Electric conductivity of two-dimensional metallic systems

R. N. Gurzhi, A. I. Kopeliovich, and S. B. Rutkevich

Physicotechnical Institute for Low Temperatures, Academy of Sciences of the USSR

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The main mechanisms of electron relaxation in two-dimensional systems are studied; these mechanisms are associated with electron-phonon, electron-electron, and phonon-phonon collisions. It is shown that under phonon-drag conditions the electric conductivity of a pure two-dimensional metal differs from that of an ordinary three-dimensional metal not only in order of magnitude but also in the nature of its temperature dependence. The temperature-dependent corrections to the residual resistance are also calculated for conditions in which electron — impurity scattering predominates.

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1. INTRODUCTION

It is well known that at sufficiently low temperatures the electron-phonon scattering is mainly at small angles. This allows us to treat the electric conductivity (and other transport processes) in terms of the diffusion of an electron over the Fermi surface (FS) with a step equal to the thermal momentum q of a phonon.[1] With phonon drag taken into account, this diffusion is described by the equation[2]

$$\operatorname{div}\hat{D}(\nabla\chi_{\mathbf{p}}-\mathfrak{a}\{\nabla\chi_{\mathbf{p}}\})=e\mathbf{E}\mathbf{n}. \qquad (1)$$

Here $\chi_{\mathbf{p}}$ is the distribution function of the nonequilibrium electrons; the diffusion tensor D is proportional to the constant of the electron-phonon interaction and depends on the temperature as T^5; div and ∇ are two-dimensional operators in the tangent plane to the FS; $\mathbf{n}$ is a unit vector along the velocity of the electron; and $\mathbf{E}$ is the electric field strength. The linear integral functional

$$\mathfrak{a}\{\nabla\chi_{\mathbf{p}}\}=\int dS_{\mathbf{p}'}\hat{A}_{\mathbf{p}\mathbf{p}'}\nabla\chi_{\mathbf{p}'},$$

is due to the phonon drag; the diffusion flux at the point p is governed by phonons emitted over the entire FS (for details see Ref. 2).

Equation (1) describes the stationary process of diffusion of electrons, which "arise" on one half of the FS (where $e\mathbf{E}\cdot\mathbf{n}>0$) and "disappear" on the other half (where $e\mathbf{E}\cdot\mathbf{n}<0$). This equation leads to Bloch's law for the temperature dependence of the electric resistance: $\rho_B\sim T^5$.

The situation is entirely different for a two-dimensional metal, in which the energy of an electron depends only on two components of its quasimomentum. It can be proved rigorously that in this case, and only in this case, the integrodifferential equation (1) is unsolvable.[3] With a two-dimensionsl FS the kernel $A_{\mathbf{p}\mathbf{p}'}$ has singularities at the points $\mathbf{p}'=\pm\mathbf{p}$, and Eq. (1) takes the form

$$\tfrac{1}{2}\operatorname{div}\hat{D}(\nabla\chi|_{\mathbf{p}}-\nabla\chi|_{-\mathbf{p}})=e\mathbf{E}\mathbf{n}.$$

This equation cannot be solved, because its left side is an even function of p and its right side is an odd function.

Let us elucidate the physical meaning of this last result. As is well known, a phonon can interact only with an electron whose velocity is almost perpendicular to the momentum $\mathbf{q}$ of the phonon (in the approximation to which the diffusion equation (1) corresponds these vectors are exactly perpendicular, $\mathbf{q}\cdot\mathbf{n}=0$). In a metal with a cylindrical FS this means that exchange of a phonon is possible only between electrons whose velocities are parallel or antiparallel to each other (Fig. 1). The electrons with a given velocity direction $\pm\mathbf{n}$ and the phonons that interact with them form a closed subsystem, which cannot exchange momentum with another subsystem corresponding to a different direction $\mathbf{n}'$. On the other hand, the diffusion flux of electrons at a given point of the FS must have a definite direction and consequently there must be constant absorption of phonon in this direction (or emission in the opposite direction). The result would be to strip the subsystem of such phonons, and so the diffusion process is locally forbidden at each point of the FS. Things are quite different in ordinary three-dimensional metals, where any two electrons can exchange a phonon (with its momentum in the direction of the vector product of their velocities), and therefore mixing of the momenta over the entire FS occurs. Accordingly, the relaxation mechanism of the electron gas in two-dimensional and in three-dimensional systems are qualitatively different, and this must result in differences between their kinetic properties.

The purpose of the present paper is to systematically study the possible conductivity mechanisms in metals in which the electron spectrum is two-dimensional and the phonon spectrum is three-dimensional.[1)] In classifying the relaxation processes we note that the conclusion that the electron-phonon system is broken up into isolated groups is only approximate, for several rea-

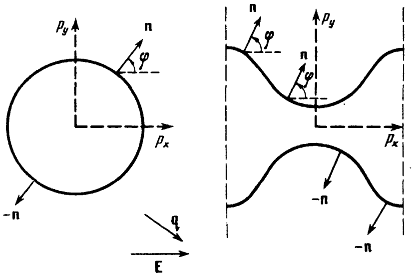

FIG. 1. Convex and nonconvex Fermi surfaces (FS).

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sons. First of all, the momentum and energy of a phonon have been regarded as infinitely small [Eq. (1) was derived in the quadratic approximation in the "inelasticity parameter" s/v, where s is the speed of sound]. Besides this, in considering only electron-phonon collisions we have neglected the contributions of other interaction processes among the quasiparticles. Owing to these facts there are three groups of relaxation mechanisms. They are as follows:

1) Scattering of electrons by phonons with small but finite momenta obviously destroys the mutual isolation of the groups of electrons. This, however, leads to relaxation only for FS of a particular structure.

We first show that in metals with convex FS taking the finite momenta of the phonons into account does not lead to relaxation. In fact, suppose that a stationary flux of electrons has been established on the FS in the external electric field. Then there are pairs of points p and k for which the transition $\mathbf{p} \to \mathbf{k}$ occurs more frequently than the opposite transition. Then there is continuous production of phonons with momentum $\mathbf{q} = \mathbf{p} - \mathbf{k}$ (or destruction of those with momentum $-\mathbf{q}$). Accumulation of such photons blocks the channel $\mathbf{p} \to \mathbf{k}$. The presence of a symmetrical pair of points does not alter this conclusion; if, for example, $\chi_\mathbf{p} > \chi_\mathbf{k}$, then $\chi_{-\mathbf{p}} < \chi_{-\mathbf{k}}$ because $\chi_\mathbf{p}$ is an odd function, and therefore photons in this direction will accumulate in the channel $-\mathbf{k} \to -\mathbf{p}$. Accordingly, relaxation in any electron distribution that is not centrally symmetrical turns out to be impossible. We emphasize that this conclusion holds only in the elastic-collision approximation (see Fig. 2).

Relaxation is possible if there are "many-channel" processes brought about by the exchange of photons among several pairs of electronic states. For this it is necessary that the FS contain at least two points with any given normal direction $\pm n$ [or else that there be none at all, see Fig. 1(b)], and that the radii of curvature be different at these points. We shall provisionally call such surfaces "nonconvex," although they include, for example, arbitrary multiplically connected FS.

In the case of a nonconvex FS the relaxation process (we call it q/p superdiffusion in analogy with the mechanism of Gurevich and Laĭkhtman[5]) is described by a local differential equation, in which, however, there exist simultaneously all states with a given direction of the normal. The corresponding resistance is decreased compared with the Bloch value by a factor $(T/\Theta)^4$: $\rho \approx \rho_B(T/\Theta)^4$, where Θ is the Debye temperature.

2) The second group of relaxation mechanisms is due to the fact that the phonon energy is finite. It is clear that in dealing with them we are concerned with the energy distribution of the electrons in the neighborhood of the FS. It turns out that the energy dependence that is established here (we call corresponding electron distribution the "local drift") is such that there is no resistance not only in zeroth order but also in the approximation quadratic in the parameter s/v. For an arbitrary structure of the FS relaxation of the local drift can be assured by two mechanisms of different nature.

The first is a nonlocal mechanism associated with the scattering of electrons by phonons whose momentum is almost parallel to a generator of the FS (angles of deviation of the order of s/v). Obviously, such glancing phonons can be exchanged by any two electronic states (as in the case of a three-dimensional FS). The corresponding resistance is $\rho \sim \rho_B(s/v)^2$.

The second is a local mechanism of superdiffusion, in which relaxation is brought about by exchange of phonons between states close together within the thermal layer. Its contribution is $\rho \sim \rho_B(T/\mu)^2$, where μ is the chemical potential.

3) Of the mechanisms of the third group, the simplest to deal with is the contribution of phonon-phonon collisions. They make possible exchanges of momentum to "isolated" groups of electrons, widely separated in their velocity directions **n**. This is a nonlocal mechanism, and is described by an equation of the form (1), $\rho \sim (T/\Theta)^9$.

Here we note that normal collisions between electrons in two-dimensional metals can be ineffective. In fact, in the case of a convex FS it follows from the conservation laws that the initial electron states $\mathbf{p}_1$ and $\mathbf{p}_2$ agree up to quantities of the order T/μ with the final states $\mathbf{p}_3$ and $\mathbf{p}_4$ (see Fig. 3). Relaxation occurs in a higher order in T/μ, corresponding to a time $\tau_{ee} \sim T^{-4}$.

The behavior of the resistance in a region where impurity scattering predominates depends on the geometry of the FS and the ratios of the characteristic parameters of the system. The resistance can also be due to small deviations of the FS from exact two-dimensional form. This gives $\rho \sim \rho_B \gamma^2$, where γ is the characteristic deviation of the FS from cylindrical form.

Finally, we note that in a number of two-dimensional

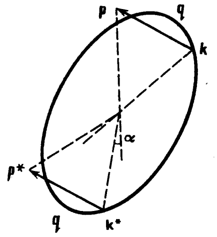

FIG. 2. The states with which a phonon with momentum **q** can interact are not, strictly speaking, centrally symmetrical: $\mathbf{p}^* \neq -\mathbf{k}$, $-\mathbf{p} \neq \mathbf{k}^*$, with the characteristic angle of deviation $\alpha \sim s/v$.

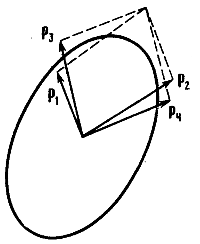

FIG. 3. The initial and final states of colliding electrons in a two-dimensional metal with convex FS are equal up to quantities of order T/μ; $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$.

metals an anomalous behavior of the resistance is observed, which is ascribed to an alteration of the spectrum or to spatial localization of the conduction electrons. Our treatment will be for conditions where these processes do not occur.

2. THE ELECTRON-PHONON INTERACTION. THE ELASTIC-COLLISION APPROXIMATION

The relaxation mechanisms that are due to scattering of electrons by phonons can be investigated by directly expanding the kinetic equation in the small parameters q/p and s/v. Even in the simplest cases, however, such calculations are very cumbersome. Here we shall describe a different scheme, based on a variation principle.$^{2)}$ It permits considerable simplification of the derivation of equations analogous to Eq. (1), and makes it relatively easy to calculate the electric conductivity of a metal in each specific case.

1. Calculation procedure

As is well known (cf. e.g., Ref. 6), the system of kinetic equations that describes the interaction between an electron and a phonon in an external electric field can be derived from the condition that the functional

$$F(\chi,\Phi)=\frac{2\pi}{\hbar}\frac{2}{h^6}\int d^3q\left(-\frac{\partial N}{\partial\omega}\right)\int d^3p\,d^3k\,G_{\mathbf{pkq}}(\chi_{\mathbf{p}}-\chi_{\mathbf{k}}-\Phi_{\mathbf{q}})^2,$$

$$G_{\mathbf{pkq}}=(n_{\mathbf{k}}-n_{\mathbf{p}})\left[\sum_{\mathbf{g}}\delta(\mathbf{p}-\mathbf{k}-\mathbf{q}-\mathbf{g})\right]\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{k}}-\omega_{\mathbf{q}})\,|b_{\mathbf{pkq}}|^2 \qquad (2)$$

be a minimum, with the supplementary condition

$$\frac{2}{h^3}\int d^3p\,\chi_{\mathbf{p}}e\mathbf{E}\,\partial n_{\mathbf{p}}/\partial\mathbf{p}=-E. \qquad (3)$$

Here the functions $\chi_{\mathbf{p}}$ and $\Phi_{\mathbf{q}}$ are associated in the usual way with the nonequilibrium corrections to the distribution functions of electrons and phonons:

$$\delta n_{\mathbf{p}}=-\chi_{\mathbf{p}}\partial n_{\mathbf{p}}/\partial\varepsilon_{\mathbf{p}},\quad \delta N_{\mathbf{q}}=-\Phi_{\mathbf{q}}\partial N_{\mathbf{q}}/\partial\omega_{\mathbf{q}},$$

where $n_{\mathbf{p}}$ and $N_{\mathbf{q}}$ are the equilibrium Fermi and Bose distribution functions, $b_{\mathbf{pkq}}$ is the matrix element of the interaction between electron and phonon, and g is a vector of the reciprocal lattice.

We note that with the normalization (3) which we have adopted for the distribution functions the variational equations differ from the kinetic equations by a numerical factor ρ/E in the right-hand side; here the electric resistance is

$$\rho=\min F(\chi,\Phi). \qquad (4)$$

We shall show that the temperature dependence of the resistance can be obtained from the variation principle. In order not to complicate the exposition, we shall regard the electron-phonon collisions as elastic and neglect quantities of order s/v. To calculate the resistance we suppose that the functions that minimize the quantity (2) satisfy the relation

$$\chi_{0\mathbf{p}}-\chi_{0\mathbf{k}}-\Phi_{0\mathbf{q}}\sim T^n. \qquad (5)$$

It is not hard to determine the exponent n. To do so we solve the system of equations

$$\begin{gathered}\chi_{\mathbf{p}}-\chi_{\mathbf{k}}-\Phi_{\mathbf{q}}=0,\\ \varepsilon_{\mathbf{p}}=\varepsilon_{\mathbf{k}}=\mu,\quad \mathbf{p}-\mathbf{k}-\mathbf{q}=0\end{gathered} \qquad (6)$$

successively in the linear, quadratic, and so on approximations in the parameter q/p, in the class of functions periodic in the reciprocal space. Let $\mathscr{L}_i$ be the set of solutions of this system in the i-th approximation. The functional F being positive-definite, it is readily seen that it is in the n-th approximation that the system (6) becomes unsolvable.

We write out the value of the functional $F(\chi,\Phi)$ in terms of functions from $\mathscr{L}_{n-1}$, in the first nonvanishing approximation in $q/p\sim T/\Theta$:

$$F^{(0)}(\chi,\Phi)=\frac{2\pi}{\hbar}\frac{2}{h^6}\int d^3q\left(-\frac{\partial N}{\partial\omega}\right)\int d^3p\,G^{(0)}_{\mathbf{pq}}\,[\hat{K}(\mathbf{q},\nabla)\chi_{\mathbf{p}}-\Phi_{\mathbf{q}}]^2, \qquad (7)$$

$$G^{(0)}_{\mathbf{pq}}=\omega_{\mathbf{q}}\delta(\varepsilon_{\mathbf{p}}-\mu)\,\delta(\mathbf{qv})\,M_{\mathbf{p}}(\mathbf{e})\,q,$$

where $\hat{K}(\mathrm{q},\nabla)$ is a differential operator of order n, and $\mathrm{e}=\mathrm{q}/q$. We have written the square of the matrix element for scattering of an electron by a long-wavelength phonon in the form

$$|b_{\mathbf{pkq}}|^2=M_{\mathbf{p}}(\mathbf{e})\,q.$$

From Eqs. (4), (5), and (7) we have as the result

$$\rho\sim T^{3+2n}. \qquad (8)$$

To obtain the approximate kinetic equation we have only to vary the functional (7) with respect to χ, Φ in the class $\mathscr{L}_{n-1}$ subject to the supplementary condition (3).

This scheme is convenient for the following reasons.

a) It is much easier to analyze the system of Eqs. (6) than the kinetic equation itself.

b) Although the system (6) is examined only to accuracy $(q/p)^n$, the result provides the kinetic equation in the $2n$-th approximation.

c) In the function $G_{\mathbf{pkq}}$ we are to confine ourselves to the main approximation, since $\chi_{\mathbf{p}}, \Phi_{\mathbf{q}}\in\mathscr{L}_{n-1}$. This is not so obvious in the direct expansion of the exact kinetic equation.

We point out that there are no umklapp processes in the integral (7); when they are not included the bracket $(\chi_{\mathbf{p}}-\chi_{\mathbf{k}}-\Phi_{\mathbf{q}})$ vanishes for the drift distribution functions

$$\chi_{\mathbf{p}}=\mathbf{up},\quad \Phi_{\mathbf{q}}=\mathbf{uq}. \qquad (9)$$

Formally, therefore, our results apply only to metals with open FS, where drift is prevented by the condition that the function $\chi_{\mathbf{p}}$ must be periodic in the reciprocal space. In the case of an open FS with an odd number of electrons and holes, on the other hand, a term which takes umklapp processes into account must be added in Eq. (7). In what follows, however, we shall not take umklapp processes into account, and accordingly stipulate that the drift distributions (9) are forbidden.

2. The diffusion approximation

In the diffusion approximation the equation (6) takes the form

$$\mathbf{q}\nabla\chi_{\mathbf{p}}=\Phi_{\mathbf{q}},\quad \varepsilon_{\mathbf{p}}=\mu,\quad \mathbf{qv}=0, \qquad (10)$$

and the differential operator $\hat{K}$ in Eq. (7) is equal to $\mathbf{q}\nabla$. If the FS is not cylindrical, the system (10) has no solutions distinct from the drift solutions (see Ref. 3). In this case the process of electric conduction is described by Eq. (1) and Bloch's law is valid.

The situation is different in metals with cylindrical FS. In fact, the equations (10) imposes a single condition on the function χ_p. Namely, the values of the quantity $\mathbf{q}\cdot\nabla\chi_p$ must be equal at points of the FS where the normals are parallel:

$$\mathbf{q}\nabla\chi|_{p_i}=\mathbf{q}\nabla\chi|_{p_j}, \qquad \varepsilon_i=\varepsilon_j=\mu, \quad \mathbf{qn}_i=\mathbf{qn}_j=0, \tag{11}$$

where a phonon with momentum q can be emitted. These conditions are satisfied by a function χ_p which is such that:

$$\sigma R^{-1}\partial\chi_p/\partial\varphi=f(\varphi)=-f(\varphi+\pi). \tag{12}$$

Here the angle φ gives the direction of the normal to the FS, and R and σ are the radius of curvature and the sign of the curvature of the FS at the point $\mathbf{p}$; $f(\varphi)$ is an arbitrary function of φ.

Accordingly, in a two-dimensional metal the diffusion of electrons on the FS cannot produce electric resistance. The process is blocked at every point by nonequilibrium phonons.

3. *q/p* superdiffusion

First let us verify that in the case of a convex FS there is no relaxation to all orders in q/p. In fact, considering only normal electron-phonon collisions, we get from Eq. (6)

$$\chi_p-\chi_k=-\chi_{-p}+\chi_{-k}.$$

This condition is satisfied by any odd function χ_p.

Let us now consider the case of a nonconvex FS. Assuming a fixed direction of $\mathbf{e}_\parallel=\mathbf{q}_\parallel/q_\parallel$ ($\mathbf{q}_\parallel$ is the component of the phonon momentum along the normal to a generator of the FS), we expand in powers of the modulus $q_\parallel$. The operator $\hat{K}(\mathbf{q},\nabla)$ contains odd powers of $q_\parallel$. Keeping the cubic terms, we have

$$\hat{K}(\mathbf{q},\nabla)\chi_p=\varkappa\left\{\frac{q_\parallel}{R}\frac{\partial\chi_p}{\partial\varphi}+\frac{1}{24}\frac{q_\parallel^3}{R^2}\left(1+\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{1}{R}\frac{\partial\chi_p}{\partial\varphi}\right)\right\}.$$

Here $\varkappa=\operatorname{sign}(\mathbf{e}_\parallel\cdot\mathbf{e}_\varphi)$, with $\mathbf{e}_\varphi$ the unit vector for the direction of the vector $\partial\mathbf{p}/\partial\varphi|_{\varepsilon=\mu}$.

For the equations analogous to (11) we now have

$$\hat{K}(\mathbf{q},\nabla)\chi_p|_{p_i}=\hat{K}(\mathbf{q},\nabla)\chi_p|_{p_j}, \qquad \varepsilon_i=\varepsilon_j=\mu, \quad \mathbf{qn}_i=\mathbf{qn}_j=0. \tag{13}$$

In this expression we must consider the coefficients of $q_\parallel$ and $q_\parallel^3$ separately, since the value of the phonon momentum is arbitrary. It is easy to verify that if $R_i\neq R_j$, then the conditions (13) are satisfied only by drift solutions, which must be discarded. Thus in this case $\rho\sim T^9$.

The approximate kinetic equation is found from the condition that the functional (7) be a minimum; for this we can assume the function χ_p in the form

$$\chi_p=\chi_p^{(0)}+(T/\Theta)^2\chi_p^{(1)},$$

where $\chi_p^{(0)}$ satisfies the requirement (12). Then for the function $f(\varphi)$ we get the superdiffusion equation

$$(1+d^2/d\varphi^2)D(\varphi)(1+d^2/d\varphi^2)f(\varphi)=a(\varphi) \tag{14}$$

with the following boundary conditions

$$D(\varphi)\left(1+\frac{d^2}{d\varphi^2}\right)f(\varphi)|_{\varphi_{\min},\varphi_{\max}}=\frac{d}{d\varphi}D(\varphi)\left(1+\frac{d^2}{d\varphi^2}\right)f(\varphi)|_{\varphi_{\min},\varphi_{\max}}=0,$$

where

$$a(\varphi)=-eE\sum_i R_i(\varphi)[\mathbf{e}_y\mathbf{p}_i(\varphi)+\lambda], \quad \varphi_{\min}<\varphi<\varphi_{\max}.$$

Here the summation is over points of the FS with the same value of the angle φ, and $\mathbf{e}_y$ is a unit vector perpendicular to the electric field. The parameter λ is determined from the condition that the boundary-value problem be solvable:

$$\int_{\varphi_{\min}}^{\varphi_{\max}} d\varphi\, a(\varphi)\mathbf{e}_y\mathbf{n}=0.$$

The function $f(\varphi)$ is subject to the supplementary condition

$$\int_{\varphi_{\min}}^{\varphi_{\max}} d\varphi f(\varphi)\sum_i R_i(\varphi)=0,$$

which corresponds to the periodicity of the electronic distribution function in the reciprocal space. We point out that the diffusion equation (14) