Relaxation oscillations in a plasma with an ultrarelativistic electron beam

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A theoretical investigation is made of a nonlinear interaction of a high-current ultrarelativistic electron beam with a cold plasma. It is shown that external modulation not only provides an effective means for the control of the spectrum of instabilities [Ya. B. Faĭnberg, Sov. J. Plasma Phys. **3**, 246 (1977)], but also largely determines the output power of a beam–plasma system, since the maximum of the oscillation energy corresponds to harmonics with small increments whose growth during the linear stage is partly suppressed by the external modulation. A nonlinear saturation of the field amplitude is due to a change in the waveguide properties of the plasma in a strong electric field and due to a considerable increase in the phase velocity of a wave (disappearance of a slow wave). Regular field pulses which appear in such a beam–plasma system are characterized by phase discontinuities near the amplitude maxima, which is typical of relaxation processes, and are described by a nonlinear equation with a small parameter in front of the highest-order derivative.

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1. INTRODUCTION

Experimental investigations of the collective interaction of high-current relativistic monoenergetic electron beams with plasmas have confirmed the theoretically predicted feasibility of control of the spectra of instabilities by the application, at the input of a beam–plasma system, of a regular signal stronger than the noise level in the plasma or by initial modulation of the electron beam.[1] The necessary power represents 10^{-4}–10^{-6} of the oscillation power and it slows down a transition of a beam–plasma system to a turbulent regime, suppresses the effects of quasilinear broadening of the oscillation spectrum,[2–4] and facilitates conversion of the beam energy into a monochromatic wave with a regular phase frequency equal to the modulation frequency.[1]

It follows from Ref. 5 that the field energy density of the most unstable mode of frequency $\omega_m = \omega_p$ (ω_m and ω_p are the modulation and plasma frequencies) grows on increase in the energy density in an electron beam proportionally to $\alpha = \nu^{1/3}\gamma_0$, where $\nu = n_b/n_p \ll 1$, n_b and n_p are the beam and plasma densities, and γ_0 is the relativistic factor, so that if $\alpha \lesssim 1$, the oscillation energy density becomes comparable with the beam energy density. However, in the case of beams with a high energy density so that $\alpha > 1$, when phase oscillations are suppressed by the relativistic increase in the mass[6] and the instability is stabilized by the plasma nonlinearity, the proportion of the energy transferred from the beam to the field decreases considerably[7]:

$$\frac{E^2}{4\pi m v_0^2 \gamma_0 n_b} \approx \begin{cases} \alpha, & \alpha \ll 1 \\ \alpha^{-2}, & \alpha \gg 1 \end{cases}. \tag{1}$$

Here, v_0 is the initial velocity in the beam, m is the electron mass, and E is the electric field.

We shall consider the possibility of increasing the efficiency of operation of a beam–plasma system in the range of ultrarelativistic energy densities by converting the beam energy into the energy of a field of harmonics with small increments. Since the complex correction to the frequency

$$\frac{\Delta\omega}{\omega_m} = \begin{cases} 2^{-4/3}(-1 \pm i\sqrt{3})\,\nu^{1/3}/\gamma_0 & |\varepsilon| \ll \nu^{1/3}/\gamma_0 \\ \pm i(\nu/\gamma_0^3|\varepsilon|)^{1/2}, & |\varepsilon| \gg \nu^{1/3}/\gamma_0 \end{cases} \tag{2}$$

decreases on increase in the detuning $\varepsilon = 1 - \omega_p^2/\omega_m^2 < 0$, the change to small increments $\delta = \operatorname{Im} \Delta\omega$ is equivalent to the use of modulation frequencies much lower than the Langmuir frequency of the plasma.[1)]

In the frequency range $|\varepsilon| \gtrsim \nu^{1/3}/\gamma_0$, we obtain estimates

$$\frac{E^2}{4\pi n_b m v_0^2 \gamma_0} \approx \begin{cases} \nu/\gamma_0\varepsilon^2, & |\varepsilon| \gg \nu^{2/3} \\ |\varepsilon|/\gamma_0\nu, & |\varepsilon| \ll \nu^{2/3} \end{cases}, \tag{3}$$

which generalize the expressions in Eq. (1) to the case of high values of $|\varepsilon|$. The dependences of the ratio of the field and beam energy densities on the value of u are plotted in Fig. 1. At low values of the parameter $\alpha \lesssim 1$ an increase in the dimensionless frequency detuning $|\varepsilon|$ is accompanied by a reduction in the oscillation energy compared with the maximum mode $\varepsilon = 0$. Conversely, in the $\alpha > 1$ range the curves have maxima at $|\varepsilon| \approx \nu^{2/3}$, where the field energy density is α

