Growth kinetics of quantum crystals

A. F. Andreev and V. G. Knizhnik

Institute of Physical Problems, Academy of Sciences of the USSR

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The growth of crystals with quantum-rough surfaces is characterized by a kinetic growth coefficient that takes account of the temporal and spatial dispersions. It is computed for the boundary between solid and liquid ^{4}He for different relations between the wave vector and the mean free paths of the thermal excitations. The damping constant of the crystallization waves is found. It is small both in the very-low-temperature ballistic regime and in the phonon-hydrodynamics region for perfect crystals, in which umklapp processes are ineffective.

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The surface of a crystal can be in two different states: atomically smooth and atomically rough. From the thermodynamic point of view, the difference between them is manifest in the presence or absence of equilibrium faceting of crystals with atomically smooth or atomically rough surfaces respectively. Furthermore, these states differ significantly in the characters of their growth or melting kinetics. An atomically smooth surface is metastable under conditions when the temperature or the pressure lies slightly off the melting curve. Therefore, the growth of a crystal with such a surface is an extremely slow process involving successive decays of the metastable states as a result of the fluctuation formation of nucleating centers for new atomic layers. In the case of an atomically rough surface, the metastability does not occur, and the growth of the crystal proceeds significantly faster.

Ordinary crystals at sufficiently low temperatures always have atomically smooth surfaces. The transition from a smooth to a rough surface occurs in them, when the temperature is raised, at quite a definite point of the melting curve, which is the critical point of a distinctive surface phase transition.[1-5]

The situation is significantly different[6] in quantum crystals of the solid-helium type, which are characterized by a large zero-point particle-vibration amplitude. The role of the quantum effects manifests itself in the fact that the majority of the faces are atomically rough even at zero temperature. Only a few most closely packed faces, which are atomically rough at low temperatures, may be an exception. The characteristics of the growth kinetics of such smooth faces in quantum crystals consist[7] only in the fact that the major role, as is generally the case in the quantum kinetics of metastable-state decay,[8,9] is played by quantum tunneling, instead of thermally-activated processes of nucleating-center formation, which are characteristic of ordinary crystals.

The peculiarity of the growth and melting processes is most clearly manifest on atomically rough surfaces of quantum crystals in equilibrium with the superfluid.[6,10,11] In this case, realized in practice in ^{4}He, we can speak of "supercrystallization" or "supermelting," since these processes occur coherently and virtually nondissipatively at low temperatures. Even a mere visual observation of the state of the surface shows[10,11] that it behaves rather like a liquid surface, continuously oscillating under the action of external perturbations. The reason is the possibility of the propagation of slowly decaying melting and crystallization waves over a quantum-rough surface.[6,10,11]

Parshin and one of us[6] have estimated the growth coefficient for quantum-rough surfaces and of the associated crystallization-wave absorption, and their results confirm qualitatively the experimental results reported by Keshishev, Parshin, and Babkin.[11] The present paper is devoted to the construction of a detailed theory, not just under the conditions of the ballistic regime considered in Ref. 6 (i.e., under the conditions of sufficiently long thermal-excitation mean free paths), but also in the opposite, hydrodynamic, limiting case. We shall see, in particular, that quantum-rough surfaces can grow nondissipatively not only at zero temperature, i.e., in the absence of thermal excitations in the crystal and the liquid, but also at finite temperatures when no umklapp processes are involved in the phonon collisions in the crystal. The results can be used to compute not only the crystallization wave absorption, but also the anomalous sound reflection from quantum-rough surfaces, which has been observed by Castaing *et al*.[12,13]

1. THE HYDRODYNAMIC REGIME

At zero temperature, in the absence of thermal excitations, because of the quantum delocalization of the growth steps at the phase boundary, it is possible for the boundary to execute free, nondissipative motion accompanied by the growth or melting of the crystal.[6] At finite temperatures, the interaction of the thermal excitations with the moving boundary of the growing crystal leads, generally speaking, to dissipation. The following circumstance will, however, be quite important hereafter.

At low temperatures the primary thermal excitations in a crystal are long-wave phonons. If the crystal is close enough to perfect, the probability for the occurrence of the umklapp process, which are the cause of the nonconservation of the total momentum of the phonons, decreases exponentially with decreasing temperature, and they can be neglected. It is easy to see

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that under these conditions nondissipative motion of the phase boundary is possible at finite temperatures as well. Indeed, let us go over into the coordinate system in which the phase boundary is at rest. In this system the thermal-excitation gas (the volume phonons of the crystal and the liquid, as well as the surface phonons and the crystallization waves) is under steady-state external conditions. Therefore, there exists an equilibrium state described by the Planck distribution function $n_0(\tilde{\varepsilon})$, where $\tilde{\varepsilon}=\varepsilon-\mathbf{p}\cdot\mathbf{V}$ is the excitation energy in the system in question, ε is the energy in the laboratory system, in which the crystal lattice is at rest, p is the momentum (quasimomentum), and **V** is the velocity of the boundary in the laboratory system in the direction parallel to the normal to the boundary. This distribution function clearly corresponds to the situation in which the excitation gas as a whole moves in the laboratory system with velocity **V**. Thus, the moving boundary of the growing crystal completely drags along the normal component of liquid He II and the phonons of the crystal. In the absence of umklapp processes, this is an equilibrium state: the mutual conversions of the excitations at the phase boundary do not destroy it.

The above result indicates satisfaction at the phase boundary of the following boundary conditions:

$$v_{nz}=v_z=V, \tag{1}$$

where v_n is the velocity of the normal component of the liquid, v is the velocity of the phonon gas of the crystal relative to the lattice, and the z axis is normal to the boundary.

The growth rate of classical crystals with atomically rough surfaces is normally characterized by the so-called growth coefficient G, given by the relation

$$V=G\Delta\mu, \tag{2}$$

where $\Delta\mu$ is the difference between the chemical potentials (per unit mass) of the contiguous phases. The coefficient G is directly related to the surface energy dissipation. Indeed, the energy dissipation $\dot{E}$ per unit surface area is equal to $\Delta\mu\dot{M}$, where $\dot{M}$ is the mass of material converted in unit time from one phase into the other. Since $\dot{M}=\rho_c V$ (ρ_c is the crystal density), we obtain by using (2)

$$\dot{E}=\Delta\mu\rho_c V=\rho_c V^2/G. \tag{3}$$

Equation (2) plays the role of a boundary condition for the determination of the temperature distribution in the crystal and the hydrodynamic motions in the liquid as a result of the boundary motion. This definition of the growth coefficient allows for the volume dissipation, besides the surface dissipation given by the formula (3).

