Localization criterion in the field theory of an electron in a random field

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This paper is devoted to a study of the general localization criterion in the field theory of an electron in a random field. We show the equivalence of the Economou-Cohen and the Berezinskiĭ-Gor'kov localization criteria. The general localization criterion is formulated as the requirement of the existence of a pole contribution in the two-particle Green function with a factorizable residue (in momentum space). We search for a solution of this kind on the basis of a study of the homogeneous Bethe-Salpeter equation and in the framework of the instanton approach. We show that the Bethe-Salpeter equation determines the point where the "normal" (metallic) phase becomes unstable. The instanton approach describes the energy region corresponding to the localized phase. In both approaches the critical energy for which the transition occurs (mobility threshold) falls in the "Ginzburg critical region" which goes substantially beyond the framework of the approximations used. Both approaches follow naturally from an effective action formalism, but they reflect different mechanisms for the instability of the normal phase.

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1. INTRODUCTION

The obvious analogy which exists between the phenomenon of the localization of electrons in disordered systems (Anderson transition) and the usual phase transitions has led to many attempts to construct a field theory for an electron in a random field (see the review[1] and Refs. 2 to 5). The results of these papers are rather contradictory and the general picture of the transition is still not at all clear. In particular, this is true of the problem of the possibility of describing the localization on the basis of some kind of order-parameter representation.

The problem of how the localization manifests itself in the basic quantities with which the theory operates, such as the Green function, has also not been studied sufficiently. This makes the final solution of the problem much more difficult. It is, for example, clear that the problem of the realization of the localization effect itself is, in general, different from the problem of the behavior of the conductivity near the mobility threshold, the solution of which may turn out ot be much more complex. The present paper is devoted to an analysis of the general criterion for localization and to some attempts to look for the corresponding solutions from the basic equations of the theory of an electron in a random field.

2. EQUIVALENCE OF ECONOMOU-COHEN AND BEREZINSKIĬ-GOR'KOV LOCALIZATION CRITERIA

We consider noninteracting electrons moving in the field of impurities which are randomly distributed (in a d-dimensional space). Following Berezinskiĭ and Gor'kov[6] we define the spectral density:

$$\langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle = \frac{1}{N(E)}\Big\langle \sum_{\nu\nu'} \varphi_\nu^*(\mathbf{r})\varphi_{\nu'}(\mathbf{r})\varphi_{\nu'}^*(\mathbf{r}')\varphi_\nu(\mathbf{r}') \times\delta(E-\varepsilon_\nu)\delta(E+\omega-\varepsilon_{\nu'})\Big\rangle, \tag{1}$$

$$N(E)=\Big\langle \sum_\nu \varphi_\nu(\mathbf{r})\varphi_\nu^*(\mathbf{r})\delta(E-\varepsilon_\nu)\Big\rangle \tag{2}$$

is the electron density of states averaged over the configurations of the random potential: $\varphi_\nu(\mathbf{r})$ and ϵ_ν are the exact wavefunctions and energy levels of the electron in the field of the impurities, ν is a set of quantum numbers characterizing these states, E is the energy of the electron, and ω is an arbitrary frequency.

According to the localization criterion proposed in Ref. 6 there arises in the range of energies E corresponding to localized states a contribution which has a δ-shape:

$$\langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle = A_E(\mathbf{r}-\mathbf{r}')\delta(\omega)+\rho_1^E(\mathbf{r}-\mathbf{r}',\omega), \tag{3}$$

or, in the momentum representation,

$$\langle\!\langle \rho_E\rho_{E+\omega}\rangle\!\rangle_{\mathbf{q}} = A_E(\mathbf{q})\delta(\omega)+\rho_1^E(\mathbf{q}\omega). \tag{4}$$

The second term in (3) or (4) is regular in ω. In the region of delocalized states $A_E(\mathbf{r}-\mathbf{r}')=A_E(\mathbf{q})=0$.

As the quantities $A_E(\mathbf{q})$ or $A_E(\mathbf{r}-\mathbf{r}')$ signal the appearance of localized states it is useful to change to their definition in the standard formalism (Green functions). Introducing retarded and advanced averaged Green functions for the electron

$$G^{R,A}(\mathbf{r}\mathbf{r}'E)=\sum_\nu (\varphi_\nu(\mathbf{r})\varphi_\nu^*(\mathbf{r}')/(E-\varepsilon_\nu\pm i\delta)) \tag{5}$$

and using the definition (1) we get immediately

$$\langle\!\langle \rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle\!\rangle = \frac{1}{\pi^2 N(E)}\langle \operatorname{Im} G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega)\operatorname{Im} G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle$$

$$= \frac{1}{2\pi^2 N(E)}\operatorname{Re}\{\langle G^R(\mathbf{r}\mathbf{r}'E+\omega)G^A(\mathbf{r}'\mathbf{r}E)\rangle - \langle G^{R,A}(\mathbf{r}\mathbf{r}'E+\omega)G^{R,A}(\mathbf{r}'\mathbf{r}E)\rangle\}, \tag{6}$$

or, in momentum space,

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$$\langle\rho_E\rho_{E+\omega}\rangle_{\mathbf{q}}=\frac{1}{\pi N(E)}\operatorname{Im}\{\phi^{RA}(E\omega\mathbf{q})-\phi^{RR}(E\omega\mathbf{q})\},\qquad(7)$$

