Theory of nonlinear saturation of stimulated Mandel'shtam–Brillouin scattering in plasma (SMBS)

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A theory of stimulated Mandel'shtam-Brillouin scattering is formulated with account of the nonlinearity of the excited sound wave. It is shown that in a rarefield plasma with density far below the critical value, the generation of the higher harmonics of the sound leads to a significant suppression of the scattering. In the limit of strong acoustic nonlinearity, when there are many harmonics of comparable amplitude in the sound spectrum, the dependence of the reflectance on the incident radiation intensity is obtained.

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1. The theory of stimulated Mandel'shtam-Brillouin scattering (SMBS) in a plasma has attracted great attention in recent years, because of practical considerations connected with possible limitations imposed by such a process on the heating of the plasma by laser radiation.[1,2] This attention has also been due to purely theoretical considerations, connected with the insignificant achievements in the search of physical processes, which has been carried out by different authors, in the hope of suppressing the SMBS (see the review[3]). Even back in Ref. 4 devoted to a discussion of the premises of Refs. 1 and 2, a hope had been raised of suppression of SMBS by the nonlinearity of ion-sound waves. However, in the years that have passed since the publication of Ref. 4, such hopes have not been realized, in spite of the fact that up to the present time a number of works that take into account the nonlinearity of the sound wave have been published on the theory of SMBS. It is necessary to note here that even before the discussions[1,2,4] of plasma heating, ideas on the importance of nonlinear hydrodynamic effects, due to the action of high-intensity radiation, were advanced in works[5] that determined in significant measure the development of the nonlinear theory of SMBS after attention had been drawn to it in Ref. 6.

A study of SMBS for a plasma layer of finite thickness was undertaken in Ref. 7, in which weak acoustic nonlinearity was described in the approximation of three harmonics. In that case the authors of Ref. 7 limited their consideration to conditions of comparatively strong sound dissipation, which did not permit them to bring to light the qualitative effect of suppression of the SMBS (see below); furthermore, the conditions of strong sound dissipation require the account of strong acoustic nonlinearity, which makes the approximation of Ref. 7 inadequate.

A modification of the approach of Ref. 7, taking into account only two harmonics of the sound wave, was proposed in Ref. 8 for an unbounded plasma. It should be noted here that, first, as is shown below in our paper in accord with Ref. 8, the restriction to only two harmonics of the sound is completely adequate for the construction of a weakly nonlinear SMBS theory, and second only by forgoing the formulation of the problem[7] on SMBS from a plasma layer and consideration of the unbounded plasma the author of Ref. 8 failed to satisfy the criteria for SMBS suppression. The development of the premises of Ref. 8 to which Ref. 9 was devoted, did not change the notion of very intense scattering.

Among the papers on weakly nonlinear theory of interaction of radiation with a plasma, mention should be made of Ref. 10, in which the generation of the first harmonics of the acoustic field under the action of a given field of beats of electromagnetic waves was studied. Such a process determines the structure of the fields in SMBS. However, even before Ref. 10, a theory of generation by such beats of a strongly nonlinear field with an unlimited number of harmonics was put forward in Ref. 11 (a direct continuation of the work of Ref. 5). The theory of Ref. 11 was further developed in Ref. 12 where, first, the generation of nonlinear, though small in amplitude, acoustical excitations was considered. These excitations had a rich spectrum. Second, a theory of SMBS from a halfspace filled with a nonlinear weakly dissipative liquid was constructed. Reference 12, which is rich in theoretical results, did not answer the question of realizing the hope of Ref. 4 of suppressing SMBS by the nonlinearity of the acoustic field. This was so because the authors of Ref. 12 considered the problem of a halfspace, the reflection from which, for example always amounted to 100% without account of dissipation. At the same time, the importance of the result of Ref. 12 should be noted. This work was also confirmed subsequently in Refs. 13 and 14 on the establishment of strongly nonlinear sawtooth distribution of acoustical excitations under the action of an electromagnetic field.

An approach that differed qualitatively from Refs. 7–14 was used in the weakly nonlinear theory of SMBS in Refs. 15, where the formation of subharmonics of the acoustic field was taken into account. Under conditions corresponding to actual experiment, such a process can lead to significant SMBS suppression only when a very intense acoustic flux flows into the region of interaction of the electromagnetic radiation with the sound waves. The presence of such a flux can be brought about, for example, in a laser plasma with developed ion-sound turbulence. However, in this case, the presence of turbulent fluctuations is itself a new cause of reflection of the radiation.[16] In this connection, in our analysis below we shall neglect the flow of the sound wave into the

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region of interaction, which makes our approach similar to that used in Ref. 7 to a certain degree.

To be precise, we formulate below a theory of SMBS from a plasma layer of finite thickness. In contrast to the currently extremely popular (see, for example, Ref. 3) approach of Refs. 17 and 18, which is used in the analysis of experiments on the action of intense radiation on a plasma, and which is restricted to the consequences of the three-wave problem of the interaction of a linear sound wave with two electromagnetic ones scattered and scattering, we consider the nonlinearity of the sound field. Inasmuch as, in contrast to Ref. 7, we consider the limit of negligible absorption, even the weakly linear theory that we have already developed predicts a qualitative suppression of the SMBS from a rarefied plasma (a preliminary communication was published in Ref. 19). We note here that the investigation of the strongly dissipative case in Ref. 20 demonstrated the significant attenuation of the weakly nonlinear effects by the strong sound absorption. As a supplement to the weakly nonlinear theory, which is suitable near the threshold of SMBS, we formulate below a strongly nonlinear theory of SMBS from a plasma layer of finite thickness with weak sound absorption. Such a theory allows us to obtain the dependence of radiation reflected in SMBS over a wide range of the intensity of the radiation incident on the plasma.

