ the ambipolar diffusion coefficient $D_{\mathrm{am}\perp}$ is

$$D_{\mathrm{am}\perp}=\langle b^2/B_0^2\rangle_{0,0}(v_{Ti}^2/\nu_{ie})\approx\langle b^2/B^2\rangle\eta_{i\parallel}.$$

Here ν_{ie} is the ion-electron collision frequency and $\eta_{i\parallel}$ the longitudinal kinematic viscosity. In the general case, however, $D_{\mathrm{am}\perp}$ depends strongly on the form of the function $\langle b^2/B_0^2\rangle_{\mathbf{r}',t'}$.

As a second example we consider the damping of Alfvén and magneto-sound waves. To do this we linearize the set of hydrodynamic Eqs. (17), (31)–(33) with respect to N_0, T_0, $\mathbf{B}_0$, assuming $\mathbf{j}_0=0$, $\mathbf{u}_0=0$, $\mathbf{E}_0=0$, $\mathbf{B}_0=\mathrm{const}$. Substituting the perturbation in the form $\propto e^{i(\omega t-\mathbf{kr})}$ into (17) we get

$$\omega N-N_0(\mathbf{ku})=ik_\perp^2\frac{D_eD_i}{D_em_i+D_im_e}(N_0T+T_0N),$$

$$N_0m_i\omega\mathbf{u}=\mathbf{k}p+\frac{1}{4\pi}[\mathbf{B}_0\times[\mathbf{k}\times\mathbf{B}]]-i\left(\frac{m_e}{m_i}\zeta_1+\zeta_2\right)\frac{\mathbf{B}_0}{B_0}\left(\mathbf{u}\frac{\mathbf{B}_0}{B_0}\right)N_0m_i$$

$$\omega\mathbf{B}=-[\mathbf{k}\times[\mathbf{u}\times\mathbf{B}_0]]+c\left[\mathbf{k}E_z\times\frac{\mathbf{B}_0}{B_0}\right],$$

$$\frac{3}{2}\omega T-T_0(\mathbf{ku})=ik_\perp^2 6\frac{D_eD_i}{D_em_i+D_im_e}\frac{T_0}{N_0}(NT_0+N_0T), \quad (38)$$

$$E_z=\frac{ik_zp}{eN_0}-\frac{m_e}{e^2N_0}\zeta_1 j_z,$$

where

$$\zeta_1=-\frac{T_0}{m_c}k_\perp^2\int_{-\infty}^{+\infty}\int_0^\infty\left(\Theta_{1e}+\frac{1}{3}\Theta_{4e}\right)\left\langle\frac{b^2}{B_0^2}\right\rangle_{r't'}d^3r'\,dt',$$

$$\zeta_2=-\frac{T_0}{m_i}k_\perp^2\int_{-\infty}^{+\infty}\int_0^\infty\left(\Theta_{1i}+\frac{1}{3}\Theta_{4i}\right)\left\langle\frac{b^2}{B_0^2}\right\rangle_{r't'}d^3r'\,dt'.$$

For the sake of simplicity we have in the set of Eqs. (38) written down only those terms which give imaginary corrections to the frequency.

From (38) we find the dispersion relation for the Alfvén waves

$$\omega=c_Ak_z+ic_Ak_\perp^2\zeta_1(m_ec/8\pi e^2N_0),$$

where $c_A=B_0/(4\pi Nm_i)^{1/2}$ is the Alfvén speed, and also for the fast and slow magnetosonic waves

$$\frac{\omega^2}{k^2}=\frac{1}{2}(c_A^2+c_s^2)\pm\left[\frac{1}{4}(c_A^2+c_s^2)^2-c_A^2c_s^2\frac{k_z^2}{k^2}\right]^{1/2}$$
$$+7iD_{\mathrm{am}\perp}\frac{k_\perp^2T_0}{\omega_0m_i}\left[1\pm\frac{{}^1/_2(c_A^2+c_s^2)-c_A^2k_z^2/k^2}{({}^1/_4(c_A^2+c_s^2)^2-c_A^2c_s^2k_z^2/k^2)^{1/2}}\right],$$

where $c_s^2=\frac{5}{3}T/m_i$ is the sound speed.

The behavior of the damping rates γ_A for the Alfvén wave and $\gamma_\pm$ for the fast and slow magentosonic waves for fixed $|\mathbf{k}|$ is shown in Fig. 1.