The above definition of the growth coefficient is not suitable for quantum crystals. We shall use the following definition.

Let $z=\zeta(x_\mu,t)$ be the equation of the crystal surface at the moment of time t; $\mu=1, 2; x_\mu=x,y$. Considering the quantity ζ to be sufficiently small, we can, without loss of generality, assume that

$$\zeta\propto\exp(i\mathbf{k}\mathbf{r}-i\omega t),$$

where k is a two-dimensional wave vector lying in the xy plane and ω is some frequency. Let us emphasize that the function $\zeta(x_\mu,t)$ is considered here to be an arbitrarily specified function, the actual form of which is formally governed by some external influence on the system. Therefore, the frequency ω and the wave vector k are in no way connected with each other. Let us write the time derivative $\dot{\mathscr{E}}$ of the total energy of the system in the form

$$\dot{\mathscr{E}}=f\dot{\zeta}. \tag{4}$$

The quantity f defined by this relation should depend linearly on $\dot{\zeta}$. Let us set

$$f=\rho_c G^{-1}(\omega,\mathbf{k})\dot{\zeta}, \tag{5}$$

where $G(\omega,\mathbf{k})$ is, by definition, the (complex) growth coefficient, which allows for the temporal and spatial dispersions. Its real part is directly connected with the energy dissipation. Indeed, averaging (4) over time and using (5) we obtain

$$\dot{\mathscr{E}}=\frac{\rho_c}{2}|\zeta|^2\operatorname{Re}G^{-1}(\omega,\mathbf{k}). \tag{6}$$

In contrast to (3), we are dealing here with the total (volume plus surface) energy dissipation.

The function $G(\omega,\mathbf{k})$ contains full information about the crystal-growth kinetics. It determines in particular, the crystallization-wave spectrum with allowance for the damping. Indeed, the natural vibrations of the system (in the absence of external influences) occur with conservation of total energy ($\dot{\mathscr{E}}=0$). Therefore the vibration frequency $\omega=\omega(\mathbf{k})$ should satisfy the equation

$$G^{-1}(\omega,\mathbf{k})=0. \tag{7}$$

The total energy $\mathscr{E}$ of the system is the sum of the volume $\mathscr{E}_v$ and surface $\mathscr{E}_s$ energies. The latter is equal to

$$\mathscr{E}_s=\int\alpha(\mathbf{n})[1+(\partial\zeta/\partial x_\mu)^2]^{1/2}dxdy, \tag{8}$$

where α, the energy of a unit surface area, depends on the orientation of the unit vector n of the normal to the surface relative to the crystallographic axes. For small ζ we have $n_z\approx1$ and $n_\mu=-\partial\zeta/\partial x_\mu$, so that

$$\alpha(n_\mu)\approx\alpha(0)-\alpha_\mu\frac{\partial\zeta}{\partial x_\mu}+\frac{1}{2}\alpha_{\mu\nu}\frac{\partial\zeta}{\partial x_\mu}\frac{\partial\zeta}{\partial x_\nu}.$$

Here the α_μ and $\alpha_{\mu\nu}$ are angle derivatives of the surface energy. Substituting the last expansion into the formula (8), we find

$$\dot{\mathscr{E}}_s=\frac{\delta\mathscr{E}_s}{\delta\zeta}\zeta=-(\alpha_{\mu\nu}+\alpha\delta_{\mu\nu})\frac{\partial^2\zeta}{\partial x_\mu\partial x_\nu}\zeta=\tilde{\alpha}_{\mu\nu}k_\mu k_\nu\zeta\zeta, \tag{9}$$

where $\tilde{\alpha}_{\mu\nu}=\alpha_{\mu\nu}+\alpha\delta_{\mu\nu}$.

The time derivative of the volume energy can clearly be written in the form

$$\dot{\mathscr{E}}_v=(E_c-E_l)\dot{\zeta}+Q_{lz}-Q_{cz}, \tag{10}$$

where E is the energy per unit volume at $z=0$, **Q** is the energy flux vector at $z=0$, and the subscripts c and l pertain respectively to the crystal and the liquid. We assume that the liquid is located in the region of positive z. In the hydrodynamic region, i.e., in the case when the wave vector k and the frequency ω are significantly smaller than the reciprocal mean free path of the thermal excitations and the collision rate respectively, we can use for $\mathbf{Q}_{l,c}$ the well-known (Ref. 14, §17) expressions

$$\mathbf{Q}_l=\mu_l\mathbf{j}+T_lS_l\mathbf{v}_n,\quad \mathbf{Q}_c=T_cS_c\mathbf{v}, \tag{11}$$

in which we have neglected the quantities that are of order higher than the second in the deviations from the equilibrium states. In the formulas (11) T, S, and μ are respectively the temperature, the entropy per unit volume, and the chemical potential; $\mathbf{j}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n$, where ρ_s and $\mathbf{v}_s$ are the density and velocity of the superfluid component of the liquid, is the mass flux vector.

Taking into consideration the boundary conditions (1), the thermodynamic relation $P=-E+TS+\mu\rho$ for the pressure, and the condition $j_z=-(\rho_c-\rho_l)\dot{\zeta}$ for the conservation of mass, we obtain from (10) and (11) the expression

$$\dot{\mathcal{E}}_v=\{P_l-P_c+\rho_c(\mu_c-\mu_l)\}\dot{\zeta}.$$

In view of the presence here of the small factor $\dot{\zeta}$, we should take into account in the expression in the curly brackets only the terms that are linear in the deviations $\delta T_{c,l}$ and $\delta P_{c,l}$ of all the quantities from equilibrium. As a result, we have

$$\dot{\mathcal{E}}_v=-\zeta\left\{S_c\delta T_c+\frac{\rho_c-\rho_l}{\rho_l}\delta P_l-\frac{\rho_c}{\rho_l}S_l\delta T_l\right\}. \tag{12}$$

Being interested first and foremost in the crystallization-wave spectrum, we shall compute the growth coefficient for frequencies ω of the order of the eigenfrequencies of the crystallization waves with a given k. Since the velocity of these waves is low as compared with the velocities of first and second sounds, we can investigate the hydrodynamic equations in the imcompressible-liquid approximation. In this case (see Landau's paper[15]) δP_l is the sum of the pressures δP_n and δP_s for the normal and superfluid motions, with

$$\delta P_s=-\rho_s\dot{\varphi}=-i\omega\rho_s\varphi,$$

where φ is the superfluid-velocity potential. It satisfies the Laplace equation $\nabla^2\varphi=0$; therefore,

$$\varphi=\varphi_0\exp(-kz+i\mathbf{kr}).$$

Here φ_0 is a constant that can be determined from the conditions $j_z=-(\rho_c-\rho_l)\dot{\zeta}$ and (1). As a result, we find the pressure of the superfluid component at $z=0$:

$$\delta P_s=\frac{i\omega}{k}(\rho_c-\rho_s)\zeta. \tag{13}$$

The temperature shift δT_l at $z=0$ is, according to Landau,[15] given by the formula

$$\delta T_l=\frac{\rho_n}{S_l}\left(\frac{\delta P_n}{\rho_n}-\frac{\delta P_s}{\rho_s}\right). \tag{14}$$

