Critical behavior of the correlation function of the Ising model with impurity bonds

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We consider a two-dimensional Ising model with a small impurity bond concentration. At the phase transition point we evaluate exactly the correlation function $\Gamma(R)\equiv\langle\sigma(0)\sigma(R)\rangle$. In contrast to the pure Ising model, where $\Gamma(R)\propto R^{-\zeta}, \zeta=\frac{1}{4}$, the exponent ζ of the correlator turns out to be equal to zero when there are impurity bonds present. At distances $R\gg r_i\propto e^{1/\nu}$, where $\nu\ll 1$ is the impurity bond concentration, $\Gamma(R)$ is a function which varies much more weakly than a power-law function.

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1. INTRODUCTION

The study of the critical properties of weakly disordered systems in the vicinity of the order-disorder phase transition point occupies an important place in the physics of unordered systems. Of special interest are here simple models of magnetic materials when there is some kind of disorder present. It has recently become clear[1] that weakly disordered magnetic systems may possess their own critical behavior differing from that of pure systems. If there are some impurities with a concentration $\nu\ll 1$ present in the magnetic material, there appears close to the phase transition point a temperature scale $\tau_\nu\propto\exp(-\text{const}/\nu)$ such that when $\tau_\nu\ll\tau\ll 1$, $\tau=(T-T_c)/T_c$, one observes the critical regime of the pure system, while for $\tau\ll\tau_\nu$ there may occur a totally different, universal, i.e., ν-independent, critical regime with a critical exponent determined by the symmetry of the initial model, the type of disorder, and so on. This was demonstrated for the first time in Refs. 1 and 2, by the example of a φ^4 model with a random mass.

The two-dimensional ($2D$) Ising model for which there is an exact analytical solution is a very convenient system if one wants to study the effect of impurities on critical behavior. We shall consider the $2D$ Ising model with impurity bonds

$$\mathcal{H}=\frac{1}{2}\sum_{i,\alpha}J_{i,\alpha}\sigma_i\sigma_{i+\alpha}, \tag{1.1}$$

where the variables $\sigma=\pm 1$ are given at the sites of a square lattice ($\alpha=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}$, are the unit vectors of the square lattice, $\mathbf{3}\equiv-\mathbf{1}$, $\mathbf{4}\equiv-\mathbf{2}$) and the bounds $J_{i,\alpha}$ are a random function of the coordinates:

$$J_{i,\alpha}=\begin{cases} J & \text{with probability } 1-\nu \\ \tilde{J} & \text{with probability } \nu \end{cases}, \tag{1.2}$$

$\tilde{J}\neq J$, $\nu\ll 1$. It is well known that near the phase transition point the pure ($\tilde{J}=J$) Ising model is equivalent to a $2D$ field of free massive fermions

$$\mathcal{L}=\int d^2x\,(\bar{\psi}\hat{\partial}\psi+m\bar{\psi}\psi), \tag{1.3}$$

where[3] $m\propto\tau$, i.e., the long-wavelength degrees of freedom of the Ising model which are important near the phase transition and which determine the critical behavior are described by the $2D$ fermion fields ψ. In this sense the fields $\psi(x)$ are near the phase transition point the "relevant" variables of the Ising model in contrast to the initial σ-variables. One can change to the continuous limit (1.3) in various ways, but in out opinion this problem is most simply solved if one starts from a formulation of the Ising model in terms of Grassmann variables[4] (see Sec. 2). In contrast to the Hamilton approach (as, e.g., in Ref. 3) the transition to the continuous limit in terms of Grassmann variables can be called Lagrangian in the sense that in that approach both $2D$ coordinates remain on a par. The transition to the continuous limit itself consists in that in the phase transition point the localized degrees of freedom are distinguished from the long-wavelength ones while the latter are described by the Lagrangian (1.3).

This is the reason why it is possible also to describe the Ising model with impurity bonds. In our earlier work[5] we showed that the presence of impurity bonds leads to the appearance of an interaction between the fermions in (1.3) and in the main order in the concentration ν the system is described by the zero-component Gross-Neveu model[6]:

$$\mathcal{L}=\int d^2x\left\{\sum_{a=1}^{N}\left[(\bar{\psi}^a\hat{\partial}\psi^a)+m(\bar{\psi}^a\psi^a)\right]+g\sum_{a,b=1}^{N}(\bar{\psi}^a\psi^a)(\bar{\psi}^b\psi^b)\right\}, \tag{1.4}$$

where $g=c\nu$ ($c\sim 1$ is a constant which depends on J and $\tilde{J}$) while in the final result one must put $N=0$. This procedure, known by the name replica method,[7] is connected with the fact that the free energy of the Ising model has the form of a sum over single-loop configuration,[8] and since in the expansion of the theory (1.4) each loop is proportional to N the condition $N=0$ leaves in the free energy just the single-loop diagrams. The two-dimensional theory (1.4) is renormalizable and renormalization-group methods[9] enable us to evaluate exactly the critical behavior (similar to how in the four-dimensional case the scalar φ^4 field theory is renormalizable[9,1]). We showed in Ref. 5 that the singularity

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in the specific heat which in the pure Ising model has the form $c \propto \ln(1/\tau)$ becomes weaker when $\tau \ll \tau_\nu$: $c \propto \ln\ln(1/\tau)$.

The aim of the present paper is the calculation of the correlation function $\Gamma(R) = \langle \sigma(0)\sigma(R)\rangle$ at the phase transition point, i.e., the calculation of the second critical exponent of the model. After this the remaining critical exponents are uniquely determined. It is well known that in the pure Ising model $\Gamma(R) \propto R^{-\zeta}$, where $\zeta = \frac{1}{4}$. It turns out that impurities weaken the singularity so much that ζ tends to zero and $\Gamma(R)$ decreases more slowly, logarithmically:

$$\Gamma(R) \sim \exp\left\{-\frac{1}{4\pi g}(\ln\ln R)^2\right\}. \tag{1.5}$$

Our paper is organized as follows. We give in the second section the formulation of the pure Ising model in terms of Grassmann variables and after that we obtain in that language an expression for the correlation function $\Gamma(R) = \langle \sigma(0)\sigma(R)\rangle$. In the third section we demonstrate how one can take in that expression the continuous limit, and after that we obtain (Sec. 4) the well known result $\zeta = \frac{1}{4}$. We show in Sec. 5 how in the spirit of Ref. 5 one can generalize the calculations given here to the case where there are impurity bonds present and we obtain the result (1.5).

