Waveguide type solutions for light beams with nonlocal self-action in the geometrical-optics approximation

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An equation having the form of the kinetic equation is derived for the Wigner function in the geometrical-optics approximation from a quasi-optics-type equation with nonlocal nonlinearity. The method proposed by Korobkin and Sazonov [Sov. Phys. JETP **54**, 636 (1981)] for finding exact solutions is generalized for this equation. As an example, the model of stationary thermal self-action of light in a medium at rest is considered. Exact solutions, which describe the propagation of light beams without self-focusing, are found in these models for certain types of beams.

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1. INTRODUCTION

There are at present no methods that allow an exact analytical description of the propagation of wave beams in nonlinear media, and therefore various approximate approaches to this problem are of interest. In Ref. 1 a procedure using the Wigner function is proposed for investigating a Schrödinger-type equation with a cubic local nonlinearity, and a class of exact solutions describing the ray distribution function of cylindrically symmetric beams propagating without self-focusing in a nonlinear medium is found in the geometrical-optics approximation. It is shown that, in principle, the beams can propagate without self-focusing even when the intensity is higher than critical.

We use the method developed in that paper to analyze equations with nonlocal nonlinearity. Such equations arise in different areas of physics and, in particular, in nonlinear optics and in plasma physics: the self-consistent field equations, the equations with unitary nonlinearity, etc.[2,3]

A fairly general scheme for describing the self-action of light beams in nonlinear media is presented in Ref. 4. The propagation of light is described in the quasi-optical approximation, the refractive index being assumed to be a function of the temperature and density. These parameters satisfy the appropriate material equations with a source which, in the case of the heat equation, is the beam intensity. Thus, for example, the model of stationary thermal self-action of light in a medium at rest is described by a constitutive equation of the form

$$-\Delta T=\alpha|E|^2;\qquad \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

The field E of the light beam satisfies the equation

$$i\frac{\partial E}{\partial z}+\frac{1}{2k}\Delta E+n(T)E=0.$$

We consider the case of self-focusing. The nonlocality of the self-action evidently manifests itself in the fact that the temperature is given by the integral of the square of the modulus of the field, $|E|^2$. In contrast to the local-nonlinearity case considered in Ref. 1, for which it has been possible to solve in the geometrical-optics approximation the equation determining the Wigner function for any initial intensity distribution in the beam, in the nonlocal-nonlinearity case the outcome of the search for an analytic expression for the solution depends most essentially on the form of the initial intensity distribution in the beam. Here we can perceive some analogy, albeit a fairly remote one, with the scheme for the inverse-problem method.[5]

In Sec. 2 we expound, following Ref. 1 as closely as possible, and referring the reader to it for details, a scheme for obtaining an equation for the Wigner function in the geometrical-optics approximation, and set forth the kinetic analogy. We also consider there the changes that the nonlocal nonlinearity introduces into the scheme for obtaining the exact solutions in this case. In Sec. 3 we find in the model of stationary thermal self-action in a medium at rest the exact solutions for certain possible forms of the initial intensity distribution in the beam.

2. DERIVATIVE OF THE EQUATION. SCHEME FOR OBTAINING THE SOLUTIONS

We shall consider equations of the form

$$i\frac{\partial E}{\partial z}+\frac{1}{2k}\Delta E+kE\int G(\mathbf{x},\mathbf{y})|E|^2(z,\mathbf{y})\,d^2\mathbf{y}=0,\tag{1}$$

where k is the wave number and $E=E(z,\mathbf{x})$. Let us introduce the Wigner function in the form

$$W(z,\mathbf{x},\mathbf{s})=\int d^2\zeta e^{-ik\mathbf{s}\zeta}E(z,\mathbf{x}+\zeta/2)\,E^*(z,\mathbf{x}-\zeta/2).\tag{2}$$

As is easy to see, it follows from (2) that

$$|E^2(z,\mathbf{x})|=(2\pi)^{-1}k^2\int d^2\mathbf{s}W(z,\mathbf{x},\mathbf{s}).\tag{3}$$

From (1) to (2) we easily obtain an equation for the Wigner function:

$$\frac{\partial W}{\partial z}+\mathbf{s}\frac{\partial W}{\partial \mathbf{x}}=\frac{ik^3}{(2\pi)^4}\int W(z,\mathbf{x},\mathbf{s}+\mathbf{s}_2)\,W(z,\mathbf{y},\mathbf{s}_1)$$
$$\times[G(\mathbf{x}+\zeta/2;\ \mathbf{y})-G^*(\mathbf{x}-\zeta/2;\mathbf{y})]\,e^{ik\zeta\mathbf{s}_2}\,d^2\zeta\,d^2\mathbf{s}_1\,d^2\mathbf{s}_2\,d^2\mathbf{y}.$$

Expanding W and G in powers of $\mathbf{s}_2$ and ζ, we obtain in the case when $G=G^*$ the equation

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$$\frac{\partial W}{\partial z}+\mathbf{s}\frac{\partial W}{\partial \mathbf{x}}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}k^{-2n}(2\pi)^{-2}}{4^n(2n+1)!}\frac{\partial^{2n+1}W}{\partial \mathbf{s}^{2n+1}}\frac{\partial^{2n+1}}{\partial \mathbf{x}^{2n+1}}$$
$$\times\int G(\mathbf{x},\mathbf{y})W(z,\mathbf{y},\mathbf{s}_1)d^2\mathbf{y}d^2\mathbf{s}_1.$$

If $G=-G^*$, then only derivatives of even order remain on the righthand side of the equation. We shall consider the $G=G^*$ case.

Retaining only the first (i.e., $n=0$) term on the righthand side, we obtain

$$\frac{\partial W}{\partial z}+\mathbf{s}\frac{\partial W}{\partial \mathbf{x}}+\frac{\partial W}{\partial \mathbf{s}}\frac{\partial}{\partial \mathbf{x}}\int G(\mathbf{x},\mathbf{y})W(z,\mathbf{y},\mathbf{s}_1)\frac{d^2\mathbf{y}d^2\mathbf{s}_1}{(2\pi)^2}=0. \quad (4)$$

This approximation implies the consideration of the diffusive divergence and the neglect of the diffraction effects at the aperture of the beam. Equation (4) has the form of the kinetic equation, and the function

$$f(z,\mathbf{x},\theta,\psi)=(2\pi)^{-2}k^2W(z,\mathbf{x},\mathbf{s}(\theta,\psi))\cos^{-3}\theta$$

has the meaning of the distribution function for the number of rays emanating from the point $\mathbf{x}$. We shall seek the solution to Eq. (4) with the condition $\partial W/\partial z=0$ and under the assumption that the system is cylindrically symmetric. Let us introduce the polar coordinates $\mathbf{x}=(\rho,\varphi)$ in the xy plane. Then the vector $\mathbf{s}$ has the components

$$s_{x_1}=u\cos\varphi-v\sin\varphi,\qquad s_{y_1}=u\sin\varphi+v\cos\varphi.$$

Here u and v are the radial and azimuthal components of the vector $\mathbf{s}$. Let us note that the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{s}$, and $\boldsymbol{\zeta}$ lie in the xy plane.

Under these assumptions Eq. (4) has the form

$$u\frac{\partial W}{\partial \rho}+\frac{\partial W}{\partial u}\frac{\partial}{\partial \rho}\int G(\rho,\mathbf{y})W(\mathbf{y},u,v)d^2\mathbf{y}\frac{du dv}{(2\pi)^2}=0. \quad (5)$$

Let us now introduce the function

$$Q(\rho)=(2\pi)^{-2}\int G(\rho,\mathbf{y})W(\mathbf{y},u,v)d^2\mathbf{y}\,du\,dv. \quad (6)$$

Now the solution to Eq. (5) can be represented as

$$W(\rho,u,v)=\Phi(Q(\rho)-u^2/2;\,v),$$

where we have imposed on the functions Q and Φ the self-consistency condition

$$Q(\rho)=(2\pi)^{-1}\int G(\rho,r)\Phi\left(Q(r)-\frac{u^2}{2};\,v\right)r dr du dv. \quad (7)$$

To simplify the calculations, let us introduce the functions

$$F(x)=\int \Phi(x,v)dv, \quad (8)$$

$$I(\rho)=(2\pi)^{-1}\int F\left(Q(\rho)-\frac{u^2}{2}\right)du. \quad (9)$$

We shall assume that the function $Q(\rho)$ decreases at the same time as $|E|^2(\rho)$ decreases. In the case of local nonlinearity $Q=|E|^2$, and this requirement is automatically fulfilled. It is also fulfilled in the case of thermal self-focusing, so that our assumption in fact implies the separating out of the self-focusing effect for Eq. (1). Let us also note that if $|E|^2(\rho)=0$ for $\rho\geqslant\rho_0$, then $W(\rho,\mathbf{s})=0$ for $\rho\geqslant\rho_0$ and all $\mathbf{s}$. Let $Q_0=\lim Q(\rho)$ for $\rho\to\rho_0$, where ρ_0 can be either finite or infinite. Then it is not difficult to deduce from (9) that

$$I(\rho)=J(Q(\rho))=\frac{\sqrt{2}}{2\pi}\int_{Q_0}^{Q(\rho)}\frac{F(x)dx}{[Q(\rho)-x]^{1/2}}. \quad (10)$$

The case considered in Ref. 1 corresponds to the situation in which $G(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})$, which implies, on the basis of (7), that $I=Q$, and thus leads to the transformation of the relation (10) into the easily soluble Abel equation. This is not so in the more general case. It follows from (7) and (10) that

$$Q(\rho)=\int G(\rho,r)I(r)r dr. \quad (11)$$

The relation (10) implies that I is a function (not a functional, but a function) of Q, and if we can determine this function, i.e., find J such that $I(\rho)=J[Q(\rho)]$, then, we can, by inverting the Abel transformation (10), determine the function

$$F(x)=\frac{\sqrt{2}}{2\pi^2}\int_{Q_0}^{x}\frac{J'(Q)dQ}{(x-Q)^{1/2}}. \quad (12)$$

On the other hand, going over in (12) to integration over ρ, where $\rho=\rho(Q)$ is determined, for example, from (11), we obtain

$$F(x)=\frac{\sqrt{2}}{2\pi^2}\int_{\rho_0}^{\rho(x)}\frac{I'(\rho)d\rho}{[x-Q(\rho)]^{1/2}};\quad I'(\rho)=\frac{dI}{d\rho}. \quad (13)$$

Thus, if we are able to invert the dependence $Q=Q(\rho)$, then the relation (13) gives us the function $F(x)$. After the function F has been found, the condition (8) gives us a class of functions Φ satisfying Eq. (5) exactly.

We shall not consider here the general problems connected with the inversion of the dependence $Q(\rho)$; instead, we shall proceed to the consideration of a specific model for the nonlinear medium. Let us only point out some similarity to the scheme for the inverse problem method: the solution of Eq. (5) amounts, for a given intensity distribution $I(\rho)$ (the "initial data"), to the solution of the inverse problem—the determination of $\rho=\rho(Q)$; further, the inversion of the Abel transformation furnishes the function F from, for example, (13); and, finally, the solution of Eq. (8) yields the sought function Φ, in the present case a class of functions.

3. THERMAL SELF-FOCUSING. EXACT SOLUTIONS

Let us now consider the thermal self-focusing model, and give exact solutions for some specific beams, i.e., solutions exactly satisfying Eq. (5). The main difficulty is the determination of the function F. The selection of the function Φ from the condition (8) is quite a free one, and we shall, for convenience of comparison with Ref. 1, decide on the same type of functions.

The model of stationary thermal self-action in a medium at rest is, as has already been noted, described by the following equations: the quasi-optical equation

$$i\frac{\partial E}{\partial z}+\frac{1}{2k}\Delta E+kTE=0$$

and the material equation

$$-\Delta T=\alpha|E|^2. \quad (14)$$

It is clear that we can, using the Green function of Eq. (14), easily reduce this system to the form considered above:

$$i\frac{\partial E}{\partial z}+\frac{1}{2k}\Delta E-\frac{1}{2\pi}\alpha kE\int \ln|\mathbf{x}-\mathbf{y}||E|^2(z,\mathbf{y})\,d^2\mathbf{y}=0.$$

Let us recall that we shall seek the solutions possessing cylindrical symmetry, and satisfying the condition $\partial W/\partial z=0$. Let us consider some possible intensity distribution in the beam.

1. Let the cylindrical beam have an intensity distribution of the form

$$|E|^2(\rho)=I_0\chi(a-\rho);\quad I_0=\frac{8\pi}{c}I;\quad \chi(x)=\begin{cases}1;\ x\geqslant 0\\ 0;\ x<0\end{cases}. \tag{15}$$

Integrating Eq. (14), we obtain the following expression for the temperature:

$$T(\rho)=T(a)+\frac{\alpha I_0}{4}(a^2-\rho^2)\chi(a-\rho)+\frac{\alpha I_0}{2}a^2\chi(\rho-a)\ln\frac{a}{\rho}. \tag{16}$$

Here $T(a)$ is the temperature at the beam boundary. It follows from (3) and (9) that $|E|^2=k^2I$, and we find from (6) and (16) that $T=k^2Q$. The relation (13) then gives the function F:

$$F(x)=\frac{\alpha I_0\sqrt{2}}{2\pi^2k^2}[x-k^{-2}T(a)]^{-1/2}\chi(x-k^{-2}T(a)).$$

It is clear that only the special form of the initial intensity distribution (15) in the beam allows us to avoid considering the inversion of the dependence (16). The class of functions Φ satisfying Eq. (8) is now known. We shall, for convenience of comparison with Ref. 1, discuss the same type of functions discussed in that paper.

1a. Solutions of the "fan" type. In this case we have

$$\Phi(x,v)=\frac{\alpha I_0\sqrt{2}}{2\pi^2k^2}[x-k^{-2}T(a)]^{-1/2}\chi(x-k^{-2}T(a))\delta(v),$$

$$f(\rho,\varphi,\theta,\psi)=\frac{\alpha I_0\sqrt{2}\delta(\sin(\psi-\varphi))}{\cos^3\theta}\times\left[k^{-2}(T(\rho)-T(a))-\frac{\operatorname{tg}^2\theta}{2}\right]^{-1/2}\times\chi\left[k^{-2}(T(\rho)-T(a))-\frac{\operatorname{tg}^2\theta}{2}\right].$$

Here, as in Ref. 1, all the rays emanating from an arbitrary point $\mathbf{x}$ in the xy plane belong to the plane passing through this point and the z axis. These rays make with the z axis an angle greater than

$$\theta_m(\rho)=\operatorname{arctg}\,[2k^{-2}(T(\rho)-T(a))]^{1/2}.$$

Here θ and ψ are the polar and azimuthal angles of a ray emanating from the point $\mathbf{x}$.

1b. Solution of the "bouquet" type. For this solution, the function Φ has the form

$$\Phi(x,v)=\frac{\alpha I_0}{k^2}\chi\left[(x-k^{-2}T(a))^{-1}-\frac{v^2}{2}\right],$$

$$f(\rho,\varphi,\theta,\psi)=\frac{\alpha I_0\cos^{-3}\theta}{(2\pi)^2}\chi\left[\left[k^2(T(\rho)-T(a))-\frac{\operatorname{tg}^2\theta\cos^2\psi}{2}\right]^{-1}-\frac{\operatorname{tg}^2\theta\sin^2\psi}{2}\right].$$

The distribution of the rays in this case differs radically from the distribution of the type 1a and the "bouquet"-type distribution considered in Ref. 1. It should be noted that the "fan"-type distribution obtained in that paper also differs greatly from ours.

2. Let us now consider an initial intensity distribution of the following form:

$$|E|^2(\rho)=I_1\chi(a-\rho)+I_2\chi(\rho-b)\chi(d-\rho);\quad a<b<d.$$

This distribution indicates that the cylindrical beam propagates in a circular beam. Proceeding as in Subsec. 1, we easily find for the function F the expression

$$F(x)=\frac{\sqrt{2}}{2\pi^2k^2}\{\alpha I_1[x-k^{-2}T(a)]^{-1/2}\chi(x-k^{-2}T(a))+\alpha I_2\times[x-k^{-2}T(b)]^{-1/2}\chi[x-k^{-2}T(b)]+\alpha I_2[x-k^{-2}T(d)]^{-1/2}\chi[x-k^{-2}T(d)]\}.$$

An increase in the number of external circular beams leads to the appearance of a corresponding number of additional terms in the expression for the function F. The expression for the function Φ can have, besides the above-considered forms, a mixed "fan-bouquet" form:

$$\Phi(x,v)=\frac{\alpha I_1\sqrt{2}}{2\pi^2k^2}[x-k^{-2}T(a)]^{-1/2}\chi[x-k^{-2}T(a)]\delta(v)+\frac{\alpha I_2}{k^2}\left\{\chi\left[[x-k^{-2}T(b)]^{-1}-\frac{v^2}{2}\right]+\chi\left[[x-k^{-2}T(d)]^{-1}-\frac{v^2}{2}\right]\right\}.$$

The function f correspondingly acquires a mixed structure.

3. The third type of initial distribution that we shall consider has the form

$$|E|^2(\rho)=I_0(a^2-\rho^2)\chi(a-\rho).$$

The parabolic intensity distribution in the beam corresponds more to reality than the rectangular distribution. For this case, we find only the function F. Choosing, for the sake of simplicity, $T(a)=0$, where $T(a)$ is the temperature at the beam boundary, we obtain

$$T(\rho)=T_0+\alpha I_0\left(\frac{1}{16}\rho^4-\frac{a^2}{4}\rho^2\right),\quad \rho\leqslant a,\quad T_0=\frac{3a^4}{16}\alpha I_0,$$

$$p(x)=\frac{\rho^2}{4},\quad p(x)=\frac{a^2-[a^4-4(T_0-x)]^{1/2}}{2},$$

$$F(x)=\frac{\alpha I_0\sqrt{2}}{\pi^2k}\chi(x)\arcsin\frac{a^2-2p(x)}{[a^4+4k^2(p(x)-k^{-2}T_0)]^{1/2}}.$$

CONCLUSION

Thus, the main result of the paper is the discovery of the possibility of the mutual canceling out of the self-focusing and diffusion-divergence effects in media with nonlocal self-action, one example of which is the considered model of self-action of light in a medium with stationary thermal nonlinearity.

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