where, for simplicity, we have introduced the notation[7]

$$\phi^{RA}(E\omega\mathbf{q})=-\frac{1}{2\pi i}\sum_{\mathbf{pp}'}\langle G^R(\mathbf{p}_+\mathbf{p}_+'E+\omega)G^A(\mathbf{p}_-'\mathbf{p}_-E)\rangle,\qquad(8)$$

where $\mathbf{p}_\pm=\mathbf{p}\pm\mathbf{q}/2$. The quantities $\varnothing^{RR}(E\omega\mathbf{q})$ or $\varnothing^{AA}(E\omega\mathbf{q})$ are defined similarly. One sees easily[7,8] that as $\mathbf{q}\to0$, $\omega\to0$ the quantities $\varnothing^{RR}$ and $\varnothing^{AA}$ behave regularly. It is clear that the singular contribution to (4) corresponding to the appearance of localized states can arise only from the first term in (8). One sees easily that

$$A_E(\mathbf{q})=\lim_{\delta\to0}\frac{1}{N(E)}\delta\operatorname{Im}\phi^{RA}(E\omega+i\delta\mathbf{q})|_{\omega=0}$$
$$=\frac{1}{2\pi N(E)}\lim_{\delta\to0}\delta\sum_{\mathbf{pp}'}\operatorname{Re}\langle G^R(\mathbf{p}_+\mathbf{p}_+'E+i\delta)G^A(\mathbf{p}_-'\mathbf{p}_-E-i\delta)\rangle,\qquad(9)$$

or, in the coordinate representation,

$$A_E(\mathbf{r}-\mathbf{r}')=\frac{1}{2\pi N(E)}\lim_{\delta\to0}\delta\langle|G(\mathbf{r}\mathbf{r}'E+i\delta)|^2\rangle.\qquad(10)$$

It is useful to introduce the quantity

$$A_E=A_E(\mathbf{r}-\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'}=\int\frac{d^d\mathbf{q}}{(2\pi)^d}A_E(\mathbf{q})$$
$$=\frac{1}{2\pi N(E)}\lim_{\delta\to0}\delta\langle|G(\mathbf{r}\mathbf{r}'E+i\delta)|^2\rangle|_{\mathbf{r}=\mathbf{r}'},\qquad(11)$$

which is proportional to the averaged probability that an electron returns to the initial point in coordinate space after infinite time.[9] Hence it is clear that the general Berezinskiĭ-Gor'kov localization criterion[6] is equivalent to the generalized Economou-Cohen localization criterion.[9]

3. LOCALIZATION FROM THE BETHE-SALPETER EQUATION

We consider the two-particle Green function

$$\phi_{\mathbf{pp}'}(E\mathbf{q}\omega)=-\frac{1}{2\pi i}\langle G^R(\mathbf{p}_+\mathbf{p}_+'E+\omega)G^A(\mathbf{p}_-'\mathbf{p}_-E)\rangle.\qquad(12)$$

It is well known that in the framework of perturbation theory it is determined by the Bethe-Salpeter integral equation[7,8]

$$\phi_{\mathbf{pp}'}(E\mathbf{q}\omega)=G^R(E+\omega\mathbf{p}_+)G^A(E\mathbf{p}_-)\Big\{-\frac{1}{2\pi i}\delta(\mathbf{p}-\mathbf{p}')$$
$$+\sum_{\mathbf{p}''}U^E_{\mathbf{pp}''}(\mathbf{q}\omega)\phi_{\mathbf{p}''\mathbf{p}'}(E\mathbf{q}\omega)\Big\},\qquad(13)$$

where $G^{R,A}(E\mathbf{p})$ is the complete averaged retarded (advanced) single-electron Green function, while the irreducible vertex part $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$ is determined by the sum of all graphs which cannot be cut along two lines—an advanced and a retarded one (see Fig. 1, where the dashed line indicates the "interaction" ρV^2, where ρ is the density of the impurities and V the Fourier transform of the potential of a single impurity, which for the sake of simplicity we assume to be a point impurity).

We consider the problem of whether the solution of Eq. (13) can lead to a two-particle Green function containing singularities corresponding to localization. Starting from

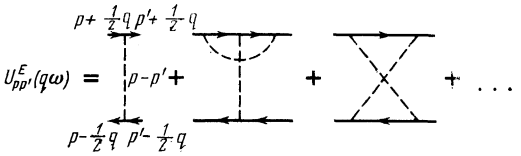

FIG. 1.

the results of the preceding section we assume that in the range of energies E where there exist localized states in the system, $\tilde{\varnothing}_{\mathbf{pp}'}(E\mathbf{q}\omega)$ has the form with a pole

$$\phi_{\mathbf{pp}'}(E\mathbf{q}\omega)=-\frac{\psi_{\mathbf{p}}{}^{\mathbf{q}}(E)\psi_{\mathbf{p}'}{}^{-\mathbf{q}}(E)}{\omega+i\delta}+\tilde{\phi}_{\mathbf{pp}'}(E\mathbf{q}\omega),\qquad(14)$$

where $\tilde{\varnothing}^{\mathbf{pp}'}(E\mathbf{q}\omega)$ is the regular part while the factorization of the residue at the pole (in momentum space) is assumed by analogy with the problem of bound states. We give a certain justification for this assumption in that follows.

From (8) and (13) we get at once

$$\phi^{RA}(E\mathbf{q}\omega)=-\frac{\chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E)}{\omega+i\delta}+\sum_{\mathbf{pp}'}\tilde{\phi}_{\mathbf{pp}'}(E\mathbf{q}\omega),\qquad(15)$$

$$\chi_{\mathbf{q}}(E)=\sum_{\mathbf{p}}\psi_{\mathbf{p}}{}^{\mathbf{q}}(E).\qquad(16)$$

It then follows from (9) that

$$A_E(\mathbf{q})=\frac{1}{N(E)}\chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E).\qquad(17)$$

One sees easily that $\chi_{\pm\mathbf{q}}(E)=\chi_{\pm\mathbf{q}}(E)$. From the general property that[6] $A_E(\mathbf{q}=0)=1$ there follows the normalization condition $\chi_0(E)=N^{1/2}(E)$. For the return probability A_E [Eq. (11)] we get

$$A_E=\frac{1}{N(E)}\sum_{\mathbf{q}}\chi_{\mathbf{q}}(E)\chi_{-\mathbf{q}}(E).\qquad(18)$$

The basic advantage of the localization criterion (14) formulated here is that when we substitude (14) into (13) the pole term dominates (as $\omega\to0$) and we get the homogeneous Bethe-Salpeter equation for $\psi^{\mathbf{q}}_{\mathbf{p}}(E)$:

$$\psi_{\mathbf{p}}{}^{\mathbf{q}}(E)=G^R(E\mathbf{p}_+)G^A(E\mathbf{p}_-)\sum_{\mathbf{p}'}U^E_{\mathbf{pp}'}(\mathbf{q}\omega=0)\psi_{\mathbf{p}'}{}^{\mathbf{q}}(E).\qquad(19)$$

It appears that a study of such an equation is appreciably simpler than the solution of the general Eq. (13). Localization would correspond to the appearance of a nontrivial solution $\psi^{\mathbf{q}}_{\mathbf{p}}(E)\neq0$ of Eq. (19) which would remain nonvanishing in the whole energy range $E\leqslant E_c$ where E_c is th mobility threshold. However, it may turn out (and we show in what follows that this is, apparently, the case) that Eq. (19) only gives the threshold E_c itself but does not describe the region $E<E_c$. We assume therefore that Eq. (19) gives a relatively simple method for finding the instability threshold of the "normal" (metallic) state.

It is obvious that an analysis of Eq. (19) in its general form is impossible. It is clear after the appearance of Refs. 10, 11 that at least in the "quasi-metallic" range of two-dimensional systems localization effects are connected with the contribution of the "maximally interesting" graphs for $U^E_{\mathbf{pp}'}(\mathbf{q}\omega)$ (Fig. 2):

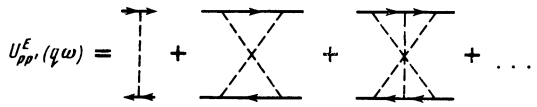

FIG. 2.

$$U_{pp'}^{E}(q\omega)=2\gamma(E)\rho V^2/(D_0^E(\mathbf{p}+\mathbf{p}')^2-i\omega), \tag{20}$$

where $D_0^E = E/md\gamma(E)$ is the classical diffusion coefficient, $\gamma(E)$ is the classical diffusion coefficient, $\gamma(E) = \pi\rho V^2 N(E)$.

In the metallic range Eq. (19) then takes the form

$$\left[E-\frac{1}{2m}\left(\mathbf{p}+\frac{1}{2}\mathbf{q}\right)^2+i\gamma(E)\right]\times\left[E-\frac{1}{2m}\left(\mathbf{p}-\frac{1}{2}\mathbf{q}\right)^2-i\gamma(E)\right]\psi_p{}^q(E)=\lambda(E)\int\frac{d^d\mathbf{p}'}{(2\pi)^d}\frac{\psi_{p'}{}^q(E)}{(\mathbf{p}+\mathbf{p}')^2}, \tag{21}$$

where $\lambda(E) = 2dm\gamma^2(E)\rho V^2/E$. After changing to dimensionless variables $\mathbf{p}\to\mathbf{p}/(2mE)^{1/2}$ we write Eq. (19) in the symmetrized from

$$\tilde{\psi}_{-p}{}^q(E)=\lambda_E\int d^d\mathbf{p}'K_q^E(\mathbf{p},\mathbf{p}')\tilde{\psi}_{p'}{}^q(E), \tag{22}$$

where

$$\tilde{\psi}_{-p}{}^q(E)=R_q^{-1/2}(\mathbf{p})\psi_{-p}{}^q(E),$$
$$R_q(\mathbf{p})=[1-(\mathbf{p}-{}^1/_2\mathbf{q})^2+i\gamma/E]^{-1}[1-(\mathbf{p}+{}^1/_2\mathbf{q})^2-i\gamma/E]^{-1},$$
$$\lambda_E=4(2\pi)^d m^2(2mE)^{d/2-3}\lambda(E), \tag{23}$$

while

$$K_q^E(\mathbf{p},\mathbf{p}')=R_q^{1/2}(\mathbf{p})R_q^{1/2}(-\mathbf{p}')\frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \tag{24}$$

is a positive-type[12] symmetric (Hermitean) kernel satisfying the inequality

$$K_q^E(\mathbf{p},\mathbf{p}')<E^2/\gamma^2|\mathbf{p}-\mathbf{p}'|^2. \tag{25}$$

Hence it is clear that for $2<d<4$ the equation considered is an integral equation with a kernel with a weak singularity[12] and certainly possesses a finite (or denumerable) eigenvalue spectrum lying on a section of the real axis with a length determined by the norm of the integral operator. From Enz's theorem[12] it follows that the first eigenvalue of this kernel is positive and simple while the corresponding eigenfunction is everywhere positive definite. Using the boundedness of the integral operator one checks easily that the equation considered does not have any trivial solutions when

$$\lambda_E<\left\{\frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{d-2}\frac{E^{-}}{\gamma^2}\right\}^{-1}, \tag{26}$$

i.e., when

$$E>\left(\frac{A_d}{d-2}\right)^{2/(4-d)}E_{sc}, \tag{27}$$

where $A_d = 2^{1-d/2}\pi^{-d/2}d/\Gamma(d/2)$ and where we have introduced the characteristic energy

$$E_{sc}=m^{d/(4-d)}(\rho V^2)^{2/(4-d)}. \tag{28}$$

Hence it is clear that for $d = 3$ the corresponding threshold energy falls in the "strong coupling" region $E_{sc} = m^3(\rho V^2)^2$, where the selection we made of diagrams is, generally speaking, invalid[1,13] and one needs to take all diagrams of the perturbation theory into account. As $d\to 2$ the range of energies for which there is no solution "takes off" to infinity which means that in that case the mobility threshold $E_c\to\infty$. In our opinion this result is a rather exact proof of the ideas of a total localization when $d = 2$ (Ref. 10). At the same time one sees easily that inequality (27) gives the analogue of the "Ginzburg critical region"[1,13] in which higher orders of the perturbation theory are important. Therefore, as $d\to 2$ simple peturbaion theory becomes inapplicable for all energies.

4. LOCALIZATION AND INSTANTONS

In view of the fact that when we describe the region of the localized states itself the approach given above, which is based upon the homogeneous Bethe-Salpeter equation, is, apparently, insufficient, we turn to an alternative approach which enables us to obtain a two-particle Green function of the form (14) in the whole energy range. It is well known[1,14,15] that the localization phenomenon is closely connected with the appearnace (in the appropriate energy range) of nonlinear solutions with a finite action (instantons) of the classical equations of an effective field theory which is associated with the problem of an electron in a random field.[1] We consider in detail the contribution of such solutions to the two-particle Green function.

To evaluate the two-particle electron Green function in a random field we can introduce[1] the following effective Lagrangian:

$$\mathscr{L}(\mathbf{r})=\frac{1}{2}\sum_{j=1}^{n}\left\{\frac{1}{2m}(\nabla\phi_j)^2-(E+\omega+i\delta)\phi_j^2\right\}+\frac{1}{2}\sum_{i=1}^{m}\left\{\frac{1}{2m}(\nabla\varphi_i)^2-(E-i\delta)\varphi_i^2\right\}-\frac{1}{8}\rho V^2\left\{\left(\sum_{j=1}^{n}\phi_j^2\right)^2+\left(\sum_{i=1}^{m}\varphi_i^2\right)^2+2\sum_{j=1}^{n}\sum_{i=1}^{m}\phi_j^2\varphi_i^2\right\}, \tag{29}$$

where at the end of the calculations one understands that one must take the limit $n\to 0$, $m\to 0$. Using the qualitative analysis of the classical field equations following from a Lagrangian[14,15] one can check that when $E<0$, $E+\omega>0$ these equations have a spherically symmetric instanton solution of the form

$$\varphi_i^{cl}(\mathbf{r})=\varphi_{cl}(r)e_i,\qquad \phi_j^{cl}(\mathbf{r})=0, \tag{30}$$

$$\varphi_{cl}(r)=\left(\frac{2|E|}{\rho V^2}\right)^{1/2}\chi_{cl}(t),\qquad r=(2m|E|)^{-1/2}t, \tag{31}$$

where $\chi_{cl}(t)\propto t^{(1-d)/2}\exp(-t)$ when $t\gg 1$, $\chi'_{cl}(0)=0$. In (30) e_i is the unit (m-component) isotopic vector of the field φ.

Considering in the corresponding functional integral contribution connected with the Gaussian fluctuations around classical solution (30) we get

$$\langle G^R(\mathbf{r}\mathbf{r}';E+\omega+i\delta)\,G^A(\mathbf{r}'\mathbf{r};E-i\delta)\rangle \sim \exp\{-S[\varphi_{cl}]\}$$
$$\times J_L^{d/2}[\varphi_{cl}]J_T^{(m-1)/2}[\varphi_{cl}]\int d^d\mathbf{R}_0\int d\mathrm{e}\varphi_{cl}(\mathbf{r}'-\mathbf{R}_0)$$
$$\times\varphi_{cl}(\mathbf{r}-\mathbf{R}_0)\int D\phi\int \widetilde{D}\varphi\phi_j(\mathbf{r})\,\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}, \qquad (32)$$

where $S[\varphi_{\mathrm{cl}}]\propto m^{-d/2}|E|^{2d/2}/\rho V^2$ is the classical action on the instaton,