2. THE CORRELATION FUNCTION OF THE ISING MODEL IN THE GRASSMANN VARIABLE FORMALISM

The transition to Grassmann variables is performed as follows (see, e.g., Ref. 4). It is well known that the partition function of the Ising model can be written as a sum over closed-contour configurations on the $2D$ lattice:

$$\sum_{\mathscr{P}} \lambda^{L_{\mathscr{P}}} \Phi(\mathscr{P}), \tag{2.1}$$

where $\lambda = \tanh J$, $L_{\mathscr{P}}$ is the length of the contour $\mathscr{P}$, and $\Phi(\mathscr{P})$ is the phase factor of the close contour:

$$\Phi(\mathscr{P}) = \prod_{\mathscr{P}} \exp\left(i\frac{\Delta\varphi}{2}\right); \tag{2.2}$$

here $\Delta\varphi$ is the angle through which the tangent vector has rotated along the path.[8] Thanks to the phase factor (2.2) each closed contour enters with a weight $(-1)^{n+1}$, where n is the number of times the contour intersects itself. One can check easily by a straightforward calculation that the same expression (2.1) is obtained from the functional integral:

$$\begin{aligned} Z &= \int D[\psi]\exp\{\mathscr{L}[\psi]\}, \\ D[\psi] &\equiv \prod_x d\psi_x^1\, d\psi_x^2\, d\psi_x^3\, d\psi_x^4; \end{aligned} \tag{2.3}$$

$$\mathscr{L}[\psi] = -\frac{1}{2}\sum_x \bar\psi_x^\alpha \psi_x^\alpha + \frac{\lambda}{2}\sum_{x,\alpha} \bar\psi_{x+\alpha}^\beta P_\alpha^{\beta\gamma}\psi_x^\gamma. \tag{2.4}$$

Here the ψ_x^α ($\alpha = 1,2,3,4$) are the Grassmann variables specified at the lattice sites and by definition have the properties (see, e.g., Ref. 4)

$$\begin{aligned} &\psi_x^\alpha \psi_y^\beta + \psi_y^\beta \psi_x^\alpha = 0, \\ &\int d\psi^\alpha = 0, \qquad \int d\psi^\alpha \psi^\beta \equiv \delta^{\alpha\beta}, \end{aligned} \tag{2.5}$$

while $\bar\psi = \psi\hat C$. The matrices $\hat P_\alpha$ and $\hat C$ have the following form:

$$\begin{aligned} P_1 &= \begin{pmatrix} 1&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix}, \qquad \hat P_2 = \begin{pmatrix} 0&1&0&0\\ 0&1&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}, \\ \hat P_3 &= \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&1&0\\ 0&0&1&0 \end{pmatrix}, \qquad \hat P_4 = \begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&1 \end{pmatrix}, \\ \hat C &= \begin{pmatrix} 0&-1&1&-1\\ 1&0&-1&1\\ -1&1&0&-1\\ 1&-1&1&0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

The Green function of the theory with (2.4) is defined as follows:

$$G^{\alpha\beta}(x,y) = \langle \psi^\alpha(x)\bar\psi^\beta(y)\rangle. \tag{2.7}$$

One can easily check by a straightforward calculation, using the Lagrangian (2.4), that the Green function (2.7) satisfies the equation

$$G^{\alpha\beta}(x,y) = \lambda\sum_{\gamma=1}^{4} G^{\alpha\gamma}(x, y+\hat\beta)P_\beta^{\gamma\beta} + \delta_{xy}. \tag{2.8}$$

We see that the matrix $\hat P_\alpha$ plays the role of the matrix which induces a transition by one step in the direction α. If we introduce the one-step-transition matrix $\hat\Lambda(x,x')$ which by definition is non-zero for neighboring points:

$$\hat\Lambda = \lambda\sum_\alpha \hat P_\alpha, \tag{2.9}$$

we can write Eq. (2.8) in the form

$$\hat G(x,y) = \sum_{x'} \hat G(x,x')\hat\Lambda(x',y) + \delta_{xy}. \tag{2.10}$$

As the functions occurring in Eq. (2.10) depends only on argument differences, the solution of that equation takes following a Fourier transformation the simple form

$$\hat G(\mathbf{p}) = (1-\hat\Lambda(\mathbf{p}))^{-1}. \tag{2.11}$$

Using (2.6) and (2.9) we can easily check that

$$\hat\Lambda(\mathbf{p}) = \lambda\begin{pmatrix} e^{ip} & e^{iq} & 0 & -e^{-iq}\\ e^{ip} & e^{iq} & e^{-ip} & 0\\ 0 & e^{iq} & e^{-ip} & e^{-iq}\\ -e^{ip} & 0 & e^{-ip} & e^{-iq} \end{pmatrix} \tag{2.12}$$

[here $\mathbf{p} = (p,q)$]. Using this expression one shows easily that in the phase transition point, at $\lambda = \lambda_c = \sqrt{2} - 1$, two of the

eigenvalues of the matrix $1-\hat{\Lambda}(\mathbf{p})$ vanish when $p=0$.