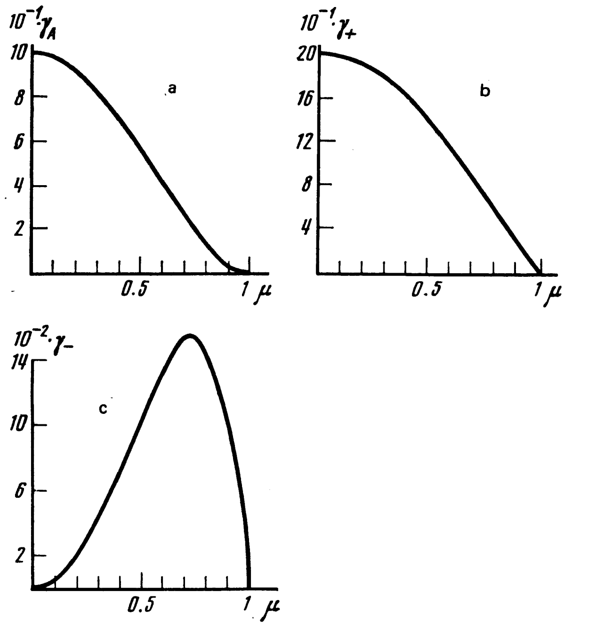

FIG. 1. Damping coefficients for the Alfvén waves, γ_A, fast magnetosonic waves, γ_+, and slow magnetosonic waves, γ_-, as functions of $\mu=k_z/|k|$ for fixed $|k|$. Because of the symmetry the graphs are given for $0<\mu<1$. The parameters were chosen as follows: a) $\zeta_1(m_eck/8\pi e^2N_0)=1$; b) and c) $c_A=c_s$; $(7D_{\mathrm{am}}/c_A^3)(T/m_i)k=1$.

§5. THE FAST PARTICLE DISTRIBUTION FUNCTION IN THE PRESENCE OF A TRANSVERSE TEMPERATURE GRADIENT

We now consider the fast particle distribution function described by the kinetic Eq. (35). We assume that the main magnetic field $\mathbf{B}_0$ in the plasma is constant and directed along the z-axis, while there is along the x-axis, at right angles to $\mathbf{B}_0$ a temperature gradient, $T(x)$. It is well known,[10] that when the temperature gradient is directed along $\mathbf{B}_0$ there occurs in the cold plasma regime a strong enrichment of the fast particle distribution. This effect is called thermal runaway. Such an effect does not occur in a direction at right angles to $\mathbf{B}_0$ if there are no fluctuations. Here we show that when there are magnetic field fluctuations present there occurs also a strong enrichment of the fast distribution function in a direction at right angles to $\mathbf{B}_0$. The reason for this effect is that at $\varepsilon>\varepsilon_k\approx T_e(\tau_e/\tau_c)^{2/3}$ the fast electrons are not contained by the main plasma and move independently of the thermal ones. They then diffuse the magnetic field appreciably faster than the slow ones.

In the case considered here we can in Eq. (35) neglect the quantity $(e/m_e)E_{0\perp}(\partial f/\partial v_z)$ compared with $v_z(\partial f/\partial r_\perp)$, since

$$\frac{e}{m}E_{0\perp}\frac{\partial f}{\partial v_z}\sim\frac{e\varphi}{T}\frac{T}{mv_f^2}\frac{v_z}{L}f,\qquad v_z\frac{\partial f}{\partial r_\perp}\sim\frac{v_z}{L}f,$$
$$\frac{e\varphi}{T}\sim 1,\quad \frac{mv_f^2}{T}\gg 1.$$

We assume here that $\mathbf{E}_0$ is the polarization field, i.e., $e\varphi/T\sim 1$, where φ is the polarization potential. Under the indicated conditions Eq. (35) takes the form

$$|v_z|\frac{1}{B_0^2}q_{xx}\frac{\partial^2 f}{\partial x^2}+S(f)=0, \quad (39)$$

where the quantity q_{xx} is given by (35). We introduce

$$D_{\mathrm{fl}}=\frac{\langle b_xb_x\rangle}{B_0^2}L_{\parallel c}v_T,\qquad u=|v|/v_T,\qquad \lambda=x(D_{\mathrm{fl}}/\nu(T_0))^{-1/2},$$
$$l_T=v_T/\nu(T_0),\qquad t=T/T_0,\quad \varkappa=(D_{\mathrm{fl}}/L_{\parallel c}^2\nu(T_0))^{1/2}, \quad (40)$$

