Phenomenological derivation of the conditions at the interface between a superfluid liquid and a solid

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The boundary conditions at the interface between a superfluid liquid and a solid are derived phenomenologically. The boundary conditions that are obtained extend, in particular, the Kapitza resistance to the case when viscous stresses are present in the liquid and the solid. They can be used to describe the melting of ^{4}He.

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The form of the boundary conditions at the interface between a solid and a liquid are of interest for many problems connected with crystal growth. Recently, this question has taken on added significance in connection with the discovery of melting waves on the boundary of solid and liquid ^{4}He at low temperatures.[1] However, there is no systematic and complete derivation of the conditions at the interface between a superfluid liquid and a solid. The present work is devoted to the phenomenological derivation of such conditions, starting from the conservation laws and the second law of thermodynamics. A similar approach was used by Landau and Khalatnikov in the derivation of the equations of the hydrodynamics of a superfluid liquid (see, for example, Ref. 2), and also by Andreev and Kompaneets[3] in the derivation of the conditions on the free surface of superfluid helium.

In this work we consider the liquid in the hydrodynamic approximation, and the solid in the approximation of elasticity theory. Processes of heat transfer in the solid are considered in the hydrodynamic approximation for a phonon gas, both in the presence and absence of umklapp processes.

The situation is somewhat more complicated in the case of umklapp processes on the interface. Near the surface of the solid, the periodic arrangement of the atoms in the crystal leads to a violation of translational symmetry in the liquid, so that the surface excitations should be classified according to the tangential quasimomentum. Here, just as in the case of the interior of the solid, the case of negligibly small number of umklapp processes can be achieved, i.e., the case in which the surfacer is smooth.

For the interface between liquid ^{4}He and an arbitrary solid, this is a strong assumption which at most can be realized only for special crystallogrphic faces. In the case of the interface between ^{4}He liquid and solid, if the melting process is dissipationless in the strict sense at low temperatures, as is assumed in Ref. 1, the surface excitations are delocalized and propagate freely. Such a quantum interface cannot itself lead to umklapp processes in the tangential direction, and the situation in which they are absent should be achieved at very low temperatures. The criteria for the existence of umklapp processes will naturally be different for the interface and for the interior of the solid.

In our phenomenological approach, we shall basically consider the case of the absence of umklapp processes. The transition to the case of intensive umklapp processes reduces to the exclusion of the quasimomentum from the number of independent variables on which the surface an volume energies depend and to a corresponding simplification of the formulas.

1. CONSERVATION LAWS ON THE INTERFACE

We choose the surface of the phase separation, as is usually done, such that surface mass is absent. This can always be done since the volume densities of the solid and the liquid are different. Then we can no longer represent the other additive quantities (energy, entropy and so on) in the form of volume integrals of the corresponding densities, but we must also take the specific surface part into consideration. Formally, this is equivalent to the addition, to the regular part of the corresponding density (for example, the energy density E) of a singular surface part:

$$E=E_V+E_s\delta_s. \tag{1.1}$$

Here δ_s is a generalized "simple layer" function, the vehicle for which is the surface of phase separation. If we assume that an arbitrary parametrization $\xi^\alpha(\alpha=1,2)$ is given at the interface, such that at an arbitrary instant of time t the surface represents the geometric set of points $\mathbf{r}'(\xi^\alpha,t)$, then the generalized function $\delta_s(\mathbf{r},t)$ can be represented in the form

$$\delta_s(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{r}'(\xi^\alpha,t))\,g^{1/2}\,d^2\xi^\alpha, \tag{1.2}$$

where g is the determinant of the metric tensor of the surface $g_{\alpha\beta}$. Here and below, the Greek symbols denote surface indices running over two values.

The flux densities of the considered additive quantities also contain similar singular parts. The energy flux Q, for example, has the form

$$\mathbf{Q}=\mathbf{Q}_V+\mathbf{Q}_s\delta_s. \tag{1.3}$$

We now consider the derivation of the differential form of the law of conservation of energy for the interface. For this purpose, we use its integral form for the arbitrary region V' with boundary Γ:

$$\frac{\partial}{\partial t}\int\limits_{V'}E\,dV+\int\limits_{\Gamma}\mathbf{Q}\,d\mathbf{S}=0. \tag{1.4}$$

Substituting formulas (1.1)–(1.3) in (1.4), transforming the

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second term by the Ostrogradskii-Gauss theorem, carrying out differentiation of the generalized functions, performing several integrations by parts, and using the identity[4]

$$\frac{\partial}{\partial \xi^\alpha}(g^{1/2}A^\alpha)=g^{1/2}\nabla_\alpha A^\alpha,$$

we obtain

$$\int_{V'}\left\{\frac{\partial E_V}{\partial t}+\operatorname{div}\mathbf{Q}_V\cdot\right.$$

$$+\left(\left(\frac{\partial E_s}{\partial t}\right)_{\xi^\alpha}-w^\alpha\frac{\partial E_s}{\partial \xi^\alpha}+\nabla_\alpha Q_s{}^\alpha+[Q_V{}^\nu-E_Vw^\nu]_s-E_sg^{\alpha\beta}K_{\alpha\beta}w^\nu\right)\cdot$$

$$\left.\times\delta_s+(E_sw^\nu-Q_s{}^\nu)\left(-\frac{\partial}{\partial\nu}\delta_s\right)\right\}dV=0. \qquad (1.5)$$