One sees easily from symmetry considerations that the random-walk matrix $\hat{\Lambda}(\mathbf{p})|_{p=0}$ is diagonalized in the so-called momentum representation—in the base of the vectors $\{S_{-1/2}, S_{1/2}, S_{3/2}, S_{-3/2}\}$, where

$$S_{\pm k/2}=\begin{pmatrix}1\\ \exp(\pm ik\pi/4)\\ \exp(\pm ik\pi/2)\\ \exp(\pm ik3\pi/4)\end{pmatrix}.$$

The corresponding transformation matrix has in this representation the form

$$U=\begin{pmatrix}1 & 1 & 1 & 1\\ \bar{\varepsilon} & \varepsilon & \varepsilon^3 & \bar{\varepsilon}^3\\ \bar{\varepsilon}^2 & \varepsilon^2 & \varepsilon^6 & \bar{\varepsilon}^6\\ \bar{\varepsilon}^3 & \varepsilon^3 & \varepsilon^9 & \bar{\varepsilon}^9\end{pmatrix}, \quad \varepsilon=e^{i\pi/4}, \quad \bar{\varepsilon}=e^{-i\pi/4}.$$

One checks easily that in the continuous limit ($p\ll 1$), close to the phase transition point $\tau\sim(\lambda-\lambda_c)/\lambda_c\ll 1$, the Green function, written in the momentum representation

$$\hat{G}'=\hat{U}^{-1}\hat{G}\hat{U},$$

acquires a 2×2 block which has the form of the Green function of a free spinor particle:

$$\hat{G}(\mathbf{p})=2(m+ip_\mu\hat{\gamma}_\mu)/(m^2+p^2), \tag{2.13}$$

while the other components are of the order of unity and therefore do not fluctuate strongly (see, e.g., Ref. 5). Here the $\hat{\gamma}_\mu$ are 2×2 Dirac matrices and $m=2(\lambda-\lambda_c)/\lambda_c\sim\tau$. Therefore, close to the phase transition point only two of the initial four degrees of freedom are important. We shall call these two degrees of freedom, which are described by the free spinor particle Lagrangian (1.3), the long-wavelength ones and the others the localized ones. We note that the Green function (2.11), (2.12) is exactly the same as the Green function of the Ising model (1.1) which in the random-walk language is given by

$$G^{\alpha\beta}(x,y)=\sum_{\mathscr{P}}\lambda^{L_{\mathscr{P}}}\Phi(\mathscr{P}).$$

The summation is here over all contours $\mathscr{P}$ which connect the points x and y with a fixed direction α for leaving the point x and a fixed direction β for entering the point y, $L_{\mathscr{P}}$ is the length and $\Phi(\mathscr{P})$ is the spinor phase factor (2.2) of the contour $\mathscr{P}$ (see, e.g., Refs. 8, 10).

We find now the expression for the correlation function $\Gamma(R)=\langle\sigma(0)\sigma(R)\rangle$ in the Lagrangian formalism (2.4). Using the identity

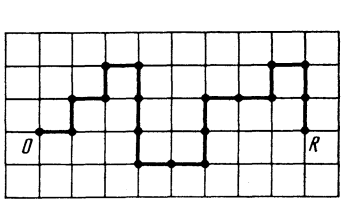

FIG. 1. Contour $\mathscr{P}_0$ connecting the sites 0 and R.

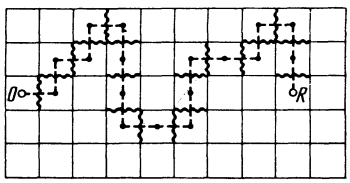

FIG. 2. The contour $\mathscr{P}_0$ on the dual lattice.

$$\exp\left(i\frac{\pi}{2}\sigma\sigma'\right)=i\sigma\sigma',$$

we can write the average

$$\langle\sigma(0)\sigma(R)\rangle\equiv Z^{-1}\sum_{\{\sigma\}}\sigma(0)\sigma(R)\exp\left\{J\sum_{i,\alpha}\sigma_i\sigma_{i+\alpha}\right\}$$

in the form

$$Z^{-1}\sum_{\{\sigma\}}\exp\left\{\sum_{i,\alpha}J'_{i,\alpha}\sigma_i\sigma_{i+\alpha}\right\}, \tag{2.14}$$

where $J'_{i,\alpha}=J+i\pi/2$, if the bond (i,α) lies on a contour $\mathscr{P}_0$ connecting the points 0 and R (Fig. 1), and $J_{i\alpha}=J$ in all other cases. It is well known that the $2D$ Ising model is self-dual (Kramers-Wannier symmetry[11]). In the dual variables, which we denote by the same letter σ, specified at the sites of the dual lattice, the partition function has the same form as in the original variables:

$$\sum_{\{\sigma\}}\exp\left\{\tilde{J}\sum_{i,\alpha}\sigma_i\sigma_{i+\alpha}\right\}, \tag{2.15}$$

while $\tilde{J}$ and J are connected through the relation

$$\operatorname{th}\tilde{J}=\exp(-2J). \tag{2.16}$$

In the phase transition point $J=\tilde{J}$.[11] When we change to the dual variables in (2.14) we obtain the same expression, except that now J must, according to (2.16), to be replaced by $-J$ on the bonds which intersect the contour $\mathscr{P}_0$ (Fig. 2). It is clear that the result is independent of the form of the contour $\mathscr{P}_0$, so that in what follows we shall assume it to be the horizontal straight line connecting the points 0 and R (Fig. 3).

Changing to the Grassmann variables we can write the expression for the correlator (2.14) as follows:

$$\Gamma(R)=Z^{-1}\int D[\psi]\exp\left\{-\frac{1}{2}\sum_x\bar{\psi}_x\psi_x+\sum_{x,\alpha}\frac{\lambda_{x,\alpha}}{2}\bar{\psi}_{x+\alpha}\hat{P}_\alpha\psi_x\right\}. \tag{2.17}$$

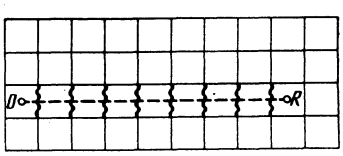

FIG. 3. The configuration of the contour $\mathscr{P}_0$, considered in what follows.

Here $\lambda_{x,\alpha} = -\lambda$, if the bond (x,α) is part of $\mathscr{P}_0$ and $\lambda_{x,\alpha} = \lambda$ in all other cases. We shall in what follows designate bonds on a contour by $(x,\mathbf{2})$, i.e., the site x of such a bond will be situated under the contour $\mathscr{P}_0$ (Fig. 3). Adding and subtracting in (2.17) the expression

$$\lambda \sum_{\mathscr{P}_0} \bar{\psi}_{x+2} \hat{P}_2 \psi_x,$$

we get

$$\Gamma(R) = \left\langle \exp\left\{ -2\lambda \sum_{\mathscr{P}_0} \bar{\psi}_{x+2} \hat{P}_2 \psi_x \right\} \right\rangle = \left\langle \prod_{\mathscr{P}_0} (1 - 2\lambda \bar{\psi}_{x+2} \hat{P}_2 \psi_x) \right\rangle . \tag{2.18}$$

In these expressions the summation and multiplication are over the points x pertaining to $\mathscr{P}_0$, and the averaging is performed with the weight (2.4).