where T_0 is the temperature of the thermal electrons, $\nu(T_0)$ the frequency of their collisions. Here $\varkappa\ll 1$ is the small parameter of the problem. We consider the fast electrons. We make (39) dimensionless, using the form of $S(f)$ given for that case in Ref. 10. We get

$$\frac{1}{u^2}\frac{\partial}{\partial u}\left\{\frac{1}{u}\left(t\frac{\partial f}{\partial u}+uf\right)\right\}+\frac{1}{u^3}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial f}{\partial\mu}\right\}$$
$$+u|\mu|\frac{\partial^2 f}{\partial\lambda^2}=0.$$

The distribution function depends here on three variables, $f=f(\lambda,u,\mu)$, while t is a given function of λ [see (40)], $\mu=v_z/v$.

We change to the variables $\tau=\varkappa\lambda$ and $g=\varkappa^{2/3}u^2$. Introducing $f=e^{-\varphi}$ we get the equation

$$2g\frac{\partial\varphi}{\partial g}\left(1-2\varkappa^{2/3}t\frac{\partial\varphi}{\partial g}\right)+4\varkappa^{2/3}tg\frac{\partial^2\varphi}{\partial g^2}-\left\{2\mu\frac{\partial\varphi}{\partial\mu}+(1-\mu^2)\left(\frac{\partial\varphi}{\partial\mu}\right)^2-(1-\mu^2)\frac{\partial^2\varphi}{\partial\mu^2}\right\}+g^2|\mu|\varkappa^{2/3}\left[-\left(\frac{\partial\varphi}{\partial\tau}\right)^2+\frac{\partial^2\varphi}{\partial\tau^2}\right]=0. \quad (41)$$

By analogy with Ref. 10 we look for the solution of Eq. (41) in the form

$$\varphi=\varkappa^{-2/3}\varphi_0+\varkappa^{-1/3}\varphi_1+\varphi_2+\dots .$$

In first approximation in the parameter $\varkappa^{-1/3}$ we have

$$\varphi_0=\varphi_0(g,\tau),\quad 2\frac{\partial\varphi_0}{\partial\tau}\left(1-2t\frac{\partial\varphi_0}{\partial g}\right)=g\left(\frac{\partial\varphi_0}{\partial\tau}\right)^2. \quad (42)$$

We now assume that the plasma temperature T changes from T_0 as $x\to-\infty$ to T_1 as $x\to+\infty$ (we shall assume that $T_0>T_1$). If we consider the solution at distances large compared to the scale on which T_0 changes to T_1, we can take this change to be a sudden one, i.e., assume that

$$t(\tau)=\begin{cases}1, & \tau<0\\ t_1, & \tau>0\end{cases} \quad (43)$$

We introduce the variable $y=2\partial\varphi_0/\partial g$. Equation (42) with $t(\tau)$ given by (43) admits the self-similar substitution $\zeta=\tau/g^{3/2}$ after which it takes the form

$$\pm\frac{dy}{d\zeta}=[y(1-ty)]^{1/2}+3\zeta\frac{d}{d\zeta}[y(1-ty)]^{1/2}. \quad (44)$$

We must add to Eq. (44) the boundary conditions following from (43)

$$y=1/t_1 \text{ at } \zeta=\infty,\quad y=1 \text{ at } \zeta=0. \quad (45)$$

The solution of Eq. (44) satisfying the conditions (45) is

$$\zeta=[y(1-t_1y)]^{-1/2}\{{}^1/_2(y^2-1)-{}^1/_3t_1(y^3-1)\}.$$

We show in Figs. 2, 3 the functions $y(g/\tau^{2/3})$ and $\ln f(g)$ in the cold plasma regime. It is clear (see Fig. 2) that the effective electron temperature $T_{\text{eff}}=T_0/y$ for low energies equals T_1 and for high energies is everywhere close to the temperature

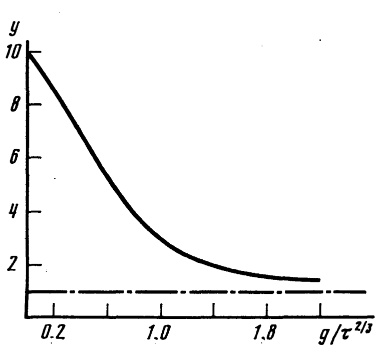

FIG. 2. The electron effective temperature y as function of $g/\tau^{2/3}$ ($t_1=0.1$).