The initial equations, which describe the interaction of the electromagnetic field with the sound field, are formulated in the second section. In this case the sound field is described by the equation of nonlinear acoustics. The conditions of applicability of the three-wave model, which describes the SMBS process with neglect of nonlinearity of the sound field, are obtained in the third section. The next section is devoted to weakly nonlinear SMBS theory with account of only the second harmonic of the sound. In spite of the comparatively narrow region of applicability, this theory allows us to demonstrate the strong effect of the acoustic nonlinearity and elucidate the effect of saturation of the SMBS in a rarefied plasma. The effect of the higher harmonics of the sound is considered in the fifth section, where it is shown that, within the framework of the weak nonlinearity approximation, account of the third harmonic does not lead to any change in the results.

A strongly nonlinear SMBS theory is formulated in the sixth section. This limit corresponds to excitation of acoustic oscillations of comparatively small amplitude by beats of electromagnetic waves. The obtained distributions of the fields in the plasma as functions of the reflection coefficient on the amplitude of the pump wave are an extension of the results obtained in the fourth section to the case of a large excess over the SMBS threshold.

It should be emphasized that the approximation in which sound dissipation is neglected, which is used in this work, has a wide range of applicability. Actually, a comparison with Refs. 7–12 shows that under the conditions of strong damping

$$|\mathbf{E}|^2/8\pi n_c \varkappa T \ll (\gamma_s/\omega_s)^2$$

(γ_s is the sound damping decrement, ω_s is its frequency, $|\mathbf{E}|^2/8\pi$ is the intensity of the electromagnetic field, n_c is the critical density of the plasma, T is the temperature) the acoustic nonlinearity is insignificant and the SMBS can be described within the framework of the three-wave model. In the opposite limiting case, the three-wave model is valid only at very small reflection coefficients

$$R < (\gamma_s/\omega_s)^2/(|\mathbf{E}|^2/8\pi n_c \varkappa T).$$

2. For the description of the sound field excited by the electromagnetic field in the plasma,

$$\vec{\mathscr{E}}(\mathbf{r}, t) = \mathrm{Re}\, \mathbf{E}(x, t)\, e^{-i\omega_0 t},$$

we use the equation of dissipation-free nonlinear acoustics (see Ref. 12)

$$\frac{\partial}{\partial x}\frac{\delta n}{n_0} = -\frac{\partial}{\partial a}\left[\frac{|\mathbf{E}|^2}{32\pi n_c \varkappa T} + \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2\right], \tag{2.1}$$

where $\delta n = n - n_0$ is the perturbation of the density. The sound waves are propagated along the x axis, the variable $a = x - v_s t$ describes the rapid change in the sound field in a system of coordinates moving with the speed of sound v_s. Equation (2.1) holds at

$$1 \gg \delta n/n_0 \gg |\mathbf{E}|^2/8\pi n_c \varkappa T.$$

The electromagnetic field $\mathbf{E}(x,t)$ consists of incident and scattered waves. Taking it into account that the polarization vectors of both waves are parallel (this corresponds to the most intense interaction), we have

$$E(x, t) = (8\pi n_c \varkappa T)^{1/2}[e_0(x)\, e^{ika} + e_1(x)\, e^{-ika}], \tag{2.2}$$

where $k = (\omega_0/c)(1 - n_0/n_c)^{1/2}$. Since $|\mathbf{E}|^2$ is a periodic function of the variable a with period π/k, the total perturbation of the density can be represented in the form

$$\delta n(x, a) = in_0 \sum_{s=1}^{\infty} (-1)^s [\nu_s(x)\, e^{2ikas} - \nu_s^*(x)\, e^{-2ikas}]. \tag{2.3}$$

Then the following set of equations follows from (2.1) for the sound harmonics:

$$\frac{d\nu_s}{dx} = \frac{1}{2} k e_0 e_1^* \delta_{s,1} - ks\left(2\sum_{s'=1}^{\infty} \nu_{s'}^* \nu_{s+s'} - \sum_{s'=1}^{s-1} \nu_{s'} \nu_{s-s'}\right). \tag{2.4}$$

Substitution of (2.2) and (2.3) in the equation for the electromagnetic field

$$c^2 \Delta E + (\omega_0^2 - 4\pi e^2 n/m)E = 0$$

leads to the appearance of terms containing not only the dependence on the variable a with period $2\pi/k$, but also the higher harmonics with period π/sk. However, the higher harmonics of the field produced by such terms turn out to be small because of the smallness of the parameter $(kL)^{-1}$, where L is the characteristic scale of change of the field in terms of the slow variable x. Such small corrections are neglected in the set of truncated equations below. Therefore, a direct effect on the electromagnetic field is exerted only by the fundamental frequency of the sound field (cf. Ref. 12):

$$de_0/dx=-i(\alpha k^2/2\pi n_0)e_1(x)\int_{a_1}^{a_1+\pi/k}da\delta n(x,a)$$

$$\times\exp(-2ika)\equiv-\alpha k\nu_1 e_1/2, \tag{2.5}$$

$$de_1/dx=i(\alpha k^2/2\pi n_0)e_0(x)\int_{a_1}^{a_1+\pi/k}da\delta n(x,a)$$

$$\times\exp(2ika)\equiv-\alpha k\nu_1^* e_0/2,$$

where $\alpha=(n_c/n_0-1)^{-1}$. The set of equations (2.4) and (2.5) is the starting set for our nonlinear SMBS theory in a plasma layer.