Using (13) and (14), we find from (12) that

$$\dot{\mathcal{E}}_v=\zeta\left\{\delta P_n-S_c\delta T_c-\frac{i\omega}{k}\frac{(\rho_c-\rho_s)^2}{\rho_s}\right\}. \tag{15}$$

A comparison of the definition (4), (5) with the sum of the expressions (9) and (15) shows that it is convenient to represent the reciprocal growth coefficient in the form of a sum:

$$G^{-1}=G_s^{-1}+G_n^{-1},$$

where

$$G_s^{-1}=\frac{i}{\rho_c\omega}\tilde{\alpha}_{\mu\nu}k_\mu k_\nu-\frac{i\omega}{k}\frac{(\rho_c-\rho_s)^2}{\rho_c\rho_s}$$

corresponds to the contribution to f from the surface energy (9) and the superfluid component [the last term in (15)] and G_n^{-1} is given by the formula

$$\rho_cG_n^{-1}\dot{\zeta}=\delta P_n-S_c\delta T_c, \tag{16}$$

and corresponds to the contribution of the normal components of the liquid and the crystal.

If we set $G_n^{-1}=0$, we obtain from the equation $G_s^{-1}=0$ the spectrum of the crystallization waves:

$$\omega_0^2(\mathbf{k})=\frac{\rho_s k}{(\rho_c-\rho_s)^2}\tilde{\alpha}_{\mu\nu}k_\mu k_\nu. \tag{17}$$

Treating G_n^{-1} as a small perturbation, we find the frequency shift $\Delta\omega$ and the coefficient γ of absorption due to the interaction with the normal component:

$$\Delta\omega=\frac{\rho_c\rho_s k}{2(\rho_c-\rho_s)^2}\operatorname{Im}G_n^{-1},\quad \gamma=\frac{\rho_c\rho_s k}{2(\rho_c-\rho_s)^2}\operatorname{Re}G_n^{-1}. \tag{18}$$

To compute G_n^{-1}, we proceed from the linearized hydrodynamic equations for the normal component. In the liquid, these equations have the following form[15]:

$$\eta_l\Delta\mathbf{v}_n+i\omega\rho_n\mathbf{v}_n-\nabla P_n=0,\quad \operatorname{div}\mathbf{v}_n=0, \tag{19}$$

where η_l is the viscosity of the normal component. Generally speaking, we should include in the hydrodynamic equations for the phonon gas of the crystal (see, for example, Ref. 16, §16) additional terms for the description of the umklapp processes:

$$\eta_c\Delta\mathbf{v}+\left(i\omega\nu-\frac{T_cS_c^2}{\varkappa}\right)\mathbf{v}-S_c\nabla T_c=0,\quad C_c\dot{T}_c+T_cS_c\operatorname{div}\mathbf{v}=0. \tag{20}$$

Here η_c is the "viscosity" of the phonon gas, ν is the normal "density," more precisely, the coefficient of proportionality between the quasimomentum of a unit volume and the velocity $\mathbf{v}$, $\varkappa$ is the thermal conductivity coefficient, and C_c is the heat capacity of a unit volume. In (20) disregard for simplicity the crystal anisotropy.

Let us assume that the wave vector k satisfies the conditions

$$kl_N\gg a/l_N,\quad kl\gg a/l,\quad kl_N\gg(l_N/l_U)^{1/2}, \tag{21}$$

where l is the mean free path of the quasiparticles of the liquid, l_N and l_U are the crystal-phonon mean free paths characterizing the normal and umklapp processes respectively, and a is the interatomic distance. Naturally, for the hydrodynamic equations to be applicable, the conditions $kl_N\ll 1$ and $kl\ll 1$ should be fulfilled. In this case the second term in the first of the equations (19) for $\omega\sim\omega_0(\mathbf{k})$ is small compared to the first term. We can neglect the second and third terms in the first of the equations (20) and the first term in the second equation. The solutions that satisfy the conditions (1) for the crystal

$$\delta T_c = T_0 e^{kz}, \quad v_z = e^{kz}\left(\zeta + \frac{S_c}{2\eta_c} z T_0\right),$$
$$v_\mu = i\frac{k_\mu}{k} e^{kz}\left\{\zeta + \frac{S_c}{2\eta_c k}(kz+1)T_0\right\} \tag{22}$$

and the liquid

$$\delta P_n = P_0 e^{-kz}, \quad v_{nz} = e^{-kz}\left(\zeta + z\frac{P_0}{2\eta_l}\right),$$
$$v_{n\mu} = -i\frac{k_\mu}{k} e^{-kz}\left\{\zeta + \frac{P_0}{2\eta_l}(kz-1)\right\} \tag{23}$$

are similar in the case under consideration. Two arbitrary constants T_0 and P_0 enter into the formulas (22) and (23). One condition for their determination can be obtained if we note that the interaction of the long-wave phonons with the phase boundary in the absence of umklapp processes conserve the tangential components of the total momentum of the phonons. The components $\Pi_{\mu z}$ of the quasiparticle momentum flux are equal to

$$\Pi_{\mu z} = -\eta(\partial v_{n\mu}/\partial z + \partial v_{nz}/\partial x_\mu),$$

and therefore the conservation condition gives at $z = 0$ the relation

$$\eta_l(\partial v_{n\mu}/\partial z + ik_\mu v_{nz}) = \eta_c(\partial v_\mu/\partial z + ik_\mu v_z). \tag{24}$$

The remaining condition is a condition on the tangential velocities themselves. Under equilibrium conditions we clearly have $v_{n\mu} = v_\mu$ at $z = 0$. On the other hand, the presence of a nonzero flux $\Pi_{\mu z}$ at the boundary gives rise to a tangential-velocity jump:

$$v_\mu - v_{n\mu} = \sigma\Pi_{\mu z}. \tag{25}$$

Here σ is a constant of the order of $(\alpha c\rho_n)^{-1}$, c is the speed of sound, $\alpha \sim (T/\Theta)^2$ is the coefficient of transmission of the phonons through the phase boundary, and Θ is Debye temperature. At low temperatures the phonons are practically completely reflected from the boundary. The above value of α is obtained (see Ref. 17) when the capillary effects are taken into account.

The solutions (22) and (23) possess the property that $v_\mu(0)$ and $v_{n\mu}(0)$ for them vanish together with $\Pi_{\mu z}(0)$. It is therefore clear that the conditions (24) and (25) are actually equivalent to the vanishing at the boundary of the tangential velocities themselves. Thus, our result does not depend at all on the character of the interaction of the quasiparticles of the liquid and crystal with the phase boundary. As a result, we find that

$$T_0 = -\frac{2\eta_c}{S_c}k\zeta, \quad P_0 = 2\eta_l k\zeta, \tag{26}$$

whence we obtain through substitution into (16) the normal part of the reciprocal growth coefficient

$$G_n^{-1} = 2(\eta_l + \eta_c)k/\rho_c. \tag{27}$$

The crystallization-wave-absorption coefficient given by (18) is proportional to k^2 in our case. The frequency of the waves is proportional to $k^{3/2}$; therefore, the Q factor of the vibrations increases with decreasing frequency.

Let us point out the following circumstance. The boundary conditions (1) hold, as we have seen, under equilibrium conditions. But the solution found above leads, as is easy to see, to the occurrence of a temperature jump $\Delta T \sim (\eta/S)k\dot{\zeta}$ at the phase boundary. In the general case, instead of the two conditions (1), we must use the continuity condition

$$T_c S_c(v_z - \dot{\zeta}) = T_l S_l(v_{nz} - \dot{\zeta}) \tag{28}$$

for the heat flux and the relation between the heat flux and the temperature spectrum

$$KT_l S_l(v_{nz} - \dot{\zeta}) = \Delta T, \tag{29}$$

where $K \sim (\alpha Sc)^{-1}$ is Kapitza's thermal resistance. The ratio $\Delta T/KTS$ corresponding to the above-given value of ΔT is of the order of $\alpha(kl)\dot{\zeta}$, i.e., it is significantly smaller than $\dot{\zeta}$. The conditions (28) and (29) are actually equivalent to the conditions (1), and this equivalence obtains not only in the considered limiting case, but whenever the crystallization waves are weakly damped.

The phonon mean free path l_N in the crystal is determined by the three-phonon processes, and increases in proportion to T^{-5} as the temperature falls. In the liquid the phonon mean free path increases in proportion to T^{-9} in the low-temperature region, where phonons are the primary excitation mode (Ref. 14, §§42 and 43). The point is that the phonon spectrum of He II does not decay on the melting curve, and the three-phonon processes are forbidden by the conservation laws. The mean free path l_U in sufficiently perfect crystals increases exponentially with decreasing temperature. Therefore, the first inequality in (21) is the first to be violated as the wave vector is decreased.

Let us assume that

$$kl_N \ll a/l_N, \quad kl \gg a/l. \tag{30}$$

In this case the old solution (23) is valid in the liquid. In the crystal we can neglect in the first equation in (20) the term with $\varkappa$ and everywhere except in a narrow region of thickness of the order of the penetration depth $\delta_c \sim [\eta_c/\nu\omega_0(\mathbf{k})]^{1/2}$ of the viscous wave, the terms with η_c. In the second equation in (20) we can neglect the first term, since the condition necessary for this purpose,

$$k \gg (\omega C_c/\varkappa)^{1/2} \quad \text{or} \quad kl_N \gg (a/l_N)(l_N/l_U)^2,$$

is fulfilled on account of the exponential smallness of l_U^{-1}. The solution that vanish as $z \to -\infty$ has the form

$$\delta T_c = \frac{i\omega\nu}{S_c k}\zeta e^{kz}, \quad v_z = \zeta e^{kz}, \quad v_\mu = i\frac{k_\mu}{k}\zeta e^{kz}. \tag{31}$$

We should, in using the boundary condition (24), bear in mind that the solution (31) is not valid in the region $|z| \lesssim \delta_c$. In this narrow region v_μ, being in order of magnitude equal to ζ, varies over distances of the order of δ_c. But because there is fulfilled along with the inequality $\eta_l \gg \eta_c$ the stronger inequality $\eta_l \gg \eta_c(k\delta_c)^{-1}$ [since

$$\eta_l \sim \rho_n cl \propto T^{-5}, \quad \eta_c \propto T^{-1}, \quad \omega \sim \omega_0(\mathbf{k}) \propto k^{3/2},$$

but $ka \gg (T/\Theta)^{18}$ on account of (30)], we can, in determining the constant P_0 with the aid of the condition (24), neglect the right-hand side of this condition, so that the earlier result (26) is obtained. As for the computation of $\delta T_c(0)$, it is not, as can be seen from

(31), essential that we know the exact value of v_μ in the narrow region $|z| \leq \delta_c$, and we obtain $\delta T_c(0) = (i\omega\nu/\mathring{S}_c k)\zeta$. The contribution of the normal components to the reciprocal growth coefficient is, on account of (16), equal to

$$G_n^{-1} = 2\eta_l k/\rho_c - i\omega\nu/k\rho_c. \tag{32}$$

Here the normal component of the liquid ensures the damping of the crystallization waves and the phonon gas of the solid ensures the frequency shift.

Now let

$$kl_N \ll a/l_N, \quad kl \ll a/l, \tag{33}$$

with k significantly greater, as before, than all the exponentially small terms. The solution for the crystal in the region outside the viscous-wave penetration region is given by the formulas (31). In the liquid in the present case, for $z \gg \delta_l$, where $\delta_l \sim (\eta_l/\rho_n\omega)^{1/2}$, the solution has a similar form

$$\delta P_n = -i\frac{\omega\rho_n}{k}\zeta e^{-kz}, \quad v_{nz} = \zeta e^{-kz}, \quad v_{n\mu} = -i\frac{k_\mu}{k}\zeta e^{-kz}. \tag{34}$$

The normal part of the reciprocal growth coefficient is equal to

$$G_n^{-1} = -i\frac{\omega}{\rho_c k}(\rho_n + \nu). \tag{35}$$

The crystallization-wave-frequency shift $\Delta\omega$ is significantly greater than the damping constant in the present case. A higher degree of computational accuracy is therefore required for the calculation of the damping constant. In the formula (10) we should take into account in the expressions for the energy fluxes $\mathbf{Q}_{c,l}$ additional dissipative terms $\mathbf{Q}'_{c,l}$ equal to

$$Q_{li}' = -\eta_l v_{nk}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i}\right),$$
$$Q_{ci}' = -\eta_c v_k\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{36}$$

and specially consider the small regions of thickness of the order of the viscous-wave-penetration depths. In these regions the expressions for v_μ and $v_{n\mu}$ contain, along with the terms given by the formulas (31)and(34), additional rapidly varying terms:

$$v_\mu = i\frac{k_\mu}{k}(\zeta e^{kz} + a_c e^{q_c z}), \quad v_{n\mu} = -i\frac{k_\mu}{k}(\zeta e^{-kz} + a_l e^{-q_l z}), \tag{37}$$

where a_l and a_c are arbitrary constants and

$$q_l = (1-i)(\omega\rho_n/2\eta_l)^{1/2}, \quad q_c = (1-i)(\nu\omega/2\eta_c)^{1/2}.$$

In the boundary condition (24) we can neglect the right-hand side on account of the inequality $\eta_l \gg \eta_c$ and the second term of the left member on account of the inequality $|q_l| \gg k$. As a result, we find that $a_l = 0$. We can also neglect the right-hand side of the boundary condition (25), so that $a_c = -2\zeta$. We can, in computing the time average $\overline{Q'_{cz}}$ of the flux Q'_{cz}, take account of only the term containing the derivative of the rapidly varying term in (37):

$$\overline{Q_{cz}'} = -\eta_c \overline{v_\mu \frac{\partial v_\mu}{\partial z}} = -(2\eta_c\omega\nu)^{1/2}|\zeta|^2.$$

A rapidly varying term does not occur in the solution for the liquid. Therefore,

$$\overline{Q_{lz}'} = -2\eta_l \overline{v_{nz}\frac{\partial v_{nz}}{\partial z}} - \eta_l \overline{v_{n\mu}\left(\frac{\partial v_{nz}}{\partial x_\mu} + \frac{\partial v_{n\mu}}{\partial z}\right)} = 2\eta_l k|\zeta|^2.$$

By comparing the expression

$$\overline{\dot{\mathscr{E}}} = \overline{Q_{lz}'} - \overline{Q_{cz}'}$$

with the formula (6), we find

$$\operatorname{Re} G_n^{-1} = \frac{4\eta_l k}{\rho_c} + \frac{2}{\rho_c}(2\eta_c\omega\nu)^{1/2}. \tag{38}$$

For the solutions (31), (34), and (37) to be applicable, it is necessary that the condition $\omega\nu \gg TS^2/\varkappa$, i.e.,

$$kl_N \gg (l_N/l_U)^{2/3}(l_N/a)^{1/3},$$

besides (33), be fulfilled. But the boundary condition (24) presupposes the negligibility of the surface umklapp processes as well. These processes are responsible for the nonconservation of the total momentum of the excitations in their interaction with the surface, a fact which can be taken into account by adding to the right-hand side of Eq. (24) an extra term that is proportional to a tangential velocity (it is not important which of the two tangential velocities v_μ and $v_{n\mu}$, since we can neglect the right member of the condition (25)]:

$$\eta_l\left(\frac{\partial v_{n\mu}}{\partial z} + ik_\mu v_{nz}\right) = \eta_c\left(\frac{\partial v_\mu}{\partial z} + ik_\mu v_z\right) + \beta v_\mu. \tag{39}$$

The constant β is perfect crystals with perfect surfaces exponentially decreases with decreasing temperature, but the exponent is, generally speaking, smaller than the corresponding exponent for the volume umklapp processes. The result (38) is valid provided $\eta_l|q_l| \gg \beta$, but in this case also it is, generally speaking, necessary to allow for the energy dissipation

$$\overline{\dot{\mathscr{E}}} = {}^1/_2\beta|v_\mu|^2 = {}^1/_2\beta|\dot{\zeta}|^2,$$

due to the surface umklapp processes, an allowance which leads to the appearance of an additional contribution to the real part of the reciprocal growth coefficient:

$$\operatorname{Re} G_n^{-1} = \beta/\rho_c. \tag{40}$$

In the opposite limiting case $\eta_l|q_l| \ll \beta$, we find from the formula (39) that $v_\mu(0) = v_{n\mu}(0) = 0$, so that the constants in (37) are equal to $a_c = a_l = -\zeta$. It is convenient to compute the energy dissipation in the present case directly with the aid of the well-known expression:

$$\overline{\dot{\mathscr{E}}}\frac{\eta_l}{4}\int dV\left|\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i}\right|^2 = \left(\frac{\eta_l\omega\rho_n}{8}\right)^{1/2}|\dot{\zeta}|^2,$$

whence we obtain

$$\operatorname{Re} G_n^{-1} = (\eta_l\omega\rho_n/2\rho_c^2)^{1/2}. \tag{41}$$

The analogous term corresponding to the crystal-phonon-induced dissipation can, on account of the inequality $\eta_l \gg \eta_c$, be neglected. Finally, let

$$\omega\nu \ll \frac{TS_c^2}{\varkappa}, \quad k \gg \left(\frac{\omega C_c}{\varkappa}\right)^{1/2} \quad \text{or} \quad \left(\frac{l_N}{l_U}\right)^{2/3}\left(\frac{l_N}{a}\right)^{1/3} \gg kl_N \gg \left(\frac{l_N}{l_U}\right)^2\frac{a}{l_N}. \tag{42}$$

This is the region of lowest frequencies where the crystallization waves still exist as slowing decaying waves. In the first equation in (20) we can neglect the first term and the term with ν, so that the velocity is expressed directly in terms of the temperature gradient. In the second equation in (20) we can, as before, neglect the first term; whence it can be seen that the temperature satisfies the Laplace equation. The solution satisfying the condition (1) has the form

$$\delta T_c = -TS_c^2\dot{\zeta}e^{kz}/\varkappa k.$$

From (16) we find

$$G_n^{-1} = TS_c^2/\rho_c\varkappa k. \tag{43}$$

In the present case we can neglect the dissipation in the liquid. The condition $\gamma \ll \omega_0(\mathbf{k})$ for weak damping of the waves is, as can be seen from (18) and (43), violated when $\omega_0 \sim (\nu/\rho_c)(TS_c^2/\varkappa\nu)$, i.e., in the frequency region (42) under consideration.

2. THE NONLOCAL HEAT CONDUCTION REGIME

If the crystal contains a large number of defects, the mean free path l_U is determined by the interaction of the phonons with the defects, and is not exponentially large at low temperatures. The damping of the crystallization waves in this case is, generally speaking, determined by the formula (43). But the important case in which we have point lattice defects or impurities to deal with requires special consideration. It is well known that the scattering of the phonons by point defects alone cannot (because of the rapid decrease of the scattering cross section with decreasing phonon energy) guarantee a finite value for the thermal-conductivity coefficient $\varkappa$ at low temperatures. As has been shown by Levinson,[18] there obtains in this case a regime of nonlocal heat conduction. The thermal conductivity coefficient in the case of an infinite crystal depends strongly on the value of the wave vector **k**. The purpose of the present section is to derive for the growth coefficient an expression that replaces (43) in the case of nonlocal heat conduction.