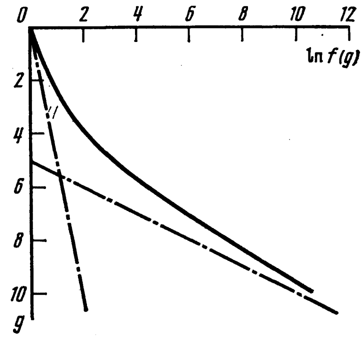

FIG. 3. The change of the distribution function from cold to hot ($t_1=0.1$; $\tau=30$). Along the horizontal axis we plot $-\ln f(g)$.

T_0 of the hot plasma. The graph of Fig. 3 shows how the distribution function changes from cold to hot. The asymptotics on Fig. 3 satisfy the following analytic expressions:

$$\varphi_0|_{g\to 0}=\frac{g}{2t_1},$$

$$\varphi_0|_{g\to\infty}=\frac{g}{2}+\frac{3}{2}\tau^{2/3}\left(\frac{1-t_1}{t_1}\right)^{1/3}\times\left\{\frac{1}{2}\left(\frac{1}{t_1}+1\right)-\frac{t_1}{3}\left(\frac{1}{t_1^2}+\frac{1}{t_1}+1\right)\right\}.$$

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APPENDIX

We consider the equation

$$\frac{\partial G}{\partial t}+v_z\frac{\partial G}{\partial z}=S(G)+\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')\delta(v_z-v_z')\delta(v_\perp-v_\perp').$$

We write G in the form

$$G=f_M(\mathbf{v})\chi(\mathbf{r},\mathbf{v},t),$$

where $f_M(\mathbf{v})$ is the Maxwell distribution function normalized to unity. We introduce

$$\chi_k=\int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}\chi\, d^3r;$$

and we get

$$\frac{\partial\chi_k}{\partial t}+ik_zv_z\chi_k=\hat{L}\chi_k, \quad (A.1)$$

where the operators $\hat{L}$ and S are connected through the relation:

$$f_M\hat{L}\chi_k=S(f_M\chi_k),$$

while the operator $S(f)$ is defined in (27). The term $ik_zv_z\propto l_{e,i}/L_c$ in Eq. (A.1) is by virtue of (11) a small correction so that we shall consider it to be a perturbation. As the form of the operator $\hat{L}$ is different for ions and for electrons we introduce $G_{e,i}=f_M^{e,i}\chi_{e,i}$. When determining χ_i and χ_e we shall assume that

$$(T_e-T_i)/T_{e,i}\ll 1,\quad (u_e-u_i)/v_T\ll 1. \quad (A.2)$$

We consider, to begin with, Eq. (A.1) for ions. In zeroth approximation in the parameters (A.2) and in l_i/L_c we have the following eigenfunctions and eigenvalues:

$$\lambda_{1i}=0,\quad \lambda_{2i}=-\nu_2,\quad \lambda_{3i}=-\nu_2,\quad \lambda_{4i}=-2\nu_2,$$
$$\chi_{1i}=1,\quad \chi_{2i}=(m_i/T)^{1/2}v_\perp,\quad \chi_{3i}=(m_i/T)^{1/2}v_z,$$
$$\chi_{4i}=(2/3)^{1/2}[-3/2+m_iv^2/2T].$$

We then can find corrections to the λ_i caused by the inhomogeneity. The first order correction caused by the inhomogeneity gives

$$\lambda_{i(1,2,3,4)}^{(1)}=-i\mathbf{ku}.$$

In second-order perturbation theory we find

$$\lambda_\alpha^{(2)}=\lim_{\varepsilon\to0}\sum_{\beta>4}^{\infty}\langle\alpha|ik_zv_z|\beta\rangle\frac{1}{\lambda_\alpha^{(1)}-\varepsilon}\langle\beta|ik_zv_z|\alpha\rangle,$$

where

$$\langle\beta|\hat{A}|\alpha\rangle=\int_{-\infty}^{+\infty}f_M(v)\chi_\beta(v)\hat{A}\chi_\alpha(v)\,d^3v.$$