3. TRANSITION TO THE CONTINUOUS LIMIT

In this section we show that the evaluation of the correlation function (2.18) at the phase transition point is equivalent to taking an average:

$$\Gamma(R) = \left\langle \exp\left\{ -2 \int_1^R dx \left(\bar{\psi}(x) \psi(x) \right) \right\} \right\rangle \tag{3.1}$$

(we set the lattice constant equal to unity), where $\psi(x)$ is a $2D$ spinor field and the averaging is performed with the weight

$$\mathscr{L} = \int d^2x \left(\bar{\psi} \hat{\partial} \psi \right). \tag{3.2}$$

When expanding the product in (2.18) we get products of all possible Green functions which are attached by their ends to the contour $\mathscr{P}_0$ (Fig. 4). Using Eq. (2.8) we can supplement these Green functions by the bonds on the contour itself and instead of (2.18) we get products of all possible closed loops (Fig. 5). For instance, the fourth-order loop (Fig. 6) will be given by the expression

$$\sum_{x_1=1}^{R} \cdots \sum_{x_4=1}^{R} G^{ab}(x_1, x_2) G^{bc}(x_2, x_3) G^{cd}(x_3, x_4) G^{da}(x_4, x_1), \tag{3.3}$$

where here $a,b,\ldots = 2,4$, as by our definition of the Green function which make up the loop (see Fig. 7) the exit and entry can only occur vertically. Moreover, exit and entry in the direction **4** are possible only at last the points $x + \mathbf{2}$ (see Fig. 7) and we include this condition also in the definition of the Green functions in (3.3).

Just as in the calculation of the free energy of the Ising model, the average (2.18) can now be written as an exponential:

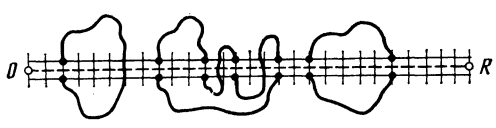

FIG. 4. Green functions, attached to the contour $\mathscr{P}_0$ arising from the expansion of the product (2.18).

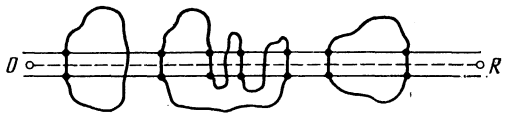

FIG. 5. Closed loops arising from the Green functions shown in Fig. 4.

$$\Gamma(\mathrm{R}) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{(-2)^n}{n} \Pi_n \right\}. \tag{3.4}$$

Here Π_n is an n-th order loop:

$$\Pi_n = \frac{1}{2} \mathrm{Sp} \sum_{x_1=1}^{R} \cdots \sum_{x_n=1}^{R} \hat{G}(x_1 - x_2) \hat{G}(x_2 - x_3) \ldots \hat{G}(x_n - x_1), \tag{3.5}$$

where

$$\hat{G} = \begin{pmatrix} G^{22} & G^{24} \\ G^{42} & G^{44} \end{pmatrix}. \tag{3.6}$$

One checks easily by straightforward calculation that in the phase-transition point

$$G^{22}(x) = G^{44}(x) = -\tfrac{1}{2}\delta(x) + \varphi_1(x), \qquad G^{24}(x) = G^{42}(x) = 1/2\pi x + \varphi_2(x), \tag{3.7}$$

where $\varphi_{1,2}(x)$ are functions which for large x decrease faster than x^{-1} while

$$\int_{-R}^{R} dx \varphi_{1,2}(x) = 0.$$

To do this we use the exact lattice expressions (2.11) and (2.12) to evaluate the appropriate components of the Green function $\hat{G}(\mathbf{p})$:

$$G^{22}(p,q) = G^{44}(p,q) = \frac{i \sin q}{4[\sin^2(p/2) + \sin^2(q/2)]} + \frac{(1/\lambda_c)\sin^2(p/2) + \sin^2(q/2)}{2[\sin^2(p/2) + \sin^2(q/2)]} - 1,$$

$$G^{24}(p,q) = G^{42}(p,q) = \frac{\sin p}{4[\sin^2(p/2) + \sin^2(q/2)]}.$$

Here $\lambda_c = \sqrt{2} - 1$. The functions in (3.5) are given by integrals of the form

$$\hat{G}(x) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{ipx} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \hat{G}(p,q), \tag{3.8}$$

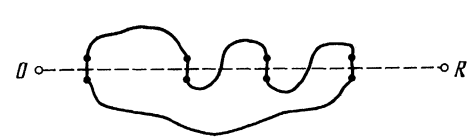

FIG. 6. Fourth-order loop.

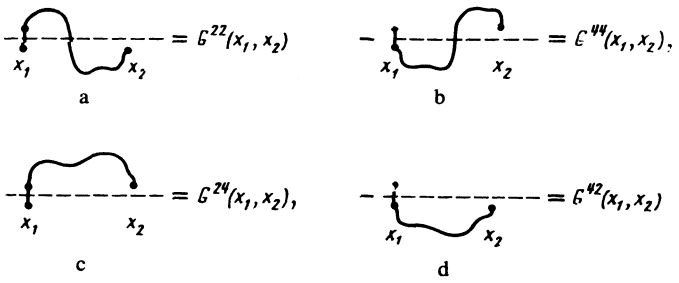

FIG. 7. Definition of the Green functions $G^{ab}(x_1,x_2)$ forming closed loops of the product (2.18).

while the off-diagonal Green functions $G^{24}(p,q)$ and $G^{42}(p,q)$ are written here with an additional weight $\exp(\pm iq)$. Integration over q gives

$$g^{22}(p)\equiv\int_{-\pi}^{\pi}\frac{dq}{2\pi}G^{22}(p,q)=-\frac{1}{2}+\frac{\sin(p/2)}{2^{1/2}[1+\sin^2(p/2)]^{1/2}}\operatorname{sign}p,$$
$$g^{24}(p)\equiv\int_{-\pi}^{\pi}\frac{dq}{2\pi}G^{24}(p,q)=-\frac{i}{2}\frac{\cos(p/2)}{[1+\sin^2(p/2)]^{1/2}}\operatorname{sign}p. \tag{3.9}$$

Substituting these expressions into (3.8) we get (3.7) for large x.