$$J_L[\varphi_{cl}]=\int d^d\mathbf{r}(\nabla\varphi_{cl})^2\sim m^{2-d/2}|E|^{(4-d)/2}/\rho V^2,$$
$$J_T[\varphi_{cl}]=\int d^d\mathbf{r}\varphi_{cl}^2(r)\sim m^{-d/2}|E|^{(2-d)/2}/\rho V^2 \qquad (33)$$

is the Jacobian of the change to integration over the collective variables $\mathbf{R}_0$ (center of the instanton) and $\mathbf{e}$ (direction in isotopic space), $S_0[\varnothing,\varphi]$ is the action describing the Gaussian fluctuations in the vicinity of the instanton solution (φ denotes now the deviation from φ_{cl})

$$S_0[\phi,\varphi]=\int d^d\mathbf{r}\{\mathscr{L}_0(\phi)+\mathscr{L}_0(\varphi)\}, \qquad (34)$$

$$\mathscr{L}_0(\varphi)=\sum_{ij}\varphi_i(M_T+i\delta)\,(\delta_{ij}-e_ie_j)\,\varphi_j+\sum_{ij}\varphi_i(M_L+i\delta)\,e_ie_j\varphi_j, \qquad (35)$$

$$\mathscr{L}_0(\phi)=\sum_{ij}\phi_i(M_T-\omega-i\delta)\,\delta_{ij}\phi_j, \qquad (36)$$

where

$$M_L=-\frac{1}{2m}\nabla^2-E-\frac{3}{2}\rho V^2\varphi_{cl}^2,$$
$$M_T=-\frac{1}{2m}\nabla^2-E-\frac{1}{2}\rho V^2\varphi_{cl}^2. \qquad (37)$$

The tilde above the symbol for the functional integration over φ indicates that the zero eigenvalues of the operators M_L and M_T (the "zero modes") which are taken into account through the integration over the collective variables $\mathbf{R}_0$ and $\mathbf{e}$ must be excluded.

Introducing the eigenfunctions and eigenvalues

$$M_L\Psi_k^L=\lambda_k^L\Psi_k^L,\qquad M_T\Psi_k^T=\lambda_k^T\Psi_k^T, \qquad (38)$$

we get easily

$$\int D\phi\phi_j(\mathbf{r})\,\phi_j(\mathbf{r}')\exp\{-S_0[\phi,\varphi]\}$$
$$\sim\sum_k\frac{\Psi_k^T(\mathbf{r}-\mathbf{R}_0)\,\Psi_k^T(\mathbf{r}'-\mathbf{R}_0)}{(\lambda_k^T-\omega-i\delta)^{1+n/2}}\xrightarrow[n\to 0]{}\frac{\Psi_0^T(\mathbf{r}-\mathbf{R}_0)\,\Psi_0^T(\mathbf{r}'-\mathbf{R}_0)}{\omega+i\delta}$$
$$+\sum_{k\neq 0}\ldots, \qquad (39)$$

where the normalized eigenfunction of the lowest level of the operator $M_T(\lambda_0^T=0$, the "rotatonal" zero mode[14,15]) has the form

$$\Psi_0^T(\mathbf{r}-\mathbf{R}_0)=J_T^{-1/2}[\varphi_{cl}]\varphi_{cl}(\mathbf{r}-\mathbf{R}_0). \qquad (40)$$

As a result we get the singular contribution to the two-particle Green function:

$$\langle G^R(\mathbf{r}\mathbf{r}';E+\omega+i\delta)\,G^A(\mathbf{r}'\mathbf{r};E-i\delta)\rangle$$
$$\sim\frac{i}{\omega+i\delta}\exp\{-S[\varphi_{cl}]\}J_L^{d/2}[\varphi_{cl}]$$
$$\times J_T^{1/2}[\varphi_{cl}]\,(|\mathrm{Det}'\,M_L|)^{-1/2}(\mathrm{Det}'\,M_T)^{1/2}$$
$$\times\int d^d\mathbf{R}_0\varphi_{cl}^2(\mathbf{r}-\mathbf{R}_0)\,\varphi_{cl}^2(\mathbf{r}'-\mathbf{R}_0). \qquad (41)$$

Here Det' M_L and Det' M_T do not contain contributions from the zero eigenvalues of the operators M_L and M_T. Cardy[14] was the first to give an expression equivalent to (41) (for $\omega=0$). Taking into account the sketchy nature of that paper we decided to perform rather detailed calculations. We note that the singular contribution turns out to be connected with the existence of a "zero" rotational mode, i.e., in fact with the symmetry of the system. One may thus expect that this contribution does not vanish even when we take into corrections to the Gaussian approximation.

Taking now the explicit form of the density of states into account which in the energy range considered is determined by a similar instanton contribution[14,15] we get at once from (10), (11), and (41)

$$A_E(\mathbf{r}-\mathbf{r}')\sim\int d^d\mathbf{R}_0\varphi_{cl}^2(\mathbf{r}-\mathbf{R}_0)\,\varphi_{cl}^2(\mathbf{r}'-\mathbf{R}_0)\left[\int d^d\mathbf{r}\varphi_{cl}^2(\mathbf{r})\right]^{-1}, \qquad (42)$$

which is valid up to dimensionless constants. For the return probability we get from this: $A_E\propto|E|^{d/2}$.

Changing to the momentum representation by using

$$\chi_{\mathbf{q}}=\int d^d\mathbf{r}e^{-i\mathbf{q}\mathbf{r}}\varphi_{cl}^2(\mathbf{r}), \qquad (43)$$

we get

$$A_E(\mathbf{q})\sim\tilde{\chi}_{\mathbf{q}}\tilde{\chi}_{-\mathbf{q}}, \qquad (44)$$

which reproduces (17). Introducing the Fourier transform of the instanton

$$\varphi_{\mathbf{q}}^{cl}=\int d^d\mathbf{r}e^{-i\mathbf{q}\mathbf{r}}\varphi_{cl}(\mathbf{r}), \qquad (45)$$

we see that

$$\tilde{\chi}_{\mathbf{q}}=\int\frac{d^d\mathbf{p}}{(2\pi)^d}\varphi_{\mathbf{p}}^{cl}\varphi_{\mathbf{q}-\mathbf{p}}^{cl}, \qquad (46)$$

and comparing this with (16) we get

$$\psi_{\mathbf{p}}^{\mathbf{q}}(E)\sim\varphi_{\mathbf{p}}^{cl}(E)\,\varphi_{\mathbf{q}-\mathbf{p}}^{cl}(E). \qquad (47)$$

The consideration given her is thus in fact a validation, in the framework of the instanton approach, of the above in (14), assumed form of the singular contribution to the two-particle Green function corresponding to localization. The residue in the pole is then expressed in terms of instantons. The region of applicability of the instanton approach is roughly determined by the condition[1,14,15] $S[\varphi_{\mathrm{cl}}]\gg 1$ which leads to the requirement $|E|\gg E_{\mathrm{sc}}$ where E_{sc} is defined in (28) (the necessary refinements will be given in what follows).

5. EFFECTIVE-ACTION FORMALISM

There arises the problem of the relation between the two approaches discussed above for finding the singular part of

the two-particle Green function. We show below that both description methods naturally arise as a manifestation of, in general, different instabilities of the system in the framework of the effective action formalism for the component fields.[16] For the system considered of the fields $\varnothing$ and φ the effective action is[16] a functional Γ of the "classical" (average) values of the fields $\varnothing_{\rm cl}$ and $\varphi_{\rm cl}$ and of the corresponding Green functions which satisfies the variational principle:

$$\frac{\delta\Gamma}{\delta\phi_{cl}(\mathbf{r})}=0,\quad \frac{\delta\Gamma}{\delta\varphi_{cl}(\mathbf{r})}=0,\quad \frac{\delta\Gamma}{\delta G(\mathbf{r},\mathbf{r}')}=0. \tag{48}$$

It is convenient to use a matrix notation

$$\Phi=\begin{pmatrix}\phi\\ \varphi\end{pmatrix},\quad \Phi^{+}=(\phi\varphi). \tag{49}$$

$$\hat{G}=\begin{bmatrix}G_{\phi\phi} & G_{\phi\varphi}\\ G_{\varphi\phi} & G_{\varphi\varphi}\end{bmatrix},\quad G_{\varnothing\varphi}=G_{\varphi\phi}. \tag{50}$$

The Lagrangian (29) can be rewritten in compact form:

$$\mathcal{L}(\mathbf{r})=\tfrac{1}{2}\,\mathrm{Sp}\int d^{d}\mathbf{r}'\,\Phi^{+}\hat{G}_{0}^{-1}\Phi-\tfrac{1}{8}\rho V^{2}(\mathrm{Sp}\,\Phi^{+}\Phi)^{2}, \tag{51}$$

$$\hat{G}_{0}^{-1}(\mathbf{r}\mathbf{r}')=\begin{bmatrix}\left\{-\frac{1}{2m}\nabla^{2}-(E+\omega+i\delta)\right\}\delta_{ij} & 0\\ 0 & \left\{-\frac{1}{2m}\nabla^{2}-(E-i\delta)\right\}\delta_{ij}\end{bmatrix}\times\delta(\mathbf{r}-\mathbf{r}'). \tag{52}$$

According to Ref. 16 with an obvious generalization to the case of two fields we have

$$\Gamma(\Phi_{cl},\hat{G})=S(\Phi_{cl})-\tfrac{1}{2}\mathrm{Tr}\ln\hat{G}^{-1}-\tfrac{1}{2}\mathrm{Tr}\,\{\hat{G}^{-1}\hat{G}-1\}+\mathcal{F}(\Phi_{cl},\hat{G}), \tag{53}$$

where Tr and ln are understood in the functional sense,[16] i.e., in particular Tr includes all necessary integrations while $\ln\hat{G}=\ln\mathrm{Det}\,\hat{G}$, $\hat{G}^{-1}$ is the reciprocal of the Green function matrix in the classical field:

$$\hat{G}^{-1}(\mathbf{r},\mathbf{r}')=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\delta(\mathbf{r}-\mathbf{r}'), \tag{54}$$

where

$$a=\left\{-\frac{1}{2m}\nabla^{2}-(E+\omega+i\delta)-\frac{1}{2}\rho V^{2}(\phi_{cl}^{2}+\varphi_{cl}^{2})\right\}\times\delta_{ij}-\rho V^{2}\phi_{cl_i}\phi_{cl_j},$$

$$b=-\rho V^{2}\phi_{cl_j}\varphi_{cl_i},\quad c=-\rho V^{2}\varphi_{cl_i}\phi_{cl_j},$$

$$d=\left\{-\frac{1}{2m}\nabla^{2}-(E-i\delta)-\frac{1}{2}\rho V^{2}(\phi_{cl}^{2}+\varphi_{cl}^{2})\right\}\times\delta_{ij}-\rho V^{2}\varphi_{cl_i}\varphi_{cl_j};$$

$$\phi_{cl}^{2}=\sum_{j=1}^{n}\phi_{cl_j}^{2},\quad \varphi_{cl}^{2}=\sum_{i=1}^{m}\varphi_{cl_i}^{2}.$$

The functional $\mathcal{F}(\Phi_{\rm cl},\hat{G})$ satisfies the conditions

$$\delta\mathcal{F}/\delta\hat{G}=\tfrac{1}{2}\Sigma \tag{55}$$

such that the equation

$$\delta\Gamma/\delta\hat{G}=\tfrac{1}{2}\hat{G}^{-1}-\tfrac{1}{2}\hat{G}^{-1}+\tfrac{1}{2}\hat{\Sigma}=0 \tag{56}$$

is simply the Dyson equation while the matrix Σ consists of the irreducible self-energy parts with dressed internal lines. One can get the formal scheme for calculation $\mathcal{F}(\Phi_{\rm cl},\hat{G})$ easily by an appropriate generalization of the prescriptions of Ref. 16.

We first consider the "normal" phase in which $\varnothing_{\rm cl}=\varphi_{\rm cl}=0$ and only the Green functions $G_{\varnothing\varnothing}$ and $G_{\varphi\varphi}$ are nonvanishing. In that case (53) simplifies

$$\Gamma(\hat{G})=\mathcal{F}(\hat{G})-\tfrac{1}{2}\mathrm{Tr}\ln\hat{G}^{-1}-\tfrac{1}{2}\mathrm{Tr}\,\{\hat{G}_{0}^{-1}\hat{G}-1\}. \tag{57}$$

The matrix (54) reduces to (52). A stable system must satisfy the condition $\delta^{2}\Gamma>0$ for any variations in $\Phi_{\rm cl}$ and $\hat{G}$. We consider the stability against arbitrary variations of the Green functions in the "normal" phase. We show graphically in Fig. 3 examples of variations of the self-energy parts when the Green functions are varied. Hence one finds, in particular, easily that

$$\frac{\delta^{2}\Gamma}{\delta G_{\phi\varphi}\delta G_{\phi\varphi}}=\frac{1}{2}\frac{\delta G_{\phi\varphi}^{-1}}{\delta G_{\phi\varphi}}+\frac{1}{2}\frac{\delta\Sigma_{\phi\varphi}}{\delta G_{\phi\varphi}}=-\frac{1}{2}G_{\phi\phi}^{-1}G_{\varphi\varphi}^{-1}+\frac{1}{2}U_{\phi\varphi\phi\varphi}, \tag{58}$$

etc., where $U_{\varnothing\varphi\varnothing\varphi}$ is the irreducible vertex part in appropriate two-particle channel. The problem of the instability of the system with respect to variations $\delta G_{\varnothing\varphi}$ is of interest to us. In a stable system

$$\mathrm{Tr}\,\delta G_{\phi\varphi}\frac{\delta^{2}\Gamma}{\delta G_{\phi\varphi}\delta G_{\varphi\phi}}\delta G_{\phi\varphi}\geqslant 0. \tag{59}$$

Using $\delta G_{\varnothing\varphi}=G_{\varnothing\varnothing}\psi_{\varnothing\varphi}G_{\varphi\varphi}$ (see Fig. 3) in (59) and (58) we see that the stability threshold of the "normal" phase is given by

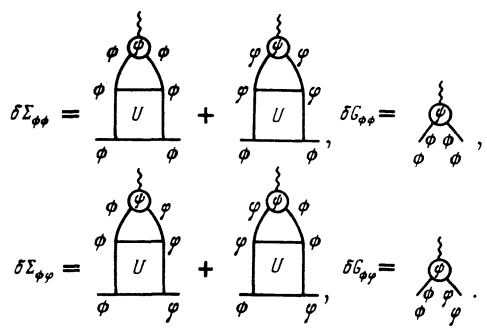

FIG. 3.

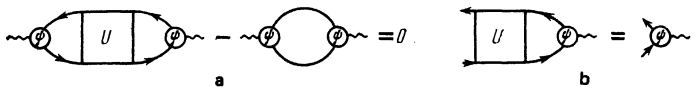

FIG. 4.

the condition

$$\mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}G_{\phi\phi}U_{\phi\varphi\varphi\phi}G_{\phi\phi}\psi_{\phi\varphi}G_{\varphi\varphi}-\mathrm{Tr}\, G_{\varphi\varphi}\psi_{\varphi\phi}\psi_{\phi\varphi}G_{\phi\phi}=0, \tag{60}$$

which is graphically represented in Fig. 4a. It is fairly obvious that when there appears a nontrivial solution of the homogeneous Bethe-Salpeter Eq. (19) the stability of the system is violated (Fig. 4b).

The analysis given here shows that the appearance of a nontrivial solution of Eq. (19) gives in the general case the threshold for the stability of the "normal" phase where we are talking about stability with respect to variations $\delta G_{\varnothing\varphi}$. An expansion of the functional $\Gamma(G)$ from (57) in powers of $\delta G_{\varnothing\varphi}\propto\psi_{\varnothing\varphi}$ gives in principle a method to consider the corresponding "condensed" phase while in that case $\psi_{\varnothing\varphi}$ plays the role of the order parameter.

The first two Eqs. (48) are in fact a generalization of the classical field equations following from the Lagrangian (29), (51). The case when they acquire nontrivial solutions of the kind (30) is important for us. The matrix (54) then reduces to

$$\bar{G}^{-1}(\mathbf{r}\mathbf{r}')=\begin{bmatrix}(M_T-\omega-i\delta)\,\delta_{ij} & 0\\ 0 & (M_L+i\delta)\,e_ie_j+(M_T+i\delta)\,(\delta_{ij}-e_ie_j)\end{bmatrix}\delta(\mathbf{r}-\mathbf{r}'), \tag{61}$$

and the simplest approximation for $\Gamma(\Phi_{\mathrm{cl}},\hat{G})$ reduces to neglecting in (53) the contribution $\mathcal{F}(\Phi_{\mathrm{cl}},\hat{G})$. In that case (53) gives

$$\begin{aligned}\Gamma(\varphi_{cl})&=S(\varphi_{cl})-{}^1/_2\,\mathrm{Tr}\ln\tilde{G}_{\phi\phi}{}^{-1}-{}^1/_2\,\mathrm{Tr}\ln_5\bar{G}_{\varphi\varphi}{}^{-1}\\&=S(\varphi_{cl})+\Gamma_1(\varphi_{cl}),\end{aligned} \tag{62}$$

and the equation $\delta\Gamma/\delta\varphi_{\mathrm{cl}}=0$ reduces to

$$-\frac{1}{2m}\nabla^2\varphi_{cl}-E\varphi_{cl}-\frac{1}{2}\rho V^2\varphi_{cl}{}^3+\frac{\delta\Gamma_1(\varphi_{cl})}{\delta\varphi_{cl}}=0, \tag{63}$$

which is the generalized equation for instantons leading to the solution (30). Here $\Gamma_1(\varphi_{\mathrm{cl}})$ is the result of summing the single-loop corrections to the classical action. Considering in it the term of first order in $\rho V^2\varphi_{\mathrm{cl}}^2$ we get

$$\begin{aligned}\Gamma_1^{(1)}(\varphi_{cl})&=-\frac{1}{2}\rho V^2\int d^d\mathbf{r}\varphi_{cl}{}^2(\mathbf{r})\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{\mathbf{p}^2/2m-E}\\&=-\frac{1}{2}\delta E\int d^d\mathbf{r}\varphi_{cl}{}^2(\mathbf{r}),\end{aligned} \tag{64}$$

where δE gives the single-loop "mass" renormalization in the original Lagrangian. Taking for E the already renormalized "mass" we shall assume that the "critical point" corresponds to $E\to 0$ so that in terms of the "bare mass"

$$E_0=E-\delta E\underset{E\to 0}{\longrightarrow}E_{0c}=-\rho V^2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{\mathbf{p}^2/2m}=-\rho V^2 2mS_d\frac{p_0^{d-2}}{d-2}, \tag{65}$$

which determines the (in the single-loop approximation) shifted band edge. Here p_0 is the cut-off momentum, $S_d=2^{-(d-1)}\pi^{-d/2}/\Gamma(d/2)$. Our definition of the shifted band edge differs from the one assumed in Ref. 17. For E we get the equation

$$\begin{aligned}E&=E_0+\delta E=E_0-E_{0c}-\rho V^2\\&\quad\times\left[\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{\mathbf{p}^2/2m-E}-\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{\mathbf{p}^2/2m}\right]\\&=E-E_{0c}+\rho V^2\pi mS_d(-2mE)^{d/2-1}\left\{\sin\pi\left(\frac{d}{2}-1\right)\right\}^{-1},\\&\quad 2<d<4.\end{aligned} \tag{66}$$

The "Ginzburg criterion" follows[18] from the requirement that the simplest formula $E\approx E_0-E_{\mathrm{oc}}$ be valid which means the equation for the renormalized electron "mass" energy reckoned from the shifted band edge. This is just the meaning of the variable E in that paper and in Refs. 1, 14, 15. It is clear that the equation is satisfied when

$$|E|\gg\left(\frac{B_d}{|\sin(\pi d/2)|}\right)^{2/(4-d)}E_{sc},\quad 2<d<4, \tag{67}$$

where $B_d=2^{-d/2}\pi^{1-d/2}/\Gamma(d/2)$ while E_{sc} is defined in (28). This inequality which determines the condition for the applicability of our approximation is equivalent, in particular, to the inequality (27) obtained earlier. In the negative energy range it delimits the region beyond which the instanton approach is valid.

From the effective action formalism there follows thus in a natural manner both the instability of the "normal" (metallic) phase which is connected with the appearance of a nontrivial solution of the homogeneous Bethe-Salpeter Eq. (19) and the instability of that phase connected with the appearance of instanton solutions. In the framework of the approximations used these two instabilities remain independent which may, in principle, indicate the existence of two kinds of electron localization. At the same time it is clear that the complete solution of the problem of the relation between the two instabilities requires one to go beyond the framework of the approximations used and to penetrate really the "strong coupling" region. The effective action formalism gives, at least in principle, a convenient apparatus for a joint consideration of these instabilities.

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