3. We limit ourselves to the consideration of SMBS. Let the plasma layer fill the space $0<x<l$, let the incident electromagnetic wave and the sound excitations propagate in the positive direction of the x axis, and let the scattered (reflected) electromagnetic wave propagate in the opposite direction. The amplitudes of the entering acoustic and the scattered electromagnetic waves will be assumed to be equal to zero and the amplitude of the incident wave to be given:

$$\nu_s(0)=0,\qquad e_1(l)=0,\qquad e_0(0)=I^{1/2}, \tag{3.1}$$

where $I=E_0^2(0)/8\pi n_c\varkappa T$ is the dimensionless intensity of the pump wave. We limit ourselves here to the weakly-nonlinear regime, assuming smallness of the sound harmonics. Then the set of equations (2.4) and (2.5), with account of only the first two sound harmonics, takes the form

$$de_0/dx=-\alpha k\nu_1 e_1/2,\qquad de_1/dx=-\alpha k\nu_1^* e_0/2,$$
$$d\nu_1/dx=ke_0e_1^*/2-2k\nu_1^*\nu_2,\qquad d\nu_2/dx=2k\nu_1^2. \tag{3.2}$$

We note that the neglect of dissipation of ion sound, whose decrement of damping by electrons has the form

$$\gamma_s=kv_s(\pi zm/8M)^{1/2},$$

is possible in every case if the thickness of the plasma is not very great:

$$kl<kv_s/\gamma_s\sim(M/zm)^{1/2}. \tag{3.3}$$

We shall assume that this condition is satisfied below.

The first three equations of the set (3.2), with the boundary conditions (3.1) at $\nu_2\equiv 0$, have been widely used previously for the analysis of SMBS. They correspond to the textbook statement and analysis of the problem of the interaction of three waves. Here a systematic description is achieved of the weakening (exhaustion) of the electromagnetic pump wave in the layer at the expense of an increase in the amplitude of the scattered wave. At the same time it is evident that the presence of the scattered wave leads to pumping of the fundamental of the sound wave. The interaction of ν_1 and e_1 determines the nonlinear scattering law in the three-wave model.

However, with increase in the fundamental of the sound, the second harmonic and its reaction on the fundamental (and, correspondingly, on the scattered electromagnetic wave) increase even more rapidly. The purpose of our further study is to make clear the conditions under which the textbook three-wave formulation turns out to be inadequate for the description of the SMBS and the role of the second harmonic of the sound become predominant in the scattering.

Along with this, we must keep it in mind that the set (3.2) was obtained under the assumption of the smallness of the amplitudes of the higher harmonics of the sound. To test this assumption, we use below the equation for the sound-field third harmonic

$$d\nu_3/dx=6k\nu_1\nu_2, \tag{3.4}$$

which arises as a result of the merging of the fundamental and the second harmonic.

Proceeding to the analysis of the solutions of the set (3.2), we first establish the region of applicability of the usual solution of the three-wave problem. We designate the following as solutions of the set (3.2) at $\nu_2=0$: $e_0^{(0)}$, $e_1^{(0)}$, $\nu_1^{(0)}$. These functions satisfy the set of equations

$$de_0^{(0)}/dx=-\alpha ke_1^{(0)}\nu_1^{(0)}/2,\qquad de_1^{(0)}/dx=-\alpha ke_0^{(0)}\nu_1^{(0)*}/2,$$
$$d\nu_1^{(0)}/dx=ke_0^{(0)}e_1^{(0)*}/2. \tag{3.5}$$

Using the integrals of this set of equations,

$$|e_0^{(0)}|^2-|e_1^{(0)}|^2=C_1^{(0)},\qquad |e_0^{(0)}|^2+\alpha|\nu_1^{(0)}|^2=C_2^{(0)},$$

$$\operatorname{Im} e_0^{(0)*}e_1^{(0)}\nu_1^{(0)}=C_3^{(0)}, \tag{3.6}$$

and also the boundary conditions (3.1), we can write down the solutions explicitly with the help of the Jacobi elliptic functions

$$e_0^{(0)}(x)=I^{1/2}\operatorname{dn}(\xi,R_0),\qquad e_1^{(0)}(x)=R_0I^{1/2}\operatorname{cn}(\xi,R_0),$$
$$\nu_1^{(0)}(x)=R_0(I/\alpha)^{1/2}\operatorname{sn}(\xi,R_0), \tag{3.7}$$

where $\xi=(1/2)kx(\alpha I)^{1/2}$, $R_0=|e_1^{(0)}(0)/e_0^{(0)}(0)|$ is the reflection coefficient, the value of which is found from the requirement of the satisfaction of the boundary condition (3.1) on the right-hand boundary of the layer $x=l$:

$$\operatorname{sn}\left(\frac{1}{2}kl(\alpha I)^{1/2},R_0\right)=1\quad\text{or}\quad\frac{2}{\pi}\mathrm{K}(R_0)=\frac{1}{\pi}kl(\alpha I)^{1/2}\equiv p, \tag{3.8}$$

where $\mathbf{K}$ is a complete elliptic integral of the first kind.

Equation (3.8) has solutions only at $p\geqslant 1$. The condition $p=1$ gives the instability threshold of the SMBS in a plasma layer of finite thickness. In the near-threshold region $p-1\ll 1$, in accord with (3.8), there is a linear dependence of the reflection coefficient $R_0^2=4(p-1)$.

In addition, we find corrections to the solution (3.7) that are connected with the account of the second harmonic of the sound. Its intensity $\nu_2^{(0)}(x)$ is found with the help of (3.7) from the last equation of the set (3.2):

$$\nu_2^{(0)}(x)=\frac{2}{\alpha}\left(\frac{I}{\alpha}\right)^{1/2}\left[kx(\alpha I)^{1/2}-2E\left(\operatorname{am}\frac{1}{2}kx(\alpha I)^{1/2},R_0\right)\right], \tag{3.9}$$

where E is an elliptic integral. Comparing $\nu_1^{(0)}$ and $\nu_2^{(0)}$ with the help of (3.7) and (3.9), we find that the assumption of weak nonlinearity of the sound wave $\nu_2\ll\nu_1$ corresponds near the instability thresholds of the SMBS to the following inequality, which is of importance for us,

$$R_0 \ll \alpha \quad \text{or} \quad p-1 \ll \alpha^2/4. \tag{3.10}$$

This same inequality arises also when finding the amplitude of the third harmonic with the help of Eq. (3.4). At large values, greatly exceeding the threshold ($p \gg 1$), the expression $kx(\alpha I)^{1/2}$ on the right-hand side of (3.9) can reach p. Therefore, under these conditions, weakness of the nonlinearity of the sound wave is realized at

$$p \ll \alpha. \tag{3.11}$$

It is taken into account here that according to (3.8), at values far exceeding threshold, the reflection coefficient is close to unity, in accord with the usual three-wave theory.