Here the index ν denotes the normal components of the three-dimensional vectors, while the index α denotes the components of their tangential parts in the curvilinear coordinates ξ^α; the square bracket with subscript s denotes the difference of the corresponding quantities in the liquid and in the solid, taken on the opposite sides of the interface; ∇_α represents coordinate differentiation;

$$\mathbf{w}=(\partial\mathbf{r}'/\partial t)_{\xi^\alpha}$$

is the velocity of motion of the surface;

$$K_{\alpha\beta}=\nu^k\partial^2r'^k/\partial\xi^\alpha\partial\xi^\beta$$

is the covariant tensor of curvature of the surface (it should not be confused with the Riemann curvature tensor or with the Ricci tensor):

$$-\frac{\partial}{\partial\nu}\delta_s=\int\left\{-\frac{\partial}{\partial\nu}\delta[\mathbf{r}-\mathbf{r}'(\xi^\alpha,t)]\right\}g^{1/2}\,d^2\xi^\alpha$$

is the generalized function of the "double layer," the vehicle for which is the interface.

Since integration in (1.5) is carried out over an arbitrary region V', the integrand should vanish identically. Setting its regular part and also the densities of the generalized functions entering into it ("simple layer" and "double layer") equal to zero, we obtain the traditional form of the law of energy conservation outside of the phase boundary:

$$\partial E_V/\partial t+\operatorname{div}\mathbf{Q}_V=0,$$

the explicit form of the part of the surface energy flux normal to the boundary:

$$Q_s{}^\nu=E_sw^\nu$$

and the desired differential form of the energy conservation law for the interface between the two phases:

$$\frac{\partial^*}{\partial t}E_s+\nabla_\alpha Q_s{}^\alpha+[Q_V{}^\nu-E_Vw^\nu]_s-E_sg^{\alpha\beta}K_{\alpha\beta}w^\nu=0. \qquad (1.6)$$

Here we have used the notation

$$\frac{\partial^*}{\partial t}=\left(\frac{\partial}{\partial t}\right)_{\xi^\alpha}-w^\alpha\frac{\partial}{\partial\xi^\alpha}$$

for the operation of differentiation with respect to the time, which is independent of the choice of the parametrization ξ^α. We note that the Eq. (1.6), introduced in the arbitrary curvilinear coordinates on the surface, has invariant form. If we choose the surface parametrization ξ_0^α such that $w^\alpha\equiv0$, then $\partial^*/\partial t=(\partial/\partial t)_{\xi\alpha}$; therefore, in what follows, we shall operate with such a parametrization and shall not make differences in the notations for the partial derivative (the final results will be valid in the case of arbitrary parametrization).

We can similarly derive for the interface the equation for mass conservation

$$\nabla_\alpha j_s{}^\alpha+[j_V{}^\nu-\rho_Vw^\nu]_s=0. \qquad (1.7)$$

Where it is taken into account that the surface mass is zero; the equation of entropy production

$$\frac{\partial}{\partial t}S_s+\nabla_\alpha F_s{}^\alpha+[F_V{}^\nu-S_Vw^\nu]_s-S_sg^{\alpha\beta}K_{\alpha\beta}w^\nu=\frac{1}{T_s}R_s \qquad (1.8)$$

with a positive-definite surface dissipation function R_s on the right side, the equation of momentum conservation and

$$\frac{\partial}{\partial t}j_s{}^k+\nabla_\alpha\Pi_s{}^{k\alpha}+[\Pi_V{}^{k\nu}-j_V{}^kw^\nu]_s-j_s{}^kg^{\alpha\beta}K_{\alpha\beta}w^\nu=0, \qquad (1.9)$$

also find the components of the corresponding fluxes normal to the boundary in explicit form:

$$j_s{}^\nu=0,\qquad F_s{}^\nu=S_sw^\nu,\qquad \Pi_s{}^{k\nu}=j_s{}^kw^\nu.$$

Here ρ is the density of the material, j^k is the mass flow (momentum density), S is the entropy density, F^k is the entropy flux, Π^{ik} is the momentum flux. In Eqs. (1.9), the tensor of the surface momentum flux is projected on the surface and is referred to curvilinear coordinates only relative to the second index.

In the following, we shall limit ourselves in the expression for the entropy to terms that are quadratic in the velocities. Accordingly, we can confine ourselves in the momentum conservation law to terms linear in the velocities. Taking it also into account that

$$j_s{}^k=j_s{}^\alpha\frac{\partial r^k}{\partial\xi^\alpha},\qquad \Pi_s{}^{l\beta}=\Pi_s{}^{\alpha\beta}\frac{\partial r^l}{\partial\xi^\alpha}+\nu^lj_s{}^\beta w^\nu,$$

and considering the projections on the normal and on the tangential plane, we obtain

$$\frac{\partial}{\partial t}j_s{}^\alpha+\nabla_\beta\Pi_s{}^{\alpha\beta}=[\Pi_V{}^{\alpha\nu}]_s, \qquad (1.10)$$