We proceed from the diffusion equation, used in Ref. 18, for the distribution function $\eta(\mathbf{r},\varepsilon)$ for long-wave phonons with energy ε significantly lower than the temperature:

$$D(\varepsilon)\Delta n = (n-n_0)/\tau. \tag{44}$$

Here $D(\varepsilon)=D_0/\varepsilon^4$ is the diffusion coefficient for phonons with energy ε; $D_0 \sim c\Theta^4/Na^2$, where N is the number of point defects per unit volume of the crystal, is a constant; $\tau=\tau_0/\varepsilon$, where $\tau_0 \sim (\Theta/T)^4$, is the relaxation time time due to the three-phonon processes; and $n_0(\varepsilon) = T(\mathbf{r})/\varepsilon$ is the equilibrium distribution function. Setting

$$T=T_0+\delta T, \qquad n=n_0(T_0)+\delta n,$$

where T_0 is the equilibrium value of the temperature, and assuming that the dependence of δT and δn on the coordinates x_μ stems from the factor $\exp ik_\mu x_\mu$, we rewrite (44) in the form

$$\partial^2\delta n/\partial z^2 - q^2\delta n = -(D\varepsilon\tau)^{-1}\delta T(z), \tag{45}$$

where $q=(k^2+1/D\tau)^{1/2}$. The solution that goes to zero in the interior of the crystal as $z \to -\infty$ is given by the formula

$$\delta n(z) = ae^{qz} + (2D\varepsilon\tau q)^{-1}\left\{e^{-qz}\int_{-\infty} e^{qz'}\delta T(z')\,dz' + e^{qz}\int_z^0 e^{-qz'}\delta T(z')\,dz'\right\}. \tag{46}$$

Here $a=a(\varepsilon)$ is an arbitrary constant. The boundary condition for its determination follows from the conservation of the number of long-wave phonons in their interaction with the phase boundary, and has the form: $D\,\partial n/\partial z = -n_0(T_0)\dot{\zeta}$. The contribution of the right-hand side of this condition to the final result is, however, small compared to the contribution of the other term, which is also proportional to $\dot{\zeta}$, but stems from the short-wave phonons. Therefore we can, assume in fact that $\partial n/\partial z = 0$ at $z=0$. As a result,

$$\delta n(z) = (2D\varepsilon\tau q)^{-1}\left\{e^{qz}\int_{-\infty}^0 e^{qz'}\delta T(z')\,dz' + e^{-qz}\right.$$
$$\left.\times\int_{-\infty} e^{qz'}\delta T(z')\,dz' + e^{qz}\int_z e^{-qz'}\delta T(z')\,dz'\right\}. \tag{47}$$

Let us formally continue the functions $\delta T(z)$ and $\delta n(z)$ into the region of positive z in an even fashion, i.e., in such a way that

$$\delta T(-z) = \delta T(z), \qquad \delta n(-z) = \delta n(z).$$

Then

$$\delta n(z) = (2D\varepsilon\tau q)^{-1}\int_{-\infty}^{+\infty} dz' e^{-q|z-z'|}\delta T(z'). \tag{48}$$

The heat flux vector **Q** is equal to

$$\mathbf{Q} = -\int_0^\infty \varepsilon D(\varepsilon)\nabla\delta n\rho(\varepsilon)\,d\varepsilon, \tag{49}$$

where $\rho(\varepsilon)=\rho_0\varepsilon^2$ ($\rho_0 = 9\rho_c/m\Theta^3$, m being the mass of the atom) is the level density.

Let us consider the heat-flux potential ψ, defined by the relation $\mathbf{Q}=\nabla\psi$, and equal, on account of (49), to:

$$\psi = -\int \varepsilon\rho(\varepsilon)D(\varepsilon)\delta n\,d\varepsilon. \tag{50}$$

From the continuity equation $\operatorname{div}\mathbf{Q}=0$ we obtain the equation $\Delta\psi=0$, so that $\psi=\psi_0 e^{kz}$, where ψ_0 is a constant. For $z=0$ the heat flux $Q_z=-k\psi_0$ through the boundary should, on account of (1), be equal to $TS_c\dot{\zeta}$, whence we find ψ_0. If we continue $\psi(z)$ into the region of positive z in an even fashion, so that the equality (50) obtains for all z, then we have

$$\psi(z) = -TS_c\dot{\zeta}e^{-k|z|}/k. \tag{51}$$

Going over from the functions $\delta n(z)$, $\delta T(z)$, and $\psi(z)$ to the Fourier transforms $\delta n(p)$, $\delta T(p)$, and $\psi(p)$, we find from (48), (50), and (51) that

$$\delta T(p) = -\frac{10}{\pi}\sin\left(\frac{\pi}{5}\right)\frac{TS_c\tau_0^{1/5}}{\rho_0 D_0^{4/5}}(p^2+k^2)^{-4/5}\zeta. \tag{52}$$

The normal part of the reciprocal growth coefficient is, on account of (16), equal to

$$G_n^{-1} = -\frac{S_c}{\rho_c\zeta}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\delta T(p) = \frac{5}{\pi^{5/2}}\sin^2\frac{\pi}{5}\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{3}{10}\right)\frac{TS_c^2\tau_0^{1/5}}{\rho_c\rho_0 D_0^{4/5}}k^{-3/5}. \tag{53}$$

Instead of the law $G_n \propto k$, (43), which is characteristic of normal thermal conduction, there obtains under nonlocal conditions the weaker dependence $G_n \propto k^{3/5}$.

3. THE BALLISTIC REGIME

For sufficiently large values of the wave vector k_μ, the wavelength $1/k$ is small compared to the mean free path of the thermal excitations. Under the conditions of such a ballistic regime, the normal component should be described with the aid of the quasiparticle distribution function.

At low temperatures the quasiparticles of the liquid and the crystal are virtually totally reflected from the phase boundary. In the case of the phonons this follows from the above-mentioned results of Castaing and Nozieres.[12] In the case of the rotons, this will be shown below. Under such conditions the quasiparticles of the liquid and the crystal clearly make additive contributions to the reciprocal growth coefficient G_n^{-1}. Therefore, we can, in computing the contribution of the liquid, neglect the thermal excitations of the crystal.