The formulas (3.10) and (3.11) allow us to perceive the interesting dependences on the plasma density that characterize the possibility of use of the assumption of weak nonlinearity of the sound. Thus, in a plasma with density close to critical, when $\alpha \gg 1$, the effects of nonlinearity of the sound are weak even far above the instability threshold. On the other hand in a rarefied plasma at $\alpha \ll 1$, the assumption of weak nonlinearity of the sound is satisfied only in the near-threshold region. Therefore, it is in the rarefied plasma that we can expect a significant manifestation of the effects due to the nonlinearity of the sound waves.

4. We now return to the solution of the set of equations (3.2), keeping in mind both the depletion of the pump and the generation of the second harmonic of the sound. The set (3.2) has three first integrals:

$$|e_0|^2-|e_1|^2=C_1, \qquad |e_0|^2+\alpha(|v_1|^2+|v_2|^2)=C_2,$$
$$\mathrm{Im}(e_0^* e_1 v_1 - 2v_1^{*2} v_2)=C_3, \tag{4.1}$$

and in accord with the boundary conditions (3.1) we have $C_1 = I(1-R^2)$, $C_2 = I$, $C_3 = 0$, where $R = |e_1(0)/e_0(0)|$ is the reflection coefficient. With account of (4.1), the set (3.2) for the four complex functions e_0, e_1, v_1, and v_2 reduces to two equations for the complex amplitude of the second harmonic:

$$s = (\alpha^{1/2}/RI^{1/2}) v_2 v_1^{*2}/|v_1|^2$$

and for the total amplitude of the soundfield

$$t=(\alpha^{1/2}/RI^{1/2})(|v_1|^2+|v_2|^2)^{1/2}:$$

$$\frac{ds}{d\xi} = 4\frac{R}{\alpha}(t^2-|s|^2+is\,\mathrm{Im}\,s), \tag{4.2}$$

$$\frac{dt}{d\xi} = \frac{1}{t}(t^2-|s|^2)^{1/2} \times \left[(1-R^2t^2)(1-t^2)-4\frac{R^2}{\alpha^2}(t^2-|s|^2)(\mathrm{Im}\,s)^2\right]^{1/2}, \tag{4.3}$$

where $\xi = 1/2kx(\alpha I)^{1/2}$, and the boundary conditions (3.1) reduce to the following:

$$s(0)=0, \qquad t(0)=0, \qquad t(\pi p/2)=1, \tag{4.4}$$

where $p = \pi^{-1}kl(\alpha I)^{1/2}$ is the quantity, introduced in the third section, that characterizes the excess over threshold. The last condition in (4.4), written down for the right-hand boundary of the region, is used to find the reflection coefficient R. The solutions of (4.2) and (4.3) determine, in particular, the distribution of the sound field at the fundamental frequency:

$$|v_1| = R(I/\alpha)^{1/2}(t^2-|s|^2)^{1/2} \tag{4.5}$$

and also the distribution of the electromagnetic fields of the incident and scattered waves:

$$|e_0| = [I(1-R^2t^2)]^{1/2}, \qquad |e_1| = R[I(1-t^2)]^{1/2}. \tag{4.6}$$

Returning to the solution of the set (4.2) and (4.3), we first show that $\mathrm{Im}s = 0$ follows from the boundary conditions (4.4). Actually, taking the imaginary part of Eq. (4.2), we have

$$\frac{d}{d\xi}\mathrm{Im}\ s = 4\frac{R}{\alpha}\mathrm{Re}\ s\cdot\mathrm{Im}\ s.$$

Consequently,

$$\mathrm{Im}\ s(\xi) = \mathrm{Im}\ s(0)\exp(4R/\alpha)\int_0^{\xi} d\xi'\mathrm{Re}\ s(\xi').$$

the reality of the function $s(\xi)$ over the entire region $0<\xi<\pi p/2$ then follows from the first condition of (4.4). We also note that, in accord with the solution (4.8) obtained below $s \sim R/\alpha$. Under conditions of weak nonlinearity of the sound wave, the only conditions under which the set (4.2) and (4.3) holds in accord with (3.10), we have $R \ll \alpha$. Therefore, even in this case, the change of $\mathrm{Im}\ s$ over the thickness of the layer can be neglected when $\mathrm{Im}\ s(0) \neq 0$.

In accord with (4.4), we set $\mathrm{Im}\ s = 0$ in what follows. We then obtain the following for $s(t)$ from the set (4.2) and (4.3)

$$\frac{ds}{dt} = 4t\frac{R}{\alpha}\left[\frac{t^2-s^2}{(1-R^2t^2)(1-t^2)}\right]^{1/2}, \tag{4.7}$$

and $s(t=0)=0$, $0<t<1$.

The smallness of the higher harmonics of sound, as applied to (4.7), corresponds to the inequality $s \ll t \lesssim 1$, which is satisfied under the conditions of smallness of the parameter $4R/\alpha \ll 1$. Here we can neglect the term s^2 in the right side of (4.7). We than have for $s(t)$

$$s(t) = \frac{4}{\alpha R}[F(\arcsin t, R) - E(\arcsin t, R)], \tag{4.8}$$

where F and E are elliptic integrals of the first and second kind, respectively. We than have, at $R^2 \ll 1$,

$$s(t) = (2R/\alpha)[\arcsin t - t(1-t^2)^{1/2}]. \tag{4.9}$$

If, however, we assume that $t \ll 1$, we get from (4.9)

$$s \approx \frac{4}{3}\frac{R}{\alpha}t^3.$$

Keeping in mind the increase of $s(t)$ with increase in t, we write down the inequality $s \ll t$ in the most "dangerous" place on the right-hand boundary of the layer at $t=1$. It then takes the form

$$\mathbf{K}(R) - \mathbf{E}(R) \ll \alpha R/4, \tag{4.10}$$

where $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals.

It follows from formula (4.10) that at R that is not small in comparison with unity the second harmonic of the sound wave can be assumed to be small only in a plasma with a density that is close to critical, when α is large. If, however, we assume smallness of the reflection coefficient $R^2 \ll 1$, then the inequality (4.10) takes a form similar to (3.10):

$$R \ll \alpha/\pi. \tag{4.11}$$

This indicates an important restriction on the applicability of a theory that neglects the effect of the second harmonic of the sound. Actually, it is just the latter inequality that should be satisfied if we apply the formula (4.9) to the case of a rarefied plasma, when α is small and when the limitation imposed by the condition (4.11) is especially significant.

The obtained dependence (4.8) allows us to solve Eq. (4.3) and to represent the dependence $\xi(t)$ in the form

$$\begin{aligned}\xi = F(\varphi, R) + (8/3\alpha^2 R^2)\{[F(\varphi, R) - E(\varphi, R)]^3 \\ - 3\operatorname{ctg}\varphi[1 - R^2\sin^2\varphi]^{1/2}[F(\varphi, R) - E(\varphi, R)]^2 \\ - (2 + R^2\cos^2\varphi - 2R^2\sin^2\varphi)[F(\varphi, R) - E(\varphi, R)] \\ + R^2 E(\varphi, R) - R^2\sin\varphi\cos\varphi[1 - R^2\sin^2\varphi]^{1/2}\}\end{aligned} \tag{4.12}$$

with accuracy up to terms $\sim (R/\alpha)^2 \ll 1$. Here $\varphi = \arcsin t$. Using the last of the boundary conditions (4.4) $\xi(t = 1) = \pi p/2$, we find the following equation for the reflection coefficient:

$$p = \frac{2}{\pi}\mathbf{K} + \frac{16}{3\pi\alpha^2 R^2}[2(\mathbf{E} - \mathbf{K}) + R^2(2\mathbf{K} - \mathbf{E}) + (\mathbf{K} - \mathbf{E})^3], \tag{4.13}$$

where $\mathbf{K} = \mathbf{K}(R)$ and $\mathbf{E} = \mathbf{E}(R)$ are complete elliptic integrals.

For a dense plasma, with density close to critical, when $\alpha \gg 1$, the dependence $R(p)$ follows from (4.13), in particular in the case $1 - R^2 \ll 1$:

$$R^2 = 1 - 16\exp[-\pi p(1 - 2\pi^2 p^2/3\alpha^2)].$$

Here the condition $2\pi^2 p^2 \ll 3\alpha^2$, which determines the applicability of the approximation of weak nonlinearity of the sound, must be satisfied (cf. (3.11)).

A qualitative effect develops in a rarefied plasma when $\alpha \ll 1$. Here we obtain the following from (4.13) under the conditions (4.11) (cf. Ref. 19)

$$R^2 = \frac{4\alpha^2}{4 + \alpha^2}(p - 1) = \frac{(\omega_0 l/\pi c)(I n_0/n_c)^{1/2} - 1}{1/4 + (n_c/n_0 - 1)^2}. \tag{4.14}$$

This formula corresponds to the condition (4.11) in the case of small excess above threshold $p - 1 \ll 1$, when the nonlinearity of the sound field can be regarded as small. At the same time, such a weak nonlinearity, according to (4.14), decreases the coefficient of reflection from the rarefied plasma by a factor $\alpha^{-2} \gg 1$ in comparison with the prediction of the three-wave theory.

The observed strong effect of the nonlinearity of the sound field in a rarefied plasma is due to the fact that under the conditions $\alpha \ll 1$ the usual effect of pump depletion turns out to be unimportant. For an illustration of this fact, we consider the energy distribution of the field of a scattered electromagnetic wave in the layer. Here, in accord with (4.11) and (4.14), we limit ourselves to the case $R \ll 1$. Then, in accord with (4.6) and (4.12), we have

$$\begin{aligned}|e_1(\xi)|^2 = R^2 I\cos^2\varphi(\xi), \\ \xi = \varphi + \frac{R^2}{4}(\varphi - \sin\varphi\cos\varphi) + \frac{R^2}{\alpha^2}(\varphi + \sin\varphi\cos\varphi - 2\varphi^2\operatorname{ctg}\varphi).\end{aligned} \tag{4.15}$$

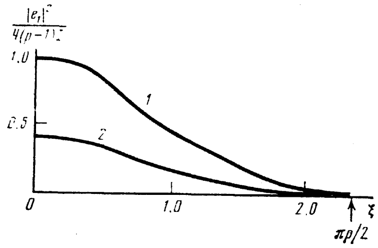

FIG. 1. Energy distribution of the scattered wave in a plasma layer at $p = 1.5$ for $\alpha = \infty$ (curve 1) and $\alpha = 1$ (curve 2).

A graph of the dependence of $|e_1|^2$ on ξ at fixed excess above threshold $p = 1.5$ is shown in Fig. 1 for $\alpha = \infty$, which corresponds to the three-wave model, and for $\alpha = 1$. It is seen that even at $\alpha = 1$ (this corresponds to a plasma with a density equal to one half of the critical value) the intensity of the scattered field falls by a factor of 2.5 upon decrease of the density of the plasma by a factor of 2. Along with this, the form of the distribution of the amplitudes of the electromagnetic waves and of the fundamental of the sound depends very weakly on the parameter α. For illustration of this fact, we show in Fig. 2 the $t(\xi)$ dependence at fixed excess above threshold $p = 1.5$ for two extreme values $\alpha = 0$ and $\alpha = \infty$. The closeness of the curves in Fig. 2 allows us to draw the conclusion that the basic effect of the sound nonlinearity in the case $R \ll 1$, $R \ll \alpha$ reduces to a change in the amplitudes of the scattered and the sound waves. Along with this, to find the distribution of the fields of the scattered and the sound waves in the plasma layer, it suffices to make use of the solutions of the three-wave problem (3.7), replacing R_0 by the value of the reflection coefficient (4.14) found above.

5. Having observed the substantial influence of the second harmonic of the sound, we must now make clear the conditions under which account of the higher harmonics will change the discussed effect of reflection attenuation. For this purpose, we consider the field of the third harmonic of the sound waves. Denoting $s_3 = (\alpha^{1/2}/RI^{1/2})v_3$ and using the expression (4.5) for the amplitude of the fundamental, we rewrite (3.4) in the form

$$\frac{ds_3}{d\xi} = 12R\frac{s}{\alpha}(t^2 - s^2)^{1/2}. \tag{5.1}$$

On the other hand, for comparison of the fields of the harmonics, in accord with the results of the previous section, it makes sense to consider the dependence of s_3 on t. Then, with the aid of (5.1) and (4.3), we obtain the equation

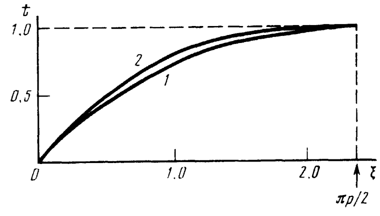

FIG. 2. Distribution of sound-wave amplitude in a plasma layer at $\alpha = \infty$ (curve 1) and $\alpha = 0$ (curve 2); $p = 1.5$.

$$\frac{ds_3}{dt}=12\frac{R}{\alpha}\frac{st}{[(1-R^2t^2)(1-t^2)]^{1/2}}, \qquad (5.2)$$

which enables us to find $s_3(t)$ directly, subject to the boundary condition $s_3(t=0)=0$, with the help of the results of the previous section. As applied to the case of small reflection $R^2 \ll 1$, using the condition (4.9), we find

$$s_3(t)=24(R/\alpha)^2[t(1-t^2/3)-(1-t^2)^{1/2}\arcsin t]. \qquad (5.3)$$

The amplitude of the third harmonic grows from the zero value at $t=0$ to a maximum at $t=1$, when it reaches the value $s_3(t=1)=16R^2/\alpha^2$. A comparison of the obtained expression (5.3) with the formula (4.9) shows that the third harmonic of the field of the sound wave turns out to be small in comparison with the second harmonic in the case $R \ll (\pi/16)\alpha$.

The effect of the third harmonic manifests itself in a weakening of the growth of the second harmonic because of the change in the last equation of the set (3.2):

$$d\nu_2/dx=2k\nu_1^2-4k\nu_1^*\nu_3.$$

The last term in this equation is negligible, as is shown by a comparison of (5.3) with (4.5), upon satisfaction of the inequality $s_3 \ll 1$ or $32R^2 \ll \alpha^2$. We note here that modification of the equation of the second harmonic and account of Eq. (3.4) lead to the following modification of the integral C_2 (4.1):

$$C_2=|e_0|^2+\alpha(|\nu_1|^2+|\nu_2|^2+2/3|\nu_3|^2).$$

It also follows from this that under the conditions of satisfaction of the inequality (4.11) the effect of the third harmonic can be neglected. All this indicates that the qualitative decrease, and realized under the conditions (4.11), of the reflection due to the SMBS from a rarefied plasma is caused only by the comparatively small second harmonic of the field of the sound wave, and the higher harmonics do not change the result (4.14) under these conditions.

6. Although the approximation of weak nonlinearity of the sound field, which was used in the preceding sections, has led us in the case of a rarefied plasma to a qualitative decrease of the reflection coefficient, it is seen at the same time that the region of applicability of the resutls is comparatively narrow. In the most interesting case of a rarefied plasma ($\alpha \ll 1$) the weak nonlinearity approximation of the sound field is satisfied in the case of comparatively small values of the reflection coefficient ($R<\alpha \ll 1$), which occurs, according to (4.14), only in the near-threshold region ($p-1 \lesssim 1$). In this connection, it is important to consider the opposite case of a large excess over threshold ($p \gg 1$), when, in correspondence with the results of the third section, the effects of nonlinearity of the sound field should become strong in a rarefied plasma.

Since, at $R \gtrsim \alpha$ in accord with the previous section, the amplitudes of the various harmonics of the sound waves are comparable in magnitude, we shall not start out with the expansion (2.3) below, but shall make use directly of the set of equations (2.1) and (2.5), which we represent in the form

$$de_0/dx=-\alpha k\nu_1 e_1/2, \quad de_1/dx=-\alpha k\nu_1^* e_0/2,$$
$$\frac{\partial \nu(x,a)}{\partial x}=-\frac{1}{2}\frac{\partial}{\partial a}[\nu^2+\mathrm{Re}(e_0e_1^*e^{2ika})], \qquad (6.1)$$

where

$$\nu_1(x)=i(k/\pi)\int_{a_1}^{a_1+\pi/k} da\,\nu(x,a)e^{-2ika}, \quad \nu(x,a)=\delta n(x,a)/n_0.$$

The set of equations of the form (6.1), supplemented by account of the sound damping, was obtained in Ref. 12 on the basis of the theory of SMBS of light in a half-space. However, since in the case of a half-space such a system of equations leads to complete reflection of the light, Ref. 12 does not contain the answer to the problem of interest to us, namely, the dependence of the reflection coefficient on the intensity of the scattered radiation, since such a problem can be posed in a spatially homogeneous plasma only for a layer of finite thickness.