$$K_{\alpha\beta}\Pi_s{}^{\alpha\beta}=[\Pi_V{}^{\nu\nu}]_s. \qquad (1.11)$$

The expressions for the volume currents in a superfluid liquid[2] have the form

$$\begin{gathered}\mathbf{j}_L=\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s,\qquad \mathbf{F}_L=S_L\mathbf{v}_n+\mathbf{q}_L/T_L,\\ \Pi_L{}^{ik}=p_L\delta^{ik}+\tau_L{}^{ik}, \qquad (1.12)\\ Q_L{}^i=(\mu_L+h)j_L{}^i+S_LT_Lv_n{}^i+\tau_L{}^{ik}v_{nk}-\rho_Lv_n{}^ih+q_L{}^i,\end{gathered}$$

where q_L^i is the dissipative part of the heat flow, h is the dissipative term in the hydrodynamic equation for the superfluid velocity, τ_L^{ik} is the viscous stress tensor, μ is the chemical potential of the liquid.

The expressions for the volume currents in the solid must be written down in Eulerian coordinates with accuracy to quadratic terms in the energy. It is simplest to obtain them by using the Galilean transformation from the system in which the given element of the solid is at rest to the laboratory system:

$$\mathbf{j}_c=\rho_c\mathbf{v}_c,\qquad \mathbf{F}_c=S_c\mathbf{v}_{ph}+\mathbf{q}_c/T_c,\qquad \Pi_c{}^{ik}=-\sigma_c{}^{ik}+\tau_c{}^{ik},$$
$$Q_c{}^i=(E_c\delta^{ik}-\sigma_c{}^{ik}+\tau_c{}^{ik})\,v_{ck}+(T_cS_c\delta^{ik}+f_c{}^{ik})\,(v_{phk}-v_{ck})+q_c{}^i. \tag{1.13}$$

Here σ_c^{ik} is the elastic stress tensor, τ_c^{ik} is the viscous stress tensor in the solid (see Ref. 5). We have introduced here the drift velocity of the phonons $\mathbf{v}_{ph}$ which, in the case of absence of umklapp, can take on an arbitrary value (in the case of the presence of umklapp processes, $\mathbf{v}_{ph}=\mathbf{v}_c$ and the entire heat flow reduces to the purely dissipative term $\mathbf{q}_c$). Further, f_c^{ik} is the viscous stress tensor in the phonon gas of the solid (see Ref. 5).

2. THERMODYNAMIC RELATIONS ON THE INTERFACE

The most important physical problem is the establishment of the number of independent thermodynamic variables on which the surface energy E_s depends. These variables correspond to the independent thermodynamic variables which can be specified in equilibrium at the interface.

As such variables, we can choose the temperature of the surface T_s, the surface elastic stress tensor σ_s^{ij} (only the tangential components of this tensor can differ from zero[6]), the tangential component $\mathbf{v}_s^t$ of the velocity of superfluid motion of the liquid, the tangential component $\mathbf{v}_c^t$ of the velocity of the solid and the drift velocity of the surface excitations $\mathbf{V}$, which at equilibrium is identical to the tangential components $\mathbf{v}_n^t$ and $\mathbf{v}_{ph}^t$ of the velocity of the normal motion of the liquid and of the drift of the phonons of the solid. The separation, as an independent variable of the velocity $\mathbf{V}$, which is not identical with $\mathbf{v}_c^t$, is equivalent to the assumption of the absence of umklapp processes on the boundary.

We limit ourselves first to the case in which the energy of the surface does not depend on its orientation relative to the crystal axes. The liquid density is not included in the number of independent variables, since under equilibrium conditions it should be determined by the remaining variables. We shall adhere to the traditional point of view that at $T\neq 0$, the flows of mass and heat through the boundary are nondissipative. In this connection, at equilibrium, the normal components of all the velocities are identical with the velocity of movement of the boundary and are not independent variables.

It is convenient to write down initially the expression for the differential energy E_s' in a system of coordinates moving with velocity $\mathbf{v}_s^t$. In this system, the energy will depend not on the velocities but on their invariant differences. As arguments of E_s', we choose quantities that are thermodynamically conjugate to those chosen above, namely: the surface entropy S_s, the tangential part of the strain tensor u_{ij}^t, the Galilean invariant surface momentum $\mathbf{j}_s$ and the surface quasimomentum $\mathbf{P}_s$. Then

$$dE_s'=T_sdS_s+\Sigma^{ij}du_{ij}{}^t+(\mathbf{V}-\mathbf{v}_c{}^t)\,d\mathbf{P}_s+(\mathbf{V}-\mathbf{v}_s{}^t)\,d\mathbf{j}_s. \tag{2.1}$$

The tensor $\boldsymbol{\Sigma}$ can be written in the form

$$\Sigma^{ij}=-A\delta_t{}^{ij}+\sigma_s{}^{ij},\qquad A=E_s-S_sT_s, \tag{2.2}$$

thus isolating in explicit fashion the diagonal part connected with the surface tension.

The relation (2.1) can be regarded as the definition of the velocity $\mathbf{V}$ as the derivative of the energy with respect to the momentum, and the surface quasimomentum $\mathbf{P}_s$ as a quantity thermodynamically conjugate to the invariant difference $\mathbf{V}-\mathbf{v}_c^t$. The physical meaning of the quantities σ_s^{ij} and $\mathbf{P}_s$, introduced by the relation (2.1), will be seen from the results that follow.