The energy per unit volume, the chemical potential, and the energy-flux vector in the liquid are, when expressed in terms of the quasiparticle distribution function $n(\mathrm{p},\mathrm{r},t)$, equal to (Refs. 14, §37 and 19, §77):

$$E_l=E_0(\rho_l)+\langle n\varepsilon\rangle,\quad \mu_l=\mu_0(\rho_l)+\left\langle n\frac{\partial\varepsilon}{\partial\rho}\right\rangle,\quad \mathbf{Q}_l=\mu_l\mathbf{j}+\left\langle nH\frac{\partial H}{\partial\mathbf{p}}\right\rangle, \tag{54}$$

where $E_0(\rho_l)$ and $\mu_0(\rho_l)$ are the values of the energy and the chemical potential at $T=0$, $\varepsilon=\varepsilon(\mathrm{p})$ is the energy spectrum of the quiescent liquid, $H=\varepsilon(\mathrm{p})+\mathrm{p}\cdot\mathbf{v}_s$ is the energy (Hamiltonian) with allowance for the superfluid motion, and the angle brackets denote multiplication by $d^3p/(2\pi\hbar)^3$ and integration over the entire momentum space.

The reflection of the quasiparticles from the phase boundary, as it moves with velocity $\dot{\zeta}$, occurs with conservation of the energy in the rest frame of the boundary, i.e., with conservation of the quantity $H-p_z\dot{\zeta}$. The excitation-energy flux through the boundary in this frame should be equal to zero:

$$\langle(H-p_z\dot{\zeta})(\partial H/\partial p_z-\dot{\zeta})n\rangle=0.$$

Hence we obtain

$$\left\langle nH\frac{\partial H}{\partial p_z}\right\rangle=\zeta\left(\left\langle p_z\frac{\partial H}{\partial p_z}n\right\rangle+\langle Hn\rangle-\zeta\langle p_z n\rangle\right). \tag{55}$$

The last term in the right member can be neglected as a term of order higher than the second in the deviation from equilibrium. We need not distinguish between H and ε in the first two terms.

Substituting (55) into (54), and using (10) and the definition (4), we obtain

$$f_v=E_c-E_0(\rho_l)-\mu_l(\rho_c-\rho_l)+\left\langle p_z\frac{\partial\varepsilon}{\partial p_z}n\right\rangle. \tag{56}$$

Let us set $n=n_0+\delta n$, where n_0 is the equilibrium distribution function corresponding to the equilibrium temperature, i.e., the temperature at which $f=0$. We should, in computing the contribution f_n made by the nonequilibrium part δn of the distribution function to f, bear the following in mind. The existence of a nonzero δn in the boundary layer of thickness of the order of the wavelength k^{-1} leads to a deviation $\delta\rho_l$ of the density from the equilibrium value at infinity. Owing to the smallness of the velocity of the crystallization waves, we can assume that the quantities δn and $\delta\rho_l$ are connected with each other by the equilibrium condition for the motion of the superfluid component, i.e., by the requirement that the chemical potential be a constant, or that $\delta\mu_l=0$, thereby neglecting the small contribution of the normal component to the real part of the wave frequency. Furthermore, we can everywhere neglect the terms of the type

$$(\mu_l-\mu_0)\delta\rho_l \text{ or } \langle np_z\partial^2\varepsilon/\partial p_z\partial\rho_l\rangle\delta\rho_l,$$

which are proportional to the product of two integrals of the distribution function. Such terms clearly decrease with decreasing temperature much faster than the single integrals.

As a result, we find from (56) that

$$f_n=\left\langle p_z\frac{\partial\varepsilon}{\partial p_z}\delta n\right\rangle. \tag{57}$$

The quantity $H(\mathrm{p},\mathrm{r})$ is conserved in the free motion of the quasiparticles in the absence of collisions. It is therefore clear that the distribution function for the quasiparticles incident at the boundary is equal to $n_0(\varepsilon+\mathrm{p}\cdot\mathbf{v}_s)$, i.e., the nonequilibrium part δn entering into (57) is, for $v_z<0$, equal to $\mathrm{p}\cdot\mathbf{v}_s\partial n_0/\partial\varepsilon$. Let $\mathrm{p}_0(\mathrm{p})$ for $v_z(\mathrm{p})>0$ be the momentum of such an incident quasiparticle, which is reflected (with probability equal to unity) into a state with momentum p. Then $n(\mathrm{p})=n(\mathrm{p}_0)$. Since

$$\varepsilon(\mathbf{p}_0)+\mathbf{p}_0\mathbf{v}_s=\varepsilon(\mathbf{p})+\mathbf{p}\mathbf{v}_s-(p_z-p_{0z})\dot{\zeta},$$

for $v_z(\mathrm{p})>0$ we have

$$\delta n=\frac{\partial n_0}{\partial\varepsilon}[\mathbf{p}\mathbf{v}_s-(p_z-p_{0z})\zeta].$$

Substitution into (57) yields

$$f_n=-\zeta\left\langle p_z(p_z-p_{0z})\frac{\partial\varepsilon}{\partial p_z}\frac{\partial n_0}{\partial\varepsilon}\right\rangle_R, \tag{58}$$

where the subscript R indicates that the integration is performed over only the reflected quasiparticles with $v_z>0$.

The normal part of the reciprocal growth coefficient is, on account of (58), equal to

$$G_n^{-1}=-\frac{1}{\rho_c}\left\langle p_z(p_z-p_{0z})\frac{\partial\varepsilon}{\partial p_z}\frac{\partial n_0}{\partial\varepsilon}\right\rangle_R. \tag{59}$$

The long-wave phonons are reflected specularly from the boundary, i.e., for them, $p_z-p_{0z}=2p_z$. In the phonon temperature region the integral (59) is equal to

$$G_n^{-1}=\frac{\pi^2T^4}{30\hbar^3\rho_c c^4}. \tag{60}$$

To compute the contribution of the rotons, let us consider their interaction with the phase boundary. Rotons are characterized by the fact that there are in their case four values of p_z for a given energy and a given momentum p_t tangential to the surface. These four states correspond to the four points 1–4 in Fig. 1, where the continuous circles represent a doubly connected constant-energy surface with $\varepsilon>\Delta$, while the dashed circle represents the $\varepsilon=\Delta$ surface (Δ is the

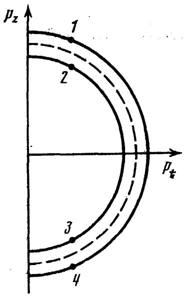

FIG. 1.