For analysis of the consequences that follow from the set of equations (6.1) it is convenient in introduce the new functions

$$e_0e_1^*=Ir(y)e^{-i\Phi(y)}, \quad \nu(x,a)=(2I)^{1/2}b(y,\theta),$$

which depend on the slow dimensionless coordinate $y=kx(I/2)^{1/2}$ and the new fast variable $\theta=2ka+\Phi$. Here, using $|e_0|^2-|e_1|^2=I(1-R^2)$ (cf. (4.1)), we can write down the following set of equations in accord with (6.1):

$$\frac{\partial b}{\partial y}+\frac{d\Phi}{dy}\frac{\partial b}{\partial\theta}=-2\frac{\partial b^2}{\partial\theta}+r\sin\theta,$$
$$\frac{dr}{dy}+ir\frac{d\Phi}{dy}=-\alpha b_1[(1-R^2)^2+4r^2]^{1/2}, \qquad (6.2)$$

where

$$b_1(y)=(i/2\pi)\int_{\theta_1}^{\theta_1+2\pi} d\theta\, b(y,\theta)e^{-i\theta}.$$

The system (6.2) holds in the interval

$$0<y<y_{\max}=kl(I/2)^{1/2}=\pi p/\sqrt{2\alpha}.$$

The variable θ corresponds to a fast, oscillatory dependence of the sound field excited by the beats of the scattered and scattering electromagnetic waves, with period 2π. The period of the driving force determines the period of the function $b(y,\theta)$. It follows from the first equation of the set (6.2) that the average value of the function b (over the period $\Delta\theta=2\pi$) turns out to be independent of the variable y, and therefore, by virtue of the vanishing of the amplitude of the sound field on the left-hand boundary of the layer, such an averaged value turns out to be equal to zero. Then the function $b(y,\theta)$ should satisfy the following conditions:

$$b(0,\theta)=0, \quad b(y,\theta+2\pi)=b(y,\theta), \quad \int_{\theta_1}^{\theta_1+2\pi} d\theta\, b(y,\theta)=0. \qquad (6.3)$$

The boundary conditions

$$r(y_{max})=0, \quad \Phi(y_{max})=0 \qquad (6.4)$$

correspond to the absence of a scattered wave at the right-hand side of the layer. Finally, the condition

$$r(0)=R \qquad (6.5)$$

determines the reflection coefficient of the electromagnetic wave. We note that the distribution of the electromagnetic fields in the layer is given by the formulas

$$|e_{0,1}|^2={}^1/_2 I\{[(1-R^2)^2+4r^2]^{1/2}\pm(1-R^2)\}. \tag{6.6}$$

An equation similar to the first equation of the set (6.2), supplemented also by the account of viscosity, was used in a number of studies[11–14] of the excitation of nonlinear sound waves by the beats of two electromagnetic waves.

We can write down the simplest solution of Eq. (6.2) if, following Ref. 12, we assume that the amplitude of the electromagnetic field $r(y)$ is such a slow function of the variable y that in the first equation of the set (6.2) we can neglect the derivatives with respect to y. Then, keeping it in mind that this equation reduces to the algebraic one $2b^2 + r\cos\theta = \text{const}$, and also taking into account the last two conditions of (6.3), we obtain (cf. Ref. 12)

$$b(y,\theta)=[r(y)]^{1/2}\sin(\theta/2-n\pi),\quad (2n-1)\pi\leqslant\theta<(2n+1)\pi, \tag{6.7}$$

where n is an arbitrary integer. Here we must note that the solution (6.7) is inaccurate near the left boundary of the layer, where the first condition of (6.3) should be satisfied, which, if (6.7) is used literally, contradicts the condition (6.5). Actually, this means that the region of rapid change of $b(y,\theta)$ is located near $y = 0$. As we shall see below, the solution (6.7) is also violated near the right boundary of the layer.

The solution (6.7) is a sawtooth wave with discontinuities at the points $\theta_n = (2n+1)\pi$. As is shown in Refs. 11–14, account of dissipation smooths out such discontinuities to a certain extent. This turns out to be unimportant for our problem of finding the field of the scattered wave and the reflection coefficient, since it is necessary for us to know only the fundamental of the sound wave. According to (6.7),

$$b_1(y)=(4/3\pi)[r(y)]^{1/2}. \tag{6.8}$$

Since $\operatorname{Im} b_1 = 0$, we have $\Phi = 0$ according to (6.2) and the boundary condition (6.4). Finally, in accord with the second equation of (6.2) and (6.8) we obtain the following simple equation which determines the field of the scattered wave:

$$\frac{dr}{dy}=-\frac{4\alpha}{3\pi}\{r[(1-R^2)^2+4r^2]\}^{1/2}. \tag{6.9}$$

Just as with formulas (6.7) and (6.8), this equation is inaccurate in the immediate vicinities of the left and right boundaries of the layer. The solution of Eq. (6.9) that satisfies the boundary condition (6.4) is written in the form

$$r(kx(I/2)^{1/2})=\frac{1}{2}(1-R^2)\frac{1-\operatorname{cn}(\zeta,1/\sqrt{2})}{1+\operatorname{cn}(\zeta,1/\sqrt{2})},$$

$$\zeta=\frac{4\sqrt{\alpha}}{3}p\left(1-\frac{x}{l}\right)(1-R^2)^{1/4}. \tag{6.10}$$

Here, in accord with the condition (6.5), the dependence of the reflection coefficient R on the excess above threshold is given by the following equation:

$$\operatorname{cn}\left(\frac{4}{3}\sqrt{\alpha}\,p(1-R^2)^{1/4},\ \frac{1}{\sqrt{2}}\right)=\frac{2-(1+R)^2}{2-(1-R)^2}. \tag{6.11}$$

Before we discuss the conclusions that follow from Eq. (6.11), we shall dwell on the conditions of applicability of the solutions (6.7) and (6.10), obtained under the assumption of satisfaction of the inequality

$$|\partial b/\partial y|\ll r|\sin\theta|, \tag{6.12}$$

which is necessary for neglect of the left side of the first equation of (6.2). When using (6.7) and (6.10), the inequality (6.12) can be represented in the form

$$\frac{2^{3/2}\alpha}{3\pi}\ll\left|\cos\left(\frac{\theta}{2}-n\pi\right)\right|\frac{1-\operatorname{cn}(\zeta,1/\sqrt{2})}{[1+\operatorname{cn}^2(\zeta,1/\sqrt{2})]^{1/2}}. \tag{6.13}$$