With the help of the Galilean transformation $E_s=E_s'+(\mathbf{v}_s^t\mathbf{j}_s)$ we can obtain from (2.1) the form of the thermodynamic identify in the laboratory system of coordinates:

$$dE_s=T_sdS_s+\Sigma^{ij}du_{ij}{}^t+(\mathbf{V}-\mathbf{v}_c{}^t)\,d\mathbf{P}_s+\mathbf{V}d\mathbf{j}_s+\mathbf{j}_sd\mathbf{v}_s{}^t. \tag{2.3}$$

We note that the attempt to introduce as independent arguments of E_s some other quantities (for example, the normal components of the strain tensor) in the considered scheme inevitably leads to the result that at thermodynamic equilibrium, the quantities that are conjugate to them turn out to be expressed in terms of other thermodynamic variables: this means that it was not possible to consider them as independent arguments of E_s.

We must differentiate the relation (2.3) with respect to t at fixed values of the parameters on the surface. Here the question arises as to how the strains u_{ij}^t on different interfaces, corresponding to different Lagrangian coordinates in the solid, are related. We adopt the viewpoint that the moving interface "inherits" the strain tensor, in spite of the crystallization from the liquid. This means that the crystallizing particles adjust themselves to the elastic strains, so that the crystal grows in a deformed manner and the strain tensor remains continuous, together with its first derivatives. In the opposite process of melting, such an assertion is obvious. Thus the relation (2.3) must be differentiated exactly as is done in elasticity theory.

It is convenient to represent the derivative of E_s in the form

$$\frac{\partial E_s}{\partial t}=\left\{\left(\frac{\partial E_s}{\partial t}\right)_\xi+v_c{}^\alpha\frac{\partial E_s}{\partial \xi^\alpha}\right\}-v_c{}^\alpha\frac{\partial E_s}{\partial \xi^\alpha}\equiv\frac{d^s}{dt}E_s-v_c{}^\alpha\frac{\partial E_s}{\partial \xi^\alpha}. \tag{2.4}$$

The surface energy E_s depends on the tangential part of the strain tensor in the Lagrangian system of coordinates and it is natural to raise the index in this set of coordinates with the help of the metric tensor of the unstrained state $g^{ij}=\delta^{ij}$. This means that the strain energy can be regarded as a function of u_{ij}^t only, in accord with the usual formula of elasticity theory $dE=\sigma^{ij}du_{ij}/\rho$. The tangential components are determined with the help of the normal $\tilde{\nu}^k$ to the surface, also in Lagrangian coordinates and, consequently,

$$u_{ij}{}^t=\Lambda_i{}^k\Lambda_j{}^lu_{kl},\qquad \Lambda_i{}^k=(\delta_i{}^k-\tilde{\nu}_i\tilde{\nu}^k).$$

It is convenient to express the Lagrangian set of coordinates such that at a given instant of time t_0 (when differentiation is carried out), it is identical to the Eulerian Cartesian set of coordinates. The transition from one set of coordinates to the other is brought about by means of the matrix

$$B_l{}^m=\partial r^m/\partial r_0{}^l,\qquad B_l{}^m(t_0)=\delta_l{}^m,$$

where r^m is the Eulerian Cartesian coordinate and r_0^l are the Lagrangian coordinates.

The expression for the derivative of the energy contains only the tangential and symmetrical part of the derivative

$$\frac{d^s}{dt} u_{ij}{}^t = \left(\frac{\partial}{\partial t} + v_c{}^\alpha \frac{\partial}{\partial \xi^\alpha}\right) \Lambda_i{}^k \Lambda_j{}^l u_{kl}.$$

From the derivative d^s/dt we can separate the material derivative d/dt along the path of the particles of the solid, so that in our special curvilinear coordinates it has the form

$$\frac{d^s}{dt} = \frac{d}{dt} + (w^\nu - v_c{}^\nu) \frac{\partial}{\partial \nu}.$$

The quantity du_{jl}/dt is expressed, according to a well-known theorem of elasticity theory, in terms of the covariant derivatives of the velocity of a solid:

$$du_{kl}/dt = {}^1/_2 (\nabla_k v_{cl} + \nabla_l v_{ck}).$$

The tangential part of the covariant derivative $\nabla_k v_{cl}$, understood in the three-dimensional sense, after separation of the covariant surface derivative connected with the Christoffel symbols with the normal index (see, for example, Ref. 4), takes in the curvilinear coordinates on the surface the form $\nabla_\alpha v_{c\beta} - v_c^\nu K_{\alpha\beta}$, where $K_{\alpha\beta}$ is the surface curvature tensor introduced previously, and ∇_α denotes covariant differentiation in the metric of the surface. The differentiation of Λ_i^k can also be carried out by following only the tangential with respect to the lower index part of the obtained expression

$$\Lambda_i{}^k \frac{d^s}{dt} \Lambda_k{}^l = -\Lambda_i{}^k \tilde{\nu}^l \frac{d^s}{dt} \tilde{\nu}_k.$$

For differentiation of the covariant components of the vector $\mathbf{v}$ in Lagrangian coordinates, it is convenient to express these components in terms of the same components in Eulerian coordinates $\tilde{v}_p = B_p^q v_q$ and then differentiate the factors separately. After transition to the curvilinear surface coordinates, this yields

$$\left(\frac{d^s \tilde{\nu}}{dt}\right)_\alpha = \frac{\partial}{\partial \xi^\alpha} (v_c{}^\nu - w^\nu).$$

This latter calculation is especially simple with our special choice of surface coordinates ξ_0^α.