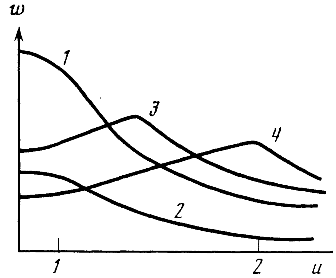

FIG. 1. Qualitative dependences of the field energy density in a plasma $w = E^2/4\pi n_b mc^2\gamma_0$, on the beam modulation frequency $u = |\varepsilon|\gamma_0\nu^{-1/3}$. Curves 1–4 correspond to the values of the parameter α = 1.0, 0.4, 1.5, and 2.0.

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times greater than the corresponding values for the maximal mode.

In the regions of negative slope of the curves the frequency of phase oscillations of beam electrons trapped by the wave field

$$\Omega_{\text{ph}}=(eE\omega_p/mv_0\gamma_0^3)^{1/2},$$

is of the order of the increment and the instability is stabilized by the nonlinearity of a low-density beam.[9,10] A postive slope corresponds to the inequality

$$\Omega_{\text{ph}}^2/\delta^2\approx|\varepsilon|^{3/2}\nu^{-1}<1,$$

and the effects of trapping of beam electrons by a wave, suppressed by external modulation, are manifested less strongly than the plasma nonlinearity. When the inequality is strong, the plasma nonlinearity is the dominant effect and it limits growth of the field amplitude, whereas the beam remains linear (modulation of the density at the field maximum is weak: $|\delta n_b|\ll n_b$).

The last case is relatively simple to investigate because there is no need to model a beam numerically.[10] The range of validity of an analytic solution given below is limited by the inequalities

$$\nu^{1/3}/\gamma_0\ll|\varepsilon|\ll\nu^{2/3}, \tag{4}$$

the first of which specifies that the increment δ should be small compared with the maximum value, whereas the second allows us to neglect the trapping of beam electrons by a wave and to assume that the electron beam is linear. Since the correction to the frequency given by Eq. (2) is pure imaginary, an increase in the field amplitude is not accompanied by a linear phase drift. Therefore, near the amplitude extrema a nonlinear "dephasing" takes place as a result of which the electron beam goes over abruptly (compared with a small parameter $\mu=\delta/|\varepsilon|\ll1$) to a state which is in antiphase with the wave and the sign of the derivative of the field amplitude is reversed.[2)]

The inequalities of Eq. (4) are compatible in the ultrarelativistic range of the beam energy densities characterized by $\alpha\gg1$, and the energy density carried by high-power regular Langmuir pulses of frequency $\omega_m=\omega_p(1+\nu^{2/3})^{-1/2}$ reaches the value $\nu^{2/3}n_pmc^2$ and is independent of the beam energy.

2. SYSTEM OF NONLINEAR EQUATIONS

It follows from Ref. 6 that an instability of an ultrarelativistic monoenergetic beam $\nu^{1/3}\gamma_0\gg1$ in a linear cold plasma is described by a system of nonlinear equations for the complex field amplitude $E(t)$ and the beam velocity $v(t)$:

$$\frac{2i}{\omega_m}\frac{d^3E}{dt^3}+\frac{d^2}{dt^2}\left[\left(\omega_p^2-\omega_m^2\frac{v^2}{v_0^2}\right)E\right]-\frac{\omega_b^2}{\gamma^3}E=0,$$
$$\gamma=\gamma_0-\frac{|E|^2}{4\pi n_b mc^2},\quad \gamma_0=\left(1-\frac{v^2}{c^2}\right)^{-1/2},\quad \omega_b^2=\frac{4\pi e^2n_b}{m}, \tag{5}$$

where $v_0=v(0)$ and $\gamma_0=\gamma(0)$. Slowing down of the beam is accompanied by a reduction in the ratio $(v/v_0)^2$ and, consequently, the beam nonlinearity does not suppress the instability. Therefore, it is necessary to allow for the plasma nonlinearity, i.e., for the dependence of the plasma frequency on the wave field amplitude.

The field amplitude is governed by a system of nonlinear hydrodynamic equations of motion of a plasma whose solution for an arbitrary electron beam density should be sought in the self-consistent approximation in the form of a wave

$$E(t,\ z)=\operatorname{Re}E(t)\exp[i\omega(t-z/c)]. \tag{6}$$

However, in the case of a low-density beam when the increment is small compared with the plasma frequency, the problem simplifies greatly because the beam makes contribution only to the increment and the waveguide properties of a beam–plasma system are governed mainly by the plasma. This makes it possible to ignore the presence of the beam in the determination of the nonlinear correction to the plasma frequency and then use Eq. (5) replacing in it the plasma frequency ω_p with its field-dependent value $\omega_p(E)$. In the case of waves of the $E(t-z/c)$ type the system of equations of motion of the plasma electrons and the Poisson equation lead to the following nonlinear equation

$$\Phi''+{}^1/_2\omega_p^2[(1-\Phi)^{-2}-1]=0, \tag{7}$$

where $\Phi=eE/mc\omega_p$, and a prime denotes a derivative with respect to the total argument.

In the case of small amplitudes $|\Phi|\ll1$ the nonlinear oscillations described by Eq. (7) are nearly harmonic and their frequency is[11]

$$\omega_p^2(E)=\omega_p^2(1-{}^3/_8|\Phi|^2). \tag{8}$$

It should be noted that the above expression is identical with the analogous formula in Ref. 7 where an allowance is made only for the relativistic correction to the plasma electron mass in the field of a wave.

Using Eq. (8) and expressing in Eq. (5) the beam velocity in terms of the field amplitude, we obtain a nonlinear equation

$$\frac{2i}{\omega_m}\frac{d^3E}{dt^3}-\frac{d^2}{dt^2}\left\{\left[\varepsilon+\frac{|E|^2}{E_p^2}\left(1-\frac{1}{\nu\gamma_0^3}\right)\right]E\right\}-\frac{\omega_b^2}{\gamma_0^3}E=0,$$
$$\varepsilon=1-\omega_p^2/\omega_m^2,\ E_p^2=(32\pi/3)n_pmc^2,\ \nu=n_b/n_p. \tag{9}$$

In the ultrarelativistic limit $\nu^{1/3}\gamma_0\gg1$ considered here the contribution of the electron beam to the nonlinear term of Eq. (9) represents a small correction to unity and the corresponding term can be omitted. For the mode with the maximum increment (2) in terms of dimensionless variables

$$y=\frac{E}{E_m},\quad E_m^2=\frac{\nu^{1/3}}{\gamma_0}E_p^2,\quad x=\frac{\nu^{1/3}}{\gamma_0}\omega_pt \tag{10}$$

Eq. (9) has no algebraic parameter and can be integrated only numerically.[12] The field amplitude varies from the initial value to E_m with a period of the order of several reciprocal increments. Hence, we obtain the second estimate of Eq. (1).

3. RELAXATION OSCILLATIONS

The presence of external modulation of frequency $\omega_m<\omega_p$, satisfying the condition $|\varepsilon|\gg\nu^{1/3}\gamma_0^{-1}$, alters drastically the nature of the instability, because in accordance with Eq. (2) the frequency correction becomes purely imaginary and the increase in the amplitude is not accompanied by a linear drift of the field phase.

Bearing in mind the presence of an additional, compared with Eq. (10), parameter $|\varepsilon|$, we shall represent Eq. (9) in the form

$$
\begin{gathered}
i\mu y'''+[(1-|y|^2)y]''-y=0,\\
y=E/E_m,\qquad E_m^2=|\varepsilon|E_p^2,\qquad x=\delta t,\\
\mu=\frac{2}{|\varepsilon|}\frac{\delta}{\omega_m}\approx 2\left(\frac{\nu}{\gamma_0^3|\varepsilon|^3}\right)^{1/2}\ll 1,
\end{gathered}
\tag{11}
$$

where $\delta=\omega_b/\gamma_0^{3/2}|\varepsilon|^{1/2}$ is the increment predicted by the linear theory.

The term containing the highest-order derivative is proportional to the small parameter μ and it is important only near extremal points of the amplitude (see Sec. 4). Omitting this term, we obtain a real second-order equation which is integrable in quadratures:

$$
\begin{gathered}
\int_{y_0}^{y}\frac{(1-3y^2)\,dy}{[(y^2-y_-^2)(y_+^2-y^2)]^{1/2}}=\pm\left(\frac{3}{2}\right)^{1/2}x,\\
y_\pm^2={}^1/_3[1\pm(1+6C)^{1/2}],\qquad C=[y_0'(1-3y_0^2)]^2-y_0^2(1-{}^3/_2y_0^2),
\end{gathered}
\tag{12}
$$

where y_0 and y_0' are the initial perturbations of a function and its derivative.

Since Eq. (11) corresponding to $\mu=0$ has singularities $y_c^2=1/3$, corresponding to a maximum of the electrical induction of a plasma, it follows that plotting the solution along the whole axis we must alter the sign of the derivative on passing through a singularity, which is equivalent to the replacement of (+) with (−) or vice versa on the right-hand side of Eq. (12).

The nature of the solution depends stongly on the constant C. If $C=0$, Eq. (12) has a solution of the soliton type (Fig. 2a):

$$
-\operatorname{Arch}\,(2/3y^2)^{1/2}+2(1-{}^3/_2y^2)^{1/2}=\pm x+C_1.
\tag{13}
$$

If $C>0$, the solution is periodic and has an alternating sign (Fig. 2b):

$$
\begin{gathered}
(1-3y_-^2)F(\varphi,k_1)-3y_-^2\left(1-\frac{y_-^2}{y_+^2}\right)\left[E(\varphi,k_1)-\frac{k_1^2\sin 2\varphi}{2\Delta(\varphi,k_1)}\right]\\
=\pm\left(\frac{3}{2}\right)^{1/2}\frac{y_+}{k_1}+C_2,\\
\frac{y_-^2}{y^2}=-\frac{\Delta(\varphi,k_1)}{k_1^2\sin^2\varphi},\qquad k_1^2=\frac{y_+^2}{y_+^2-y_-^2}.
\end{gathered}
\tag{14}
$$

If $C<0$, there is a constant-sign periodic solution (Fig. 2c):

$$
\begin{gathered}
F(\varphi,k_2)-3\frac{y_-^2}{y_+^2}\left[E(\varphi,k_2)-\frac{k_2^2\sin 2\varphi}{2\Delta(\varphi,k_2)}\right]=\pm\left(\frac{3}{2}\right)^{1/2}y_+x+C_3,\\
y_-^2/y^2=\Delta(\varphi,k_2),\qquad k_2^2=(y_+^2-y_-^2)/y_+^2.
\end{gathered}
\tag{15}
$$

In Eqs. (14) and (15) the quantities $F(\varphi,k)$ and $E(\varphi,k)$ are elliptic integrals of the first and second kind,[13] and $\Delta(\varphi,k)=1-k^2\sin^2\varphi$. The integration constants C_1, C_2, and C_3 are determined by the initial conditions or by the continuity of the function $y(x)$ at singularities.

It should be noted that retention of the sign of the derivative on passing through a singularity makes it possible to plot smooth (free of discontinuities of the derivatives) solutions with the maximum amplitude $y^2_{\max}=\frac{2}{3}$. However, these solutions do not satisfy the initial equation. This can be demonstrated by integrating Eq. (11) between the limits 0 and ∞, and then adopting zero initial conditions.[3)]

4. NONLINEAR PHASE DYNAMICS NEAR AMPLITUDE SINGULARITIES

Near amplitude singularities the derivatives of the function $y(x)$ increase and it is necessary to allow for the term $i\mu y'''$ in Eq. (11), because this term is no longer small. The appearance of an imaginary correction is an indication of a phase shift, because the field amplitude becomes complex.

In an investigation of the phase dynamics near singularities it is convenient to use a system of equations for the field amplitude in a plasma and a variable component of the beam

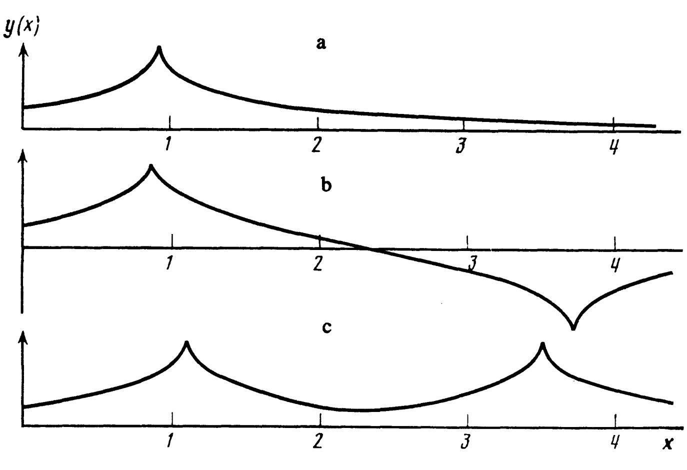

FIG. 2. Graphs of the function $y(x)$: a) $C=0$; b) $C=0.013$; c) $C=-0.013$. The initial value of the function is $y_0=0.15$, and the maxima and minima correspond to $\pm 3^{-1/2}$.

density $\tilde{n}_b$:

$$-\mu y'+i(1-|y|^2)y=\tilde{N},$$
$$\tilde{N}''=iy,\quad \tilde{N}=4\pi e v_0 \tilde{n}_b/\omega_m|\varepsilon|E_m, \tag{16}$$

which is equivalent to Eq. (11). Substituting in Eq. (16) the expressions $y = iae^{-i\theta}$ and $\tilde{N} = Ne^{-i\varphi}$, we obtain a system of nonlinear equations for the amplitudes and phases:

$$\mu a'=-N\sin(\theta-\varphi),\quad \varphi''N+2\varphi'N'=-a\sin(\theta-\varphi),$$
$$\mu a\theta'=-a(1-a^2)-N\cos(\theta-\varphi), \tag{17}$$
$$N''-\varphi'^2N=-a\cos(\theta-\varphi).$$

The first two equations of the system (17) yield an integral

$$\varphi'=\frac{\mu}{2}\frac{a^2}{N^2}, \tag{18}$$

which allows us to drop from the equations the terms originating from the derivative of the beam phase and small compared with the others in the ratio μ^2. In this approximation the system (17) has another integral

$$N'^2=-a^2(1-a^2/2)-2aN\cos\eta+C,$$
$$\eta=\theta-\varphi. \tag{19}$$

Using Eq. (19) and eliminating the variable N by means of the third equation in the system (17), we obtain

$$\left[\frac{a(1-a^2)+\mu a\eta'}{\cos\eta}\right]'$$
$$=\left[C+a^2\left(1-\frac{3}{2}a^2\right)+2\mu a^2\eta'\right]^{1/2}, \tag{20}$$
$$\mu a'=[a(1-a^2)+\mu a\eta']\operatorname{tg}\eta.$$

The system (20) generalizes Eq. (12) to the case $\mu > 0$, and the integration constants are the same in both cases.

Far from singularities the determining factor is the growth of their amplitude, whereas the phase plays a secondary role and is governed by the second equation of the system (20). Conversely, near singularities the phase motion predominates and it determines the behavior of the derivatives when the field amplitude remains actually constant.

We shall now consider the solution of the system (20) in the vicinity of a singularity $y_c = 3^{-1/2}$ where derivatives of the function $y(x)$ become infinite for $\mu = 0$. Assuming that $a_1 = a - a_c$ and $\xi = x - x_c$ $(a_1 \ll a_c, |\xi| \ll x_c)$, and also that $|\eta| \ll 1$, we shall reduce the system (20) to the form

$$(\mu\eta'-3a_1^2+{}^1/_3\eta^2)'=({}^1/_2+2\mu\eta')^{1/2},$$
$$\mu a_1'=-3^{-1/2}({}^2/_3+\mu\eta')\eta. \tag{21}$$

The phase $\eta(\xi)$ reaches its maximum at the point $-\xi\approx\mu$, where η' changes its sign. Since in the $\eta' > 0$ case we have $\mu\eta' \ll 1$, the system (21) simplifies to

$$\mu\eta'=3a_1^2-{}^1/_3\eta^2+2^{-1/2}\xi,\quad \mu a_1'=-2\cdot3^{-3/2}\eta. \tag{22}$$

Sufficiently far from a singularity $|\xi| \gg \mu^{4/5}$ it follows from Eq. (22) that

$$a_1^2=-\xi/3\sqrt{2}, \tag{23}$$

which corresponds to the approximation of Eq. (12). The formula (23) describes more accurately the nature of a singu-

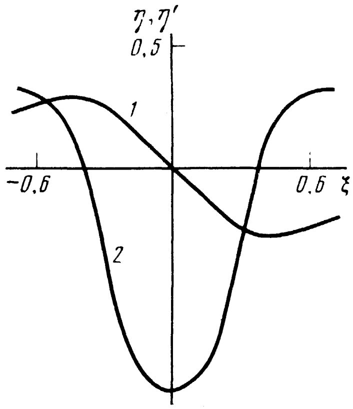

FIG. 3. Graph showing the function $\eta(\xi)$ (curve 1) and its derivative $\eta'(\xi)$ (curve 2) near an amplitude maximum corresponding to $\mu = \frac{1}{4}$.

larity of Eq. (11) at $\mu = 0$: we have $a' \approx |\xi|^{-1/2}$ in the limit $|\xi| \to 0$.

It follows from the formulas (22) and (23) that the term $\mu\eta'$ becomes comparable with a_1^2 when $|\xi| \gtrsim \mu^{4/5}$, and that beginning from this point the increase in the phase slows down the growth of the amplitude. It is clear from the first expression in the system (22) that a further shift of the field phase relative to the beam alters the sign of η' when $a_1^2 \approx \eta_m^2 \approx |\xi_m| \approx \mu$. This estimate can be obtained by introducing a new scale $\xi_1 = \xi/\mu$ into the system (22).

In the region $|\xi| < |\xi_m|$, where ξ_m corresponds to the maximum phase η_m, the phase rapidly decreases and at the point $\xi = 0$ the system (21) has a solution

$$a_1=a_1'=\eta=0,\quad \eta'=-1/4\mu. \tag{24}$$

Therefore, it is clear that near the zero point the equations in the system (21) are dominated by the terms proportional to η':

$$\eta'=\frac{9\mu}{4\eta^2}\left[1-\left(1+\frac{2\eta^2}{9\mu^2}\right)^{1/2}\right]. \tag{25}$$

Integrating Eq. (25), we obtain

$$\eta\left[2+\left(1+\frac{2\eta^2}{9\mu^2}\right)^{1/2}\right]+\frac{3\mu}{2^{1/2}}\operatorname{Arsh}\left(\frac{2^{1/2}\eta}{3\mu}\right)=-\frac{\xi}{\mu}. \tag{26}$$

If $|\xi| \ll 4\mu^2$, then Eq. (26) becomes identical with Eq. (24) and in the region $|\xi| \gg 4\mu^2$ we have $\eta^2 = 3|\xi|/\sqrt{2}$, which is of the same order of magnitude as the value given by Eq. (22) when $|\xi| \approx \mu$.

It follows from the second expression in Eq. (22) that

$$a(\xi)=a_c(1-\xi^2/12\mu^2),\quad |\xi|\ll4\mu^2, \tag{27}$$

and, consequently, the field amplitude reaches its maximum when $\xi = 0$. The kinks in the curves in Fig. 2 therefore coincide with maxima or minima of the function $y(x)$.

The nonlinear variation of the field phase and of its derivative near an amplitude singularity are illustrated in Fig. 3.

5. DISCUSSION OF RESULTS

We shall conclude by considering the physical mechanism of nonlinear stabilization of the instability and of the appearance of phase discontinuities in a plasma with an ultrarelativistic electron beam. The effect can be followed qualitatively even in the case of the initial formulas (8) and (9), and it is related to a nonlinear reduction in the plasma frequency because of the relativistic increase in the electron mass and because of a change in the waveguide properties of the plasma. However, even at the field amplitude maximum the plasma permittivity retains its sign at the modulation frequency $\varepsilon(E) = \varepsilon + |E|^2/E_p^2$ and the nonlinear saturation of the amplitude cannot be explained by a simple change in the sign of $\delta^2 = -\omega_b^2/\gamma_0^3\varepsilon(E)$ (by vanishing of the increment). In reality, this process is more complex because of the nonlocal nature of the nonlinearity [the nonlinear term occurs in Eq. (9) in the second derivative] and it is manifested by an increase in the phase velocity of a wave $v_{ph} = v_0(1 - \delta\theta'/\omega_m)$ near extremal points of the field amplitude. During the nonlinear stage of the instability we have $\theta'_L = \mu/2$ and a retarded wave with $v_{ph} < v_0$, amplified by the beam under the Cherenkov effect conditions, forms in a plasma. However, on increase in the amplitude the value of θ' decreases and vanishes, and then it becomes negative in the region $|\xi| \lesssim \mu$. Correspondingly, the wave becomes fast ($v_{ph} > v_0$) and interacts weakly with the beam. Next, when θ' changes the sign again, the beam is already in antiphase with the retarded wave and the Cherenkov deceleration changes to acceleration and the energy density of oscillations in the plasma decreases. The process is repeated periodically as a function of the initial conditions.

In the optimal modulation case the energy density of oscillations $\nu^{2/3}n_p mc^2$ is independent of the beam energy and, consequently, the efficiency of beam–plasma systems can be increased further by increasing the beam density and not its energy.

[1)]Our analysis presupposes that up to the moment of development of an instability the electron beam is modulated uniformly over its length and the external modulating signal acts as the initial perturbation. However, if the modulation takes place at the entry of the beam to the plasma, then the perturbation evolves in space rather than in time. The problem formulated in this way for a nonrelativistic beam injected continuously into a plasma half-space is solved in Ref. 8.

[2)]In the equation for the complex field amplitude there is a small parameter in front of the highest-order derivative, which is typical of relaxation processes.[11] The possibility of appearance of relaxation oscillations in beam–plasma systems was pointed out by Ya. B. Faĭnberg in 1967.

[3)]This property of the equation was drawn to our attention by L. I. Rudakov.

[1]Ya. B. Faĭnberg, Fiz. Plazmy **3**, 442 (1977) [Sov. J. Plasma Phys. **3**, 246 (1977)].

[2]L. I. Rudakov, Zh. Eksp. Teor. Fiz. **59**, 2091 (1970) [Sov. Phys. JETP **32**, 1134 (1971)].

[3]B. N. Breĭzman and D. D. Ryutov, Zh. Eksp. Teor. Fiz. **60**, 408 (1971) [Sov. Phys. JETP **33**, 220 (1971)].

[4]A. A. Ivanov, Fizika sil'noneravnovesnoĭ plazmy (Physics of a Strongly Nonequilibrium Plasma), Atomizdat, M., 1977, p. 293.

[5]Ya. B. Faĭnberg, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **57**, 966 (1969) [Sov. Phys. JETP **30**, 528 (1970)].

[6]V. B. Krasovitskiĭ, Zh. Eksp. Teor. Fiz. **66**, 154 (1974) [Sov. Phys. JETP **39**, 71 (1974)].

[7]V. B. Krasovitskiĭ, Nonlinear Collective Oscillations in Nonequilibrium Systems: Doctoral Thesis, Moscow State University, 1980.

[8]V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Eksp. Teor. Fiz. **51**, 445 (1966) [Sov. Phys. JETP **24**, 300 (1967)].

[9]V. B. Krasovitskiĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **13**, 1902 (1970); Pis'ma Zh. Eksp. Teor. Fiz. **15**, 348 (1972) [JETP Lett. **15**, 244 (1972)].

[10]N. G. Matsiborko, I. N. Onishchenko, Ya. B. Faĭnberg, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **63**, 874 (1972) [Sov. Phys. JETP **36**, 460 (1973)].

[11]N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ, Fizmatgiz, M., 1963, p. 107 (Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York, 1964).

[12]V. I. Kurilko, A. P. Tolstoluzhskiĭ, and Ya. B. Faĭnberg, Zh. Tekh. Fiz. **44**, 985 (1974) [Sov. Phys. Tech. Phys. **19**, 622 (1974)].

[13]E. Jahnke, F. Emde, and F. Lösch, Tables of Higher Functions, McGraw-Hill, New York, 1960 (Russ. Transl. Nauka, M., 1964, p. 94).

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