It then follows that our approximation is violated near the points of discontinuity θ_n. However, the size of the region in which the relation (6.13) is violated is smaller the lower the density of the plasma. This is precisely why the discontinuous approximation (6.7) can be used at $\alpha\ll1$. Another region of violation of the inequality (6.13) is located near the right edge of the layer, $\operatorname{cn}(\zeta,1/\sqrt{2})\to1$. Here (6.13) takes the form

$$3/2\pi p^2\ll(1-R^2)(1-x/l)^2|\cos(\theta/2-n\pi)|. \tag{6.14}$$

It then follows, that at values of the reflection coefficient that are not too close to unity, the size of the region near the right edge, in which the condition (6.12) is violated, turns out to be smaller the higher the excess above threshold ($p\gg1$). In discussing the condition for the applicability of the approximation (6.7)–(6.10), we must consider the inaccuracies of the solution (6.7) near the left boundary of the layer, as mentioned above. The corresponding approximate expression for $b(y,\theta)$, which satisfies the first of the boundary conditions (6.3), can easily be written down at small values of y, when $r = R$. Here we have

$$b(y,\theta)=yR\sin\theta-{}^2/_3y^3R^2\sin2\theta+O(y^5R^3). \tag{6.15}$$

This formula describes the growth of the sound field near the left boundary of the layer, when the condition

$$(y/y_{max})^2\ll3\alpha/2\pi^2p^2R|\cos\theta| \tag{6.16}$$

is satisfied. Since the transition region (6.15) and (6.7) corresponds to the values $y\sim R^{-1/2}$, the comparative narrowness of the transition layer is assured by the inequality

$$y_{max}^2=\pi^2p^2/2\alpha\gg1, \tag{6.17}$$

which holds because of the sufficiently large excess above threshold ($p>1$) and the rarefaction of the plasma ($\alpha<1$) (see Fig. 3).

We now proceed to a discussion of the consequences of the Eq. (6.11) which determines the reflection coefficient. We shall first show that, in the limit αp^2, we obtain

$$R=\frac{2}{9}\alpha p^2=\frac{2Il^2\omega_0^2}{9\pi^2c^2}\frac{n_0/n_c}{n_c/n_0-1}, \tag{6.18}$$

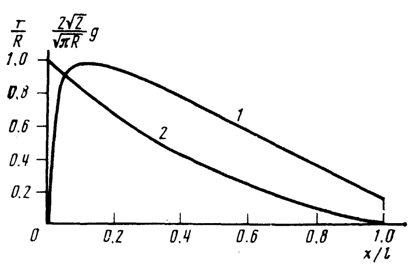

FIG. 3. Distribution of sound amplitude (curve 1) and a scattered electromagnetic (curve 2) wave in a layer according to the solution of Eqs. (6.2) at a value of the parameter $\alpha/R = 0.23$.

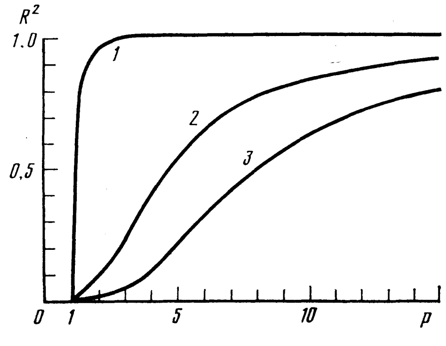

FIG. 4. Dependence of the reflection coefficient R^2 on the excess over threshold of SMBS in the three-wave model (curve 1) and with account of the acoustic nonlinearity for $n_0/n_c = 0.25$ (curve 2) and $n_0/n_c = 0.1$ (curve 3).

from (6.11). Here, in the first place, the reflection coefficient is small compared with unity; second, the reflection coefficient increases with increase in the intensity of the scattered wave and in proportion to that intensity; third, at $p \sim 1$ Eq. (6.18) is approximately the same as (4.14) obtained above as a result of the theory of weak nonlinearity of the sound. Thus, the dependence of the small reflection coefficient $R(p) \ll 1$ for large and small superthreshold is given by formulas (4.14) and (6.18), respectively. At values of $R(p)$ that are not small, we must use the formula (6.11). At the same time, for very intense scattered fields, when $\alpha p^2 \gg 1$ or, what amounts to the same thing,

$$\left(\frac{\omega_0 l}{\pi c}\right)^2 I \gg \frac{n_c}{n_0}\left(\frac{n_c}{n_0}-1\right), \tag{6.19}$$

when the reflection coefficient approaches unity, we have the following simple formula:

$$R^2 = 1 - \frac{9}{4p^2\alpha}\mathrm{K}^2\left(\frac{1}{\sqrt{2}}\right)$$
$$= 1 - \frac{9\pi c^2}{64\omega_0^2 l^2 I}\left[\Gamma\left(\frac{1}{4}\right)\right]^4 \frac{n_c}{n_0}\left(\frac{n_c}{n_0}-1\right). \tag{6.20}$$

The power law of the approach of the reflection coefficient to unity distinguishes the result (6.20) qualitatively from the exponential law that appears in the theory neglecting the nonlinearity of the sound field.

An illustration of the dependence for the reflection coefficient that has been obtained is given in Fig. 4, where the dependence $R^2(p)$ is given both for not taking into account the nonlinearity of the sound field of the three-wave model (curve 1) and for our theory at $\alpha = 1/3$ (curve 2), which corresponds to a plasma with density one fourth that of critical, and at $\alpha = 1/9$ (curve 3), which corresponds to the condition $n_0/n_c = 0.1$. We note here that the numerical calculations indicate the comparatively broad range of application of the asymptotic formulas (6.18) and (6.20). That is, the first of these is applicable at $\alpha^2 \lesssim R^2 \lesssim 0.1$, and the second at $R^2 \gtrsim 0.6$. A comparison of the curves of Fig. 4 shows that, in spite of the fact that in our dissipation-free theory, the reflection coefficient tends to a value equal to unity in the limit of high excess above threshold, our allowance for the nonlinearity of the soundfield significantly suppresses the growth of $R(p)$. With decrease in the density of the plasma, this suppression of the SMBS becomes ever more substantial.

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