We see that the functions $G^{22}(x-x')$ and $G^{44}(x-x')$ contain a contact term which goes into operation when the points x and x' coincide. Indeed, when evalating the loop Π_n there occurs yet another contact term due to the fact that

$$\int_{-\infty}^{+\infty}dxG^{24}(x_1-x)G^{42}(x-x_2)=-{}^1/_4\delta(x_1-x_2). \tag{3.10}$$

The transition to the continuous limit consists just in summing these contact terms which, firstly, lead to a linear divergence of each loop Π_n and, secondly, can lead to the renormalization of the coefficient (-2) for the "long-wavelength" Green functions $1/2\pi x$.

First of all we now verify that indeed the series

$$\sum\frac{1}{n}(-2)^n\Pi_n$$

does not contain a linear divergence (as should be the case at the phase transition point), and after that we show that the coefficient (-2) is not renormalized. It is very simple to establish the first. The linearly divergent part of the series occurring in (3.4) is given by the expression

$$-R\sum_{n=1}^{\infty}\frac{(-2)^n}{n}\frac{1}{2}\operatorname{Sp}\times\int_{-\infty}^{+\infty}dx_1\dots dx_{n-1}\hat G(x_1)\dots\hat G(x_{n-1})\hat G(-x_1-\dots-x_{n-1}).$$

Next, changing to the functions (3.9) we have

$$-R\frac{1}{2}\operatorname{Sp}\left[\int_{-\pi}^{\pi}\frac{dp}{2\pi}\sum_{n=1}^{\infty}\frac{(-2)^n}{n}(\hat g(p))^n\right]$$
$$=-R\int_{-\pi}^{\pi}\frac{dp}{2\pi}\frac{1}{2}\operatorname{Sp}\ln(1+2\hat g(p))$$
$$=-\frac{1}{2}R\int_{-\pi}^{\pi}\frac{dp}{2\pi}\ln[\det(1+2\hat g(p))].$$

Substituting here (3.9) we see easily that

$$\det(1+2\hat g(p))\equiv 1.$$

To explain the second problem we substitute instead of the coefficient (-2) the factor $-a$ and assume for the present that $a\neq 2$ and afterwards take the limit $a\to 2$ in the final result. Due to the diagonal (localized) Green functions G^{22} and G^{44}, loops consisting of "long-wavelength" Green functions G^{24} and G^{42}

$$(-a)^n\int_1^R dx_1\dots dx_nG^{24}(x_1-x_2)G^{42}(x_2-x_3)\dots G^{42}(x_n-x_1), \tag{3.11}$$

arise not only from the loop

$$\Pi_n=\frac{1}{2}\operatorname{Sp}\int_1^R dx_1\dots dx_n\hat G(x_1-x_2)\dots\hat G(x_n-x_1),$$

where all $\hat G$ must be replaced by G^{24} and G^{42}, but from the higher-order loops in which there are some diagonal (localized) Green functions. One sees easily that as a result there arises for the loop (3.11) a coefficient $(-aA)^n$ with

$$A=\sum_{k=0}^{\infty}\left[(-a)\int_{-\infty}^{+\infty}dxG^{22}(x)\right]^n=\frac{1}{1-a/2}. \tag{3.12}$$

The loops Π_n consisting of G^{24} and G^{42} are non-vanishing only when $n=2k$. To take into account the contact term (3.10) we integrate in the loops Π_{2k} over k staggered variables. Then

$$\int_1^R dxG^{24}(x_1-x)G^{42}(x-x_2)=-{}^1/_4\delta(x_1-x_2)-f(x_1,x_2). \tag{3.13}$$

Here $f(x_1,x_2)$ is a regular function of x_1 and x_2:

$$f(x_1,x_2)=\frac{1}{(2\pi)^2(x_1-x_2)}\ln\frac{x_1(R-x_2)}{x_2(R-x_1)}. \tag{3.14}$$

We thus obtain loops consisting of the functions f and one sees easily that due to the contact term in (3.14) the coefficient of each function f in the loop is given by the expression

$$(-a)^2\sum_{k=1}^{\infty}(-1)^{k+1}A^{2k}\left(\frac{1}{4}\right)^k=(-a)^2\frac{A^2}{4+A^2}.$$

Substituting here (3.12) and taking the limit $a\to 2$ we get the coefficient $(-2)^2$.

On the other hand, the expansion (3.1) has the form (3.4), (3.5) with continuous Green functions

$$S(x)=\langle\psi(x)\psi^{+}(0)\rangle=1/2\pi x.$$

In this case the expression for the loop (3.5) has the form

$$\Pi_n=\frac{1}{(2\pi)^n}\int_1^R dx_1\dots dx_n\frac{1}{(x_1-x_2)\dots(x_n-x_1)}. \tag{3.15}$$

If we now integrate over every other variable, we again obtain the above-mentioned series of loops which consist of the functions $F(x,x')$ of (3.14) with a coefficient $(-2)^2$ in front of each function f. We are thus led to the statement (3.1), (3.2) made at the beginning of this section.

4. CALCULATION OF THE EXPONENT OF THE CORRELATION FUNCTION OF THE PURE ISING MODEL

In this section we obtain the well known result

$$\Gamma(R)\approx R^{-1/4}. \tag{4.1}$$

We showed in the preceding section that

$$\Gamma(R)=\exp\Big\{-\sum_{n=1}^{\infty}\frac{(-2)^n}{n}\Pi_n\Big\}, \tag{4.2}$$

where

$$\Pi_n=\frac{1}{(2\pi)^n}\int_1^R dx_1\dots dx_n\frac{1}{(x_1-x_2)\dots(x_n-x_1)}. \tag{4.3}$$

In the even-order loop Π_{2k} (odd-order loops vanish) we integrate alternatively over k points:

$$\Pi_{2k}=\frac{1}{(2\pi)^{2k}}\times\int_1^R dx_1\dots dx_k\frac{\ln\left[\frac{x_1(R-x_2)}{x_2(R-x_1)}\right]\dots\ln\left[\frac{x_k(R-x_1)}{x_1(R-x_k)}\right]}{(x_1-x_2)\dots(x_k-x_1)}. \tag{4.4}$$

Below we see that the main (logarithmic) contribution to this integral comes from integrating close to the endpoints of the interval $[1,R]$ when

$$\Pi_{2k}\approx\frac{2}{(2\pi)^{2k}}\int_1^R dx_1\dots dx_k\frac{\ln(x_1/x_2)\ln(x_2/x_3)\dots\ln(x_k/x_1)}{(x_1-x_2)(x_2-x_3)\dots(x_k-x_1)}. \tag{4.5}$$

In this integral we make the substitution

$$x_2/x_1=e^{\xi_1},\quad x_3/x_2=e^{\xi_2},\quad\dots\dots,\quad x_k/x_{k-1}=e^{\xi_{k-1}},$$

after which we obtain

$$\Pi_{2k}=\frac{2}{(2\pi)^{2k}}\int_1^R\frac{dx_1}{x_1}\int_{-\infty}^{+\infty}d\xi_1\dots d\xi_{k-1}\frac{\xi_1}{2\operatorname{sh}(\xi_1/2)}\dots\frac{\xi_{k-1}}{2\operatorname{sh}(\xi_{k-1}/2)}\frac{\xi_1+\dots+\xi_{k-1}}{2\operatorname{sh}[(\xi_1+\dots+\xi_{k-1})/2]}. \tag{4.6}$$

We thus see that the integration over x_1 is logarithmic. The other integrals converge so that we could extend the integration to infinity. Changing to the Fourier representation

$$\mathcal{F}(p)=\int_{-\infty}^{+\infty}d\xi\frac{\xi}{2\operatorname{sh}(\xi/2)}e^{ip\xi}=\frac{\pi^2}{\operatorname{ch}^2(\pi p)} \tag{4.7}$$

and substituting the expression for Π_{2k} into (4.2) we get

$$\ln\Gamma(R)\approx-(\ln R)\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{\pi^2}\mathcal{F}(p)\right)^k$$

$$=(\ln R)\int_{-\infty}^{\pm\infty}\frac{dp}{2\pi}\ln\left(1-\frac{1}{\operatorname{ch}^2(\pi p)}\right)=-\frac{1}{4}\ln R. \tag{4.8}$$

Thus $\Gamma(R)\approx R^{-1/4}$.

Before we go over to the impurity Ising model we make the following remark. The calculation of the exponent of the correlation function of the pure Ising model formulated in the form (3.1), (3.2) is similar to calculation made in Ref. 12 where the authors started from the beginning from a Hamilton formulation of the continuous limit (1.3) and reduced the problem to calculating the average:

$$\left\langle\exp\left\{-\pi\int_1^R dx\,(\bar{\psi}(x)\psi(x))\right\}\right\rangle \tag{4.9}$$

(apart from the substitution $2\to\pi$ this is the average (3.1)). However, in such a formulation it is necessary, when expanding the exponential in integrals along the contour $\mathcal{P}_0$, to go correctly around the pole $(x-x')^{-1}$ of the Green function. It then turns out that only the second-order loop gives a logarithmic contribution. However, because we are applying this in what follows to the impurity model, it was important to learn how to obtain the continuous limit of (3.1), (3.2) in the symmetric (Lagrangian) form, starting directly from the initial lattice formulation of the Ising model.

5. CORRELATION FUNCTION OF THE ISING MODEL WITH IMPURITY BONDS

In the case of the Ising model with impurity bonds all we have said in Sec. 2, leading to Eq. (2.18), remains valid, with the sole difference that now the factor λ depends on the coordinates and that we must add to the averaging in (2.18) averaging over the impurities:

$$\Gamma(R)=\overline{\left\langle\prod_{\mathcal{P}_0}(1-2\lambda_{x2}\bar{\psi}_{x+2}\hat{P}_2\psi_x)\right\rangle}. \tag{5.1}$$

The averaging over the $\lambda_{x\alpha}$ after the expansion of the product in (5.1) is done in exactly the same way as for the evaluation of the free energy.[5]

It is well known that the free energy of the Ising model is given by an expression such as (2.1) where one must sum over the configurations of *one* closed contour.[8] When there are impurity bonds present one must write this expression in the form

$$\sum_{\mathcal{P}}\overline{\left(\prod_{\mathcal{P}}\lambda_{x\alpha}\right)}\Phi(\mathcal{P}), \tag{5.2}$$

i.e., one must average the λ-product along a closed path. The fact that this averaging is non-trivial is connected with the fact that there are amongst all possible configurations of the closed contour in (5.2) configurations with overlapping paths. If, e.g., in one of the configurations the bond $\lambda_{x_0\alpha_0}$ is passed through twice, this bond will occur with an addi-

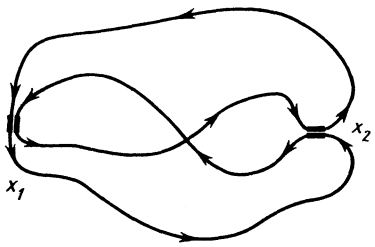

FIG. 8. Diagram of second order in ν in the free energy expansion.

tional weight $\overline{(\lambda^2)} - (\overline{\lambda})^2$. If we consider simultaneously two contacts as, e.g., in Fig. 8, we get an additional contribution of the form

$$\sim \frac{1}{2!}\nu^2 \int\int d^2x_1\, d^2x_2 \times \sum_{\alpha,\beta=1}^{4} G^{\alpha\beta}(x_1,x_2)\, G^{\beta\alpha}(x_2,x_1)\, G^{\alpha\bar{\beta}}(x_1,x_2)\, G^{\bar{\beta}\alpha}(x_2,x_1) \tag{5.3}$$

($\bar{\beta}$ is the direction opposite to β). Continuing thus we can construct a perturbation theory in terms of the number of contacts in the loop or in powers of ν.