The corresponding values $\lambda^{(2)}$ are connected with the classical transfer coefficients found in Refs. 8, 11, and are equal to

$$\lambda_1^{(2)}=\lambda_2^{(2)}=-\frac{\eta_\parallel}{Nm_i}k_z^2,\quad \lambda_3^{(2)}=-\frac{2}{3}\frac{\eta_\parallel}{Nm_i}k_z^2,$$
$$\lambda_4^{(2)}=-\frac{2}{3}\frac{\varkappa_\parallel}{Nm_i}k_z^2,$$

where $\eta_\parallel$ is the longitudinal viscosity, $\varkappa_\parallel$ the longitudinal heat conductivity. When we take in Eq. (A.1) into account the terms caused by drift we find corrections proportional to $k_\perp^2$; we find

$$\lambda_{1i}=-i\mathbf{ku}-\eta_1k_z^2-\eta_2k_\perp^2,\quad \lambda_{3i}=-i\mathbf{ku}-\nu_2-{}^2/_3(\eta_1k_z^2+\eta_2k_\perp^2),$$
$$\lambda_{2i}=-i\mathbf{ku}-\nu_2-\eta_1k_z^2-\eta_2k_\perp^2,$$
$$\lambda_{4i}=-i\mathbf{ku}-2\nu_2-{}^2/_3(\varkappa_1k_z^2+\varkappa_2k_\perp^2).$$

Here

$$\varkappa_1=\varkappa_\parallel/Nm_i,\quad \varkappa_2=\varkappa_\perp/Nm_i,\qquad \eta_1=\eta_\parallel/Nm_i,\quad \eta_2=\eta_\perp/Nm_i. \tag{A.3}$$

We now consider the equation for the electrons. We take the first four orthogonal functions as for the ions and expand the operator $\hat{L}_{ee}+\hat{L}_{ei}$ in terms of them. Such an approach is here analogous to the hydrodynamic approximation. We obtain the modes corresponding to the hydrodynamic relaxation:

$$\lambda_{1e}=0,\quad \lambda_{2,3e}=-2/\tau_e,\quad \lambda_{4e}=-2\nu_2,$$
$$\chi_{1e}=1,\quad \chi_{2,3e}=\left(\frac{m_e}{T}\right)^{1/2}v_\perp,v_z,$$
$$\chi_{4e}=\left(\frac{2}{3}\right)^{1/2}\left[-\frac{3}{2}+\frac{m_ev^2}{2T}\right].$$

The calculation of the corrections to the λ_e caused by the inhomogeneity is similar to the procedure of calculating them for the ions. We can thus write for the ions and the electrons the eigenvalues λ_i and λ_e in the form

$$\lambda_1^{e,i}=-i\mathbf{ku}_{e,i}-\eta_1^{e,i}k_z^2-\eta_2^{e,i}k_\perp^2,$$
$$\lambda_3^{e,i}=-i\mathbf{ku}_{e,i}-\nu^{e,i}-{}^2/_3(\eta_1^{e,i}k_z^2+\eta_2^{e,i}k_\perp^2),$$
$$\lambda_2^{e,i}=-i\mathbf{ku}_{e,i}-\nu^{e,i}-\eta_1^{e,i}k_z^2-\eta_2^{e,i}k_\perp^2,$$
$$\lambda_4^{e,i}=-i\mathbf{ku}_{e,i}-2\nu_2-{}^2/_3(\varkappa_1^{e,i}k_z^2+\varkappa_2^{e,i}k_\perp^2).$$

We can thus write the solution of Eq. (A.1) in the following form [see (29)]:

$$G(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}',t,t')=\sum_{\alpha=1}^{4}f_M(\mathbf{v})\chi_\alpha(\mathbf{v})\chi_\alpha(\mathbf{v}')\Theta_\alpha(\mathbf{r},\mathbf{r}',t,t')+\sum_{\beta=5}^{\infty}G_\beta, \tag{A.4}$$

where the functions Θ_α are given by Eqs. (30).

The first four terms in (A.4) correspond to the relaxation of the main moments: of the density, momentum, and energy. As we consider hereafter the solution of (A.1) for times $\tau_c\gg\tau_{e,i}$, (11), the largest effect on the fluctuations is due just to the first, hydrodynamic terms.

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