minimum roton energy). The points 1 and 3 correspond to the reflected rotons; the points 2 and 4, to the rotons incident at boundary. Since $\varepsilon - \Delta \ll \Delta$, the pairs of points 1,2 and 3,4 are close to each other. The transitions 2 → 1 and 4 → 3 therefore correspond to reflections with a small change in the momentum. It is interesting that, here, as in the case of the reflection of electron-like quasiparticles from a superconducting-normal metal interface,[20] all the three components of the velocity of the quasiparticle change sign. It is easy to see that, for sufficiently small values of $\varepsilon - \Delta$, reflections of this type occur with overwhelming probability from an arbitrary liquid boundary, i.e., under conditions when we could, generally speaking, expect the $\mathbf{p}_t$-conservation law to be violated because of the roughness of the boundary. Indeed, let us consider the energy ε of the roton as a function of p_z for a given $\mathbf{p}_t$ in small neighborhoods of the pairs of points 1,2 and 3,4. Let q_z be the value of p_z measured from $p_z^{(0)}$, the middle of the straight line joining the points 1 and 2 (or 3 and 4). Expanding the roton-energy spectrum in powers of q_z, we obtain

$$\varepsilon(q_z) = \Delta + (q_z^2/2\mu)\cos^2\theta, \tag{61}$$

where $\cos\theta = p_z^{(0)}/p_0$, p_0 and μ being the roton momentum and effective mass. The corresponding Schrödinger equation for the wave function ψ describing the "channel," under consideration here, of reflection involving a small change in the momentum has the form

$$\frac{\partial^2\psi}{\partial z^2} + \frac{2\mu}{\hbar^2\cos^2\theta}[\varepsilon - \Delta - U(z)]\psi = 0.$$

Here $U(z)$ is the energy of interaction of a roton with the surface at large distances $|z|$. Arguments that are literally a repetition of the arguments adduced in Ref. 21 for the case of the interaction of grazing electrons with a metal boundary show that, if $U(z)$ decreases as $|z| \to \infty$ faster than $|z|^{-2}$ (which is virtually always the case), then the 2 → 1 or 4 → 3 reflection probability tends to unity as $\varepsilon \to \Delta$.

For the two points 1 and 3 the difference $p_z - p_{0z}$ in (59) is equal to

$$p_z - p_{0z} = 2|q_z| = 2[2\mu(\varepsilon - \Delta)]^{1/2}/|\cos\theta|.$$

Substitution into (59) uields after simple calculations the expression

$$G_n^{-1} = \frac{2p_0^2\mu T}{\pi^2\hbar^3\rho_c}e^{-\Delta/T}\int\frac{d\cos\theta}{\cos\theta}.$$

The logarithmically diverging angle integral should be trancated at the limit

$$q_z \sim (\mu T)^{1/2}/\cos\theta \ll p_z \sim p_0\cos\theta$$

of the region of applicability of the expansion of (61) in powers of q_z. The lower limit of the integral is therefore the quantity $(\mu T/p_0^2)^{1/4}$. As a result, the roton part of the reciprocal growth coefficient is equal to

$$G_n^{-1} = \frac{p_0^2\mu T}{2\pi^2\hbar^3\rho_c}e^{-\Delta/T}\ln\frac{p_0^2}{\mu T}. \tag{62}$$

Under the conditions of the ballistic regime, the phonons of the crystal make to G_n^{-1} a contribution of the same order of magnitude as the contribution (60) of the phonons of the liquid. Since in the case of the crystal phonons the exact computation of the coefficient of T^4 would require allowance for the anisotropy of the elastic moduli of the crystal, we shall not consider this problem here, especially as there exists a broad frequency region where the phonons of the liquid are ballistic, but, for the crystal phonons, because of their significantly shorter mean free path, the regime is hydrodynamic.

4. DISCUSSION OF THE RESULTS

Let us track the frequency dependence of the dimensionless crystallization-wave absorption coefficient $\tilde{\gamma} = \gamma/\omega_0(\mathbf{k})$ (the reciprocal Q factor of the vibrations). In the region of extremely low frequencies, where the waves still exist as slowly decaying waves, $\tilde{\gamma}$ decreases, according to (18) and (43), in proportion to ω as the frequency increases. For perfect crystals, the umklapp processes become ineffective as the frequency is increased in the hydrodynamic region, and the decrease of $\tilde{\gamma}$ is replaced by an increase first according to the law $\omega^{1/6}$ [(41), (38)], and then like $\omega^{1/3}$ [(38), (32), (27)] (in the intermediate frequency region the law $\omega^{-1/3}$ is possible as a result of the surface umklapp processes). Finally, in the ballistic region the coefficient $\tilde{\gamma}$ again decreases in proportion to $\omega^{-1/3}$ [(60), (62)] as the frequency increases. In the region of still higher frequencies the decay of the spectrum of the crystallization waves should have an effect, and the coefficient $\tilde{\gamma}$ should increase, as the frequency increases, in proportion to $\omega^{10/3}$ as a result of the processes involving the decay of one vibration quantum into two quanta.

The maximum value of the coefficient $\tilde{\gamma}$ at the boundary between the ballistic, (60), and hydrodynamic, (27), regimes in the phonon temperature region is of the order of $(\rho_n/\rho)(l/a)^{1/2}$. For $l \sim a(\theta/T)^9$, this value is large, and, therefore, the slowly decaying waves exist either at lower, or at higher, frequencies only. In contrast, in the high-temperature roton region, the peak value of $\tilde{\gamma}$ is proportional to $\exp(-\Delta/2T)$, i.e., it is small, and, consequently, the transition from the hydrodynamic to the ballistic regime occurs under weak-damping conditions.

In the case of sufficiently imperfect crystals, only the laws (43) and (53) can exist besides the ballistic laws (60) and (62). In the former case $\tilde{\gamma}$ is proportional to $\omega^{-11/15}$. There is no hydrodynamic damping minimum.

Judging by the frequency dependence of the damping, the experimental data reported by Keshishev, Parshin, and Babkin[11] pertain to the ballistic regime. In the roton region the observed temperature dependence of the damping is in good agreement with the formula (62), but the experimental value for the damping constant itself is roughly an order of magnitude higher than the theoretical value (62). In the phonon region the experimentally obtained value of the coefficient attached to T^4 is roughly 5 times higher than the value given by the formula (60), which corresponds to the situation in which only the phonons of the liquid are taken into account. This discrepancy can hardly be ascribed to the crystal phonons, since, according to the latest data,[22] their mean free path in the temperature region investigated by Keshishev *et al.*[11] is significantly shorter than the wavelength of the vibrations. These discrepancies can be attributed to the state of the crystal surface. As in the case of heat exchange between solid and liquid helium,[23] the limiting low-temperature laws of reflection of quasiparticles from the boundary can actually be valid at temperatures lower than those at which the experiment was performed.

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