We note that in the continuous limit under the condition $\nu \ll 1$ we can take into account only two-fold overlapping of the contours. We can thus represent the n-th order of the perturbation theory as follows: n "doubled-up" bonds (bonds on which contact occurs) must in all possible ways be connected by free Green functions $G^{\alpha\beta}(x-x')$ in such a way that then only a single closed loop is formed (Fig. 9). If we now sum over the directions of these bonds and after that replace the lattice Green functions by the spinor ones (2.13), there occurs in the points of contact a "splicing" of Green functions, four-fermion vertices arise here (Fig. 10), and we get the n-th order of the perturbation theory of the Lagrangian theory

$$\mathcal{L} = \int d^2x \left[\sum_{a=1}^{N} (\bar{\psi}^a \hat{\partial} \psi^a) + g \sum_{a,b=1}^{N} (\bar{\psi}^a \psi^a)(\bar{\psi}^b \psi^b) \right] \tag{5.4}$$

in the charge $g = c\nu$ [c is a number depending on the quantities J and $\bar{J}$ in (1.2)]. Here, $a,b = 1,2,\ldots,N$ are replica exponents and in the final result we must put $N = 0$. This last method, known as the replica method,[7] leaves in the perturbation theory only single-loop diagrams.

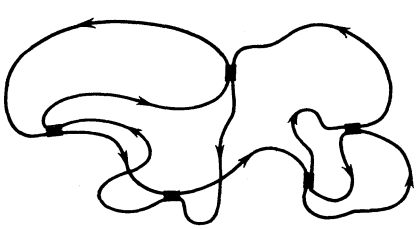

FIG. 9. Example of a fifth-order diagram in the free energy expansion.

FIG. 10. Four-fermion vertex $(\bar{\psi}\psi)(\bar{\psi}\psi)$.

We note that there is in fact yet another effect of the overlapping of the contour which occurs at the level of the localized degrees of freedom. This effect is connected with the overlapping of the contour caused by the localized closed loops at each bond (Fig. 11). Taking these loops into account leads to a simple mass renormalization in the Green function, i.e., simply to a shift in the phase transition temperature.[5] We shall asssume that we are at the phase transition point of the impurity model and that the renormalized mass is zero. There are no other effects connected with the localized degrees of freedom to first order in ν. In view of this, the transition to the continuous limit (taking the localized degrees of freedom into account) in Eq. (5.1) proceeds exactly in the same way as in the case of the "pure" Ising model (Sec. 3).

In contrast to the free energy (5.2) we get in the present case from (5.1) a many-loop expansion in which vacuum loops, i.e., the loops not attached to the contour $\mathcal{P}_0$ (Fig. 5), are forbidden. However, by itself the averaging over the impurities does not differ at all from the earlier problem. Due to this averaging there will now be a "splicing" of Green functions both inside separate loops and between different loops. It thus looks ass if we had again Eq. (3.1):

$$\left\langle \exp\left\{ -2 \int_1^R dx\, (\bar{\psi}\psi) \right\} \right\rangle , \tag{5.5}$$

in which, however, the averaging occurs with the weight (5.4). However, now in the expansion of (5.5) the loops attached to the contour $\mathcal{P}_0$ should not, in contrast to the vacuum loops, contain the replica factor N. This is attained by introducing the $N \times N$ replica matrix:

$$\hat{A} = \begin{pmatrix} 2 & 0 & 0 \ldots 0 \\ 0 & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \\ 0 & 0 & 0 \ldots 0 \end{pmatrix}, \tag{5.6}$$

and, in fact, instead of (5.5) we must average the quantity

$$\exp\left\{ -\int_1^R dx \sum_{a,b=1}^{N} (\bar{\psi}^a(x) A^{ab} \psi^b(x)) \right\}. \tag{5.7}$$

The calculation of the correlation function of the impurity Ising model thus is reduced to the following form:

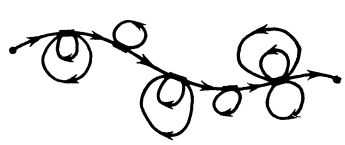

FIG. 11. Localized closed loops leading to mass renormalization.

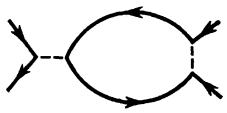

FIG. 12. Diagram renormalizing the charge g.

$$\Gamma(R)=Z^{-1}\int D[\psi]\exp\Big\{\int d^2x\Big[\sum_{a=1}^{N}(\overline{\psi}^a\hat{\partial}\psi^a)$$
$$+g\sum_{a,b=1}^{N}(\overline{\psi}^a\psi^a)(\overline{\psi}^b\psi^b)\Big]-\int_1^R dx\sum_{a,b=1}^{N}\overline{\psi}^aA^{ab}\psi^b\Big\}. \qquad (5.8)$$

The matrix $\hat{A}$ selects from the N replicas a single one for the closed contours on the contour $\mathscr{P}_0$. All other closed loops vanish when $N=0$. We note that in this case it is important that real spinors [by $\overline{\psi}$ in (5.8) we understand the quantity $\psi\hat{\gamma}_5$] occur in (5.8). The reason is that the problem formulated in the language of complex spinors in fact gives the square of the correlator, while we must average the correlator itself, and not its square, over the impurities.

As in (1.4), the theory (5.8) is renormalized (renormalization group method; see, e.g., Ref. 9). As before[5] the renormalized charge is given by the diagram shown in Fig. 12. The corresponding equation has the form

$$dg/d\xi=-2g^2/\pi. \qquad (5.9)$$

Here $\xi=\ln(\Lambda/\Lambda')$ is the renormalization parameter and Λ and Λ' are the old and the new length scales at small distances. It follows from (5.9) that

$$g(\xi)=\left(1+\frac{2g_0}{\pi}\xi\right)^{-1}. \qquad (5.10)$$

To renormalize the matrix $\hat{A}$ we write it in the form

$$\hat{A}(\xi)=\begin{pmatrix} a(\xi) & 0 & 0 & \dots & 0\\ 0 & a'(\xi) & 0 & \dots & 0\\ 0 & 0 & a'(\xi) & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & a'(\xi)\end{pmatrix}, \qquad (5.11)$$

where

$$a(0)=2, \quad a'(0)=0. \qquad (5.12)$$

The renormalization of $\hat{A}$ is done by the diagrams shown in Fig. 13. It is important that the diagrams of higher order in a such as the one in Fig. 14 are not logarithmic and therefore do not contribute to the renormalization [notwithstanding

FIG. 13. Diagrams renormalizing the matrix $\hat{A}$.