Finally, transforming to arbitrary curvilinear coordinates on the surface, we obtain

$$\Sigma^{ij} \frac{d^s}{dt} u_{ij}{}^t$$
$$= \Sigma^{\alpha\beta} \left[\nabla_\alpha v_{c\beta} - v_c{}^\nu K_{\alpha\beta} - (v_c{}^\nu - w^\nu) \frac{\partial}{\partial \nu} u_{\alpha\beta} - 2 u_{\nu\beta} \frac{\partial}{\partial \xi^\alpha} (v_c{}^\nu - w^\nu)\right]. \tag{2.5}$$

Substituting (2.3) in (2.4) and using (2.5), we obtain the expression for arbitrary surface energy:

$$\left(\frac{\partial E_s}{\partial t}\right)_{\xi_0{}^\alpha} = V_\alpha \left(\frac{\partial j_s{}^\alpha}{\partial t}\right)_{\xi_0{}^\alpha} + j_s{}^\alpha \left(\frac{\partial v_{s\alpha}}{\partial t}\right)_{\xi_0{}^\alpha} + (-A g^{\alpha\beta} + \sigma_s{}^{\alpha\beta})$$
$$\times \left[\nabla_\alpha v_{c\beta} - v_c{}^\nu K_{\alpha\beta} - (v_c{}^\nu - w^\nu) \frac{\partial}{\partial \nu} u_{\alpha\beta} - 2 \left(\frac{\partial}{\partial \xi^\alpha} (v_c{}^\nu - w^\nu)\right) u_{\nu\beta}\right]$$
$$+ T_s \left[\left(\frac{\partial S_s}{\partial t}\right)_{\xi_0{}^\alpha} + v_c{}^\alpha \frac{\partial S_s}{\partial \xi^\alpha}\right] + (V_\alpha - v_{c\alpha}) \left(\frac{\partial P_s{}^\alpha}{\partial t}\right)_{\xi_0{}^\alpha} - v_c{}^\alpha \frac{\partial E_s}{\partial \xi^\alpha}. \tag{2.6}$$

Since we are limiting ourselves to terms that are quadratic in the velocities, the index ξ_0^α is unimportant for the derivatives of j_s^α, P_s^α, and $v_{s\alpha}$.

3. FORM OF THE SURFACE CURRENTS AND OF THE DISSIPATION FUNCTION

Substituting in the equation of energy conservation (1.6) the equations of continuity (1.7), of momentum conservation (1.9), of entropy production (1.8), the expression (2.6) for $\partial E_s/\partial t$, the explicit expressions for the volume flows (1.12) and (1.13), using the usual equation for the superfluid velocity[2]

$$\frac{\partial}{\partial t} v_s{}^k + \frac{\partial}{\partial r^k} (\mu_L + h) = 0 \tag{3.1}$$

and transforming to the curvilinear surface coordinates in all terms, we can rewrite the energy conservation equation in the form

$$\nabla_\alpha Q_s{}^\alpha + R_s = \nabla_\alpha \{T_s F_s{}^\alpha + (\mu_L + h) j_s{}^\alpha$$
$$+ (A g^{\alpha\beta} - \sigma_s{}^{\alpha\beta}) (v_{c\beta} - 2 (v_c{}^\nu - w^\nu) u_{\nu\beta})$$
$$+ (\Pi_s{}^{\alpha\beta} + \sigma_s{}^{\alpha\beta}) V_\beta\} + (j_c{}^\nu - \rho_c w^\nu) \left[\left(\tilde{\mu}_c + \frac{\tau_c{}^{\nu\nu} - f_c{}^{\nu\nu}}{\rho_c}\right) - (\mu_L + h)\right.$$
$$\left. - \frac{1}{\rho_c} (A g^{\alpha\beta} - \sigma_s{}^{\alpha\beta}) \left(K_{\alpha\beta} + \frac{\partial}{\partial \nu} u_{\alpha\beta}\right) + \frac{2}{\rho_c} \nabla_\alpha [(A g^{\alpha\beta} - \sigma_s{}^{\alpha\beta}) u_{\beta\nu}]\right]$$
$$+ \left[T_s - \left(T_L + \frac{\tau_L{}^{\nu\nu} - \rho_L h}{S_L}\right)\right] [S_L (v_n{}^\nu - w^\nu)]$$
$$+ \left[\left(T_c + \frac{f_c{}^{\nu\nu}}{S_c}\right) - T_s\right] [S_c (v_{ph}{}^\nu - w^\nu)] + [V_\alpha - v_{n\alpha}] \tau_L{}^{\alpha\nu}$$
$$+ [v_{ph\alpha} - V_\alpha] f_c{}^{\alpha\nu} - \frac{\partial T_s}{\partial \xi^\alpha} (F_s{}^\alpha - S_s V^\alpha) - (V_\alpha - v_{c\alpha})$$
$$\times \left[\frac{\partial P_s{}^\alpha}{\partial t} + S_s \nabla^\alpha T_s - f_c{}^{\nu\alpha} + (-\sigma_s{}^{\alpha\nu} + \tau_c{}^{\alpha\nu} + \nabla_\beta \sigma_s{}^{\alpha\beta})\right]$$
$$- (\Pi_s{}^{\alpha\beta} + \sigma_s{}^{\alpha\beta}) (\nabla_\alpha V_\beta - w^\nu K_{\alpha\beta}), \tag{3.2}$$

where

$$\tilde{\mu}_c = (E_c - T_c S_c - \sigma_c{}^{\nu\nu}) / \rho_c.$$

In the right side of (3.2) we have left our terms connected with dissipative heat flows in the liquid and the solid, which are known to be small in comparison with the other terms.

Requiring that the dissipation-free part of the entropy flow be determined by the drift velocity of the excitations $\mathbf{V}$, and that the dissipative function not contain terms proportional to $(V - v_c^t)$, we can note that (3.2) is satisfied identically if P_s^α satisfies the equation

$$\frac{\partial P_s{}^\alpha}{\partial t} + S_s \nabla^\alpha T_s - f_c{}^{\alpha\nu} + (-\sigma_c{}^{\alpha\nu} + \tau_c{}^{\alpha\nu} + \nabla_\beta \sigma_s{}^{\alpha\beta}) = -\nabla_\beta f_s{}^{\alpha\beta}, \tag{3.3}$$

and the surface energy flux Q_s^α and the surface dissipative function R_s have respectively the forms

$$Q_s{}^\alpha = (E_s g^{\alpha\beta} - \sigma_s{}^{\alpha\beta}) v_{c\beta} - 2 (A g^{\alpha\beta} - \sigma_s{}^{\alpha\beta})$$
$$\times (v_c{}^\nu - w^\nu) u_{\beta\nu} + (\mu_L + h) j_s{}^\alpha$$
$$+ T_s \psi^\alpha + \tau_s{}^{\alpha\beta} V_\beta + f_s{}^{\alpha\beta} (V_\beta - v_{c\beta}) + T_s S_s (V^\alpha - v_c{}^\alpha), \tag{3.4}$$

$$R_s=-(j_c^\nu-\rho_c w^\nu)\left\{(\mu_L+h)-\mu_c(p_L)-\frac{\tau_L^{\nu\nu}-f_c^{\nu\nu}}{\rho_c}\right.$$
$$+\frac{1}{\rho_c}Ag^{\alpha\beta}\left(K_{\alpha\beta}+\frac{\partial u_{\alpha\beta}}{\partial\nu}\right)-\frac{\sigma_s^{\alpha\beta}}{\rho_c}\frac{\partial u_{\alpha\beta}}{\partial\nu}$$
$$\left.-\frac{2}{\rho_c}\nabla_\alpha[u_{\beta\nu}(Ag^{\alpha\beta}-\sigma_s^{\alpha\beta})]\right\}$$
$$-[S_L(v_n^\nu-w^\nu)]\left\{\left(T_L+\frac{\tau_L^{\nu\nu}-\rho_L h}{S_L}\right)-T_s\right\}$$
$$-[S_c(v_{ph}^\nu-w^\nu)]\left\{T_s-\left(T_c+\frac{f_c^{\nu\nu}}{S_c}\right)\right\}$$
$$-\tau_L^{\nu\alpha}\{v_{n\alpha}-V_\alpha\}-f_c^{\nu\alpha}\{V_\alpha-v_{ph\alpha}\}$$
$$-\psi^\alpha\left\{\frac{\partial T_s}{\partial\xi^\alpha}\right\}-\tau_s^{\alpha\beta}\{\nabla_\alpha V_\beta-w^\nu K_{\alpha\beta}\}-f_s^{\alpha\beta}\{\nabla_\alpha(V_\beta-v_{c\beta})\}. \tag{3.5}$$

Here

$$\mu_c(p_L)=(E_c-T_cS_c+p_L)/\rho_c,$$

and the dissipation-free and the dissipation terms are separated in the entropy and momentum flows:

$$F_s^\alpha=S_sV^\alpha+\psi^\alpha,\quad \Pi_s^{\alpha\beta}=-\sigma_s^{\alpha\beta}+\tau_s^{\alpha\beta}.$$

We note that the surface flows enter into the conservation laws under the integral sign. Therefore, retention in (3.4) and (3.5) of the dissipative surface terms $\psi^\alpha,\tau_s^{\alpha\beta},f_s^{\alpha\beta}$ is an exaggeration of the accuracy, since it requires, in the volume flows, account of terms of higher order in the small free path length of the excitations than in the usual hydrodynamic approximation. However, at low temperatures, as is noted in Ref. 3, when the excitations are frozen in the volume, these terms can be important.

Equation (3.3) does not have the form of a conservation law. This can be attributed to an inappropriate choice of the quantity P_s. If we choose the quantity $P_s^{\prime\alpha}=P_s^\alpha+j_s^\alpha$ and use the law of momentum conservation (1.10), we obtain the modified equation

$$\frac{\partial P_s^{\prime\alpha}}{\partial t}+S_s\nabla^\alpha T_s+\nabla_\beta f_s^{\prime\alpha\beta}=f_c^{\alpha\nu}-\tau_L^{\alpha\nu}, \tag{3.3'}$$
$$f_s^{\prime\alpha\beta}=f_s^{\alpha\beta}+\tau_s^{\alpha\beta}.$$

According to this equation, the quantity $P_s^{\prime\alpha}$ is conserved if we can neglect the dependence of the entropy on the strain (similar to the law of conservation of the quasimomentum in the volume of the crystal[5]) and, in addition, viscous stresses are lacking in the liquid at the interface. Correspondingly, this equation takes the form

$$\frac{\partial P_s^{\prime\alpha}}{\partial t}+\nabla_\beta G_s^{\alpha\beta}=f_c^{\alpha\nu}-\tau_L^{\alpha\nu}, \tag{3.6}$$

for the undeformed solid. Here the term

$$G_s^{\alpha\beta}=-Ag^{\alpha\beta}+f_s^{\prime\alpha\beta}$$

can be called the tensor of the flow of tangential quasimomentum on the surface, and the quantity $P_s^{\prime\alpha}$ the surface density of quasimomentum. Equation (3.6) shows that the separation boundary is a source of quasimomentum with density—$\tau_L^{\alpha\nu}$. On the interface of a solid with a vacuum, $\tau_L^{\alpha\nu}=0$, i.e., the quasimomentum is conserved.

The formulas obtained above correspond to the case of the absence of umklapp processes. In order to proceed to the case of intense umklapp processes, we must omit Eq. (3.3), since P_s^α is large and is not a thermodynamic variable, and set $f_s^{\alpha\beta}\equiv 0$ for the same reason. Moreover, at equilibrium, $\mathbf{V}=\mathbf{v}_c$ can be connected only with dissipative processes. As a result, for intense umklapp processes on the surface, we shall have, in place of (3.4) and (3.5),

$$Q_s^\alpha=(E_sg^{\alpha\beta}-\sigma_s^{\alpha\beta})v_{c\beta}-2(Ag^{\alpha\beta}-\sigma_s^{\alpha\beta})$$
$$(v_c^\nu-w^\nu)u_{\alpha\beta}+(\mu_L+h)j_s^\alpha+T_s\psi^\alpha+\tau_s^{\alpha\beta}v_c^\beta, \tag{3.4'}$$

$$R_s=-(j_c^\nu-\rho_c w^\nu)\left\{(\mu_L+h)-\mu_c(p_L)-\frac{\tau_L^{\nu\nu}-f_c^{\nu\nu}}{\rho_c}\right.$$
$$+\frac{1}{\rho_c}Ag^{\alpha\beta}\left(K_{\alpha\beta}+\frac{\partial u_{\alpha\beta}}{\partial\nu}\right)-\frac{1}{\rho_c}\sigma_s^{\alpha\beta}\frac{\partial u_{\alpha\beta}}{\partial\nu}$$
$$\left.-\frac{2}{\rho_c}\nabla_\alpha[u_{\nu\beta}(Ag^{\alpha\beta}-\sigma_s^{\alpha\beta})]\right\}-S_L(v_n^\nu-w^\nu)$$
$$\times\left\{\left(T_L+\frac{\tau_L^{\nu\nu}-\rho_L h}{S_L}\right)-T_s\right\}+S_c(v_{ph}^\nu-w^\nu)\left\{T_s-\left(T_c+\frac{f_c^{\nu\nu}}{S_c}\right)\right\}$$
$$-\tau_L^{\nu\alpha}\{v_{n\alpha}-V_\alpha\}-f_c^{\nu\alpha}\{v_{c\alpha}-v_{ph\alpha}\}-\psi^\alpha\left\{\frac{\partial T_s}{\partial\xi^\alpha}\right\}$$
$$-\tau_s^{\alpha\beta}\{\nabla_\alpha V_\beta-w^\nu K_{\alpha\beta}\}-(-\sigma_c^{\alpha\nu}+\tau_c^{\alpha\nu}+\nabla_\beta\sigma_s^{\alpha\beta})\{V_\alpha-v_{c\alpha}\}. \tag{3.5'}$$

The entropy flux in this case will be equal to $F_s^\alpha=S_sv_c^\alpha+\psi^\alpha$.

The transition to the case of intense umklapp processes in the volume of a solid is trivial under these conditions. It is necessary to set the stress in the phonon gas of the solid equal to zero, $f_c^{\nu\alpha}=0$, and regard the heat flow as a purely dissipative process, substituting in (3.5′)

$$S_c(v_{ph}^\nu-w^\nu)\to S_c(v_c^\nu+q_c^\nu/S_cT_c-w^\nu),$$

where q_c is the heat flow connected with the thermal conductivity of the solid. In what follows, we shall consider only the case of the absence of umklapp processes, keeping it in mind that the formulas for the opposite limiting case can be obtained by the method shown above.

We can obtain similar but more complicated expressions for Q_s^α and R_s if we take into account the dependence of E_s on the orientation of the surface relative to the crystal axes. We shall not write them out in the general case, but limit ourselves to the remark that if we neglect the strain terms (which can almost always be considered as small), then the entire difference reduces to the fact that there is an added term $-(\partial A/\partial\varphi_\alpha)(v_c^\nu-w^\nu)$, in the energy flux, and in the first curly bracket the expression for R_s is replaced by

Here

$$\frac{\partial}{\partial\varphi_\alpha}\equiv g^{\alpha\beta}\frac{\partial r^i}{\partial\xi^\beta}\frac{\partial}{\partial\nu^i},$$

and differentiation of A with respect to the normal vector ν^i that defines the orientation of the surface, is by implication carried out at constant positions of the crystallographic axes.

In lowest order theory of irreversible processes,[8] the positiveness of the dissipation function R_s requires the existence of a linear connect ion between the expressions in the curly brackets in formula (3.5) (which can be understood as the quantities $X = -\partial S/\partial x$) and the coefficients that stand in front of them (which play the role of the quantities $\dot{x}$). This leads to kinetic equations of the form

$$\dot{x}_i = -C_{ij}X_j, \tag{3.7}$$

where the C_{ij} satisfy the conditions of positiveness of the quadratic form $C_{ij}X_iS_j$ and are symmetric.

The total number of kinetic coefficients C_{ij}, whose introduction is required by the dissipation function of the form (3.5), is close to one hundred in the case of an anisotropic surface. In the case of an isotropic surface, their number is 24, while the boundary conditions (in which we omit terms connected with the derivatives of surface quantites) have the form

$$\begin{aligned} j_c^\nu - \rho_c w^\nu &= -C_{11}\Delta\mu - C_{12}\Delta T_{L-s} - C_{13}\Delta T_{s-c}, \\ S_L(v_n^\nu - w^\nu) &= -C_{12}\Delta\mu - C_{22}\Delta T_{L-s} - C_{23}\Delta T_{s-c}, \\ S_c(v_{ph}^\nu - w^\nu) &= -C_{13}\Delta\mu - C_{23}\Delta T_{L-s} - C_{33}\Delta T_{s-c}, \end{aligned} \tag{3.8}$$

where

$$\Delta\mu = (\mu_L + h) - \mu_c(p_L) - \frac{\tau_L^{\nu\nu} - f_c^{\nu\nu}}{\rho_c} + \frac{1}{\rho_c}\left[\left(Ag^{\alpha\beta} + \frac{\partial^2 A}{\partial\varphi_\alpha\,\partial\varphi_\beta}\right)K_{\alpha\beta}\right],$$

$$\Delta T_{L-s} = \left(T_L + \frac{\tau_L^{\nu\nu} - \rho_L h}{S_L}\right) - T_s, \qquad \Delta T_{s-c} = T_s - \left(T_c + \frac{f_c^{\nu\nu}}{S_c}\right)$$

for scale quantities and

$$\begin{aligned} \tau_L^{\alpha\nu} &= -C_{44}(v_n^\alpha - V^\alpha) - C_{45}(V^\alpha - v_{ph}^\alpha), \\ f_c^{\alpha\nu} &= -C_{45}(v_n^\alpha - V^\alpha) - C_{55}(V^\alpha - v_{ph}^\alpha) \end{aligned} \tag{3.9}$$

for vector quantities.

The conditions (3.7) [written down partially in (3.8) and (3.9)] together with the conservation laws (1.6), (1.7), and (1.10) and Eq. (3.3′), which describe the change in the surface quasimomentum, form a complete set of boundary conditions. If, as it is assumed in the theory of crysallization waves, $C_{11} = \infty$ at low temperatures, we obtain the condition $\Delta\mu = 0$, which generalizes the well-known condition of Herring.

The two remaining equations of (3.8) generalize the relations which describe the Kapitza jump. This generalization consists of introducing the surface temperature, which differs both from the temperature of the liquid and from the temperature of the solid. Moreover, the expression with temperature jumps contains terms that are connected with the components $\tau_L^{\nu\nu}$ of the viscous stress tensor and with the dissipative term h from Eq. (3.1), and also with the components $f_c^{\nu\nu}$ of the dissipative part of the flow of quasimomentum of the phonon gas of the solid. The appearance in (3.8) of the component of the viscous stress tensor is by its nature a thermomechanical effect.

If there is no flow of mass through the boundary and we can neglect effects connected with surface excitations (the classical face of the ^{4}He crystal or the interface between He II and an ordinary solid), then the expressions (3.9) reduce to the condition

$$Q^\nu = -C\left[\left(T_L + \frac{\tau_L^{\nu\nu} - \rho_L h}{S_L}\right) - \left(T_c + \frac{f_c^{\nu\nu}}{S_c}\right)\right],$$

which must replace the ordinary relation for the Kapitza jump $Q^\nu = -C(T_L - T_c)$, which is valid only in the absence of dissipative stresses.

CONCLUSION

The phenomenological approach to the derivation of the boundary conditions on the interface between a solid and a superfluid liquid allows us to establish their general form without leading to any conflict with the conservation laws and the second law of thermodynamics. In the equilibrium limit, when the velocity of motion of the boundary is vanishingly small and dissipation is absent, the Herring formula for the equilibrium of the melt and the crystal follows from the obtained formulas, as well as the usual conditions of equality of temperature and trangential velocities. In the presence of viscous stresses, both in the liquid and in the gas of phonons of the solid, the obtained formulas take into account both the dissipative additions to the Herring formula and the additional terms in the coupling of the temperature jump with the thermal flow through the interface (the Kapitza jump). An assumption as to the presence or absence of umklapp processes is not essential and the corresponding boundary conditions can be written down in both limiting cases.

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