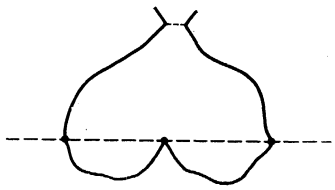

FIG. 14. Example of a diagram which does not contribute to the renormalization of the matrix $\hat{A}$.

the fact that $a(0)$ is not small]. The corresponding equations of the renormalization group have the form

$$\frac{dA^{ab}}{d\xi}=-\frac{1}{\pi}[A^{ab}-\delta^{ab}(\mathrm{Sp}\,\hat{A})]g(\xi) \qquad (5.13)$$

or (when $N=0$):

$$\frac{da}{d\xi}=-\frac{1}{\pi}a'g, \quad \frac{da'}{d\xi}=-\frac{2}{\pi}a'g+\frac{1}{\pi}ag. \qquad (5.14)$$

One can easily solve these equations and they lead when we use the initial conditions (5.12), to the following functions:

$$a'(\xi)=\left(1+\frac{2g_0}{\pi}\xi\right)^{-1/2}\ln\left(1+\frac{2g_0}{\pi}\xi\right), \qquad (5.15)$$
$$a(\xi)=2\left(1+\frac{2g_0}{\pi}\xi\right)^{-1/2}+\left(1+\frac{2g_0}{\pi}\xi\right)^{-1/2}\ln\left(1+\frac{2g_0}{\pi}\xi\right).$$

When evaluating the correlator $\langle\sigma(0)\sigma(R)\rangle$ we are interested in distances $R\gg r_i\propto\exp(1/\nu)$, i.e., length scales at which the critical regime of the impurity Ising model begins. We shall therefore in what follows be interested only in the asymptotic expression at $\xi\gg\pi/g_0$:

$$a'(\xi)\approx\left(\frac{\pi}{2g_0\xi}\right)^{1/2}\ln\left(\frac{2g_0\xi}{\pi}\right)\to 0, \quad \xi\to\infty, \qquad (5.16)$$
$$a(\xi)\approx\left(\frac{\pi}{2g_0\xi}\right)^{1/2}\ln\left(\frac{2g_0\xi}{\pi}\right)+2\left(\frac{\pi}{2g_0\xi}\right)^{1/2}\to 0, \quad \xi\to\infty.$$

In order to evaluate the correlator $\Gamma(R)$ we must now average the exponential (5.7), which depends on the renormalized $a(\xi)$ and $a'(\xi)$ of (5.16), over non-interacting fermions. When we expand the exponential we get again a single-loop expansion. To second order we have

$$\frac{1}{2}\int_1^R dx_1\,dx_2[a^2-(a')^2]G^2(x_1-x_2). \qquad (5.17)$$

From (5.16) we find:

$$a^2-(a')^2\approx\frac{2\pi}{g_0\xi}\ln\left(\frac{2g_0\xi}{\pi}\right)$$
$$\approx 2\pi\ln\left(\frac{2g_0}{\pi}\ln|x_1-x_2|\right)\Big/\,g_0\ln|x_1-x_2|. \qquad (5.18)$$

One checks easily that due to the factors $a(\xi)$ and $a'(\xi)$, leading in loops of higher order than the n-th to an additional decrease at large distances $\propto\xi^{-n}$, all loops of higher than

second order converge. The main logarithmic contribution to $\Gamma(R)$ therefore comes only from the second-order loop (5.17):

$$\Gamma(R)\approx\exp\left\{-\frac{1}{2}\int_1^R dx_1\,dx_2\frac{1}{(2\pi)^2(x_1-x_2)^2}\right.$$

$$\left.\times 2\pi\ln\left(\frac{2g_0}{\pi}\ln|x_1-x_2|\right)\Big/\,(g_0\ln|x_1-x_2|)^{-1}\right\}.$$

The calculation leads to the result

$$\Gamma(R)\approx\exp\left\{-\frac{1}{4\pi g_0}[\ln\ln R]^2\right\}(\ln R)^{(2\pi g_0)^{-1}\ln(c/g_0)} \quad (5.19)$$

$$(c=\text{const}\sim 1).$$

6. CONCLUSION

We have studied in this and in earlier papers[5] the critical properties of a $2D$ Ising model with impurity bonds. We have shown that in accordance with general ideas about the critical behavior of weakly disordered systems[1,2] the Ising model with impurity bonds changes in a narrow temperature range $\tau_\nu \propto \exp(-1/\nu)\ll 1$ near the phase transition point into another (universal) critical regime where, as compared to the "pure" Ising model, the singularities in the phase-transition point become weaker. The effect of the impurities turns out to be so strong that the exponent of the correlation function becomes zero.

According to the scale-invariance hypothesis, the five critical exponents α of the specific heat, β of the spontaneous magnetization, ν of the correlation radius, ζ of the correlation function, and γ of the susceptibility are connected by two relations (see, e.g., Ref. 10):

$$\alpha+2\beta+\gamma=2, \quad \nu(2-\zeta)=\gamma. \quad (6.1)$$

Since for the Ising model with impurity bonds we now know three exponents, viz., $\alpha=0$, $\nu=1$, and $\zeta=0$, Eqs. (6.1) enable us to determine the other two: $\gamma=2$ and $\beta=0$. One can, independent of (6.1), by the method developed in this paper evaluate the spontaneous moment below the transition point. One sees easily that the calculations in that case are similar to those for the correlator and the spontaneous moment has a "logarithmic" form, analogous to (1.5), where the inverse mass plays the role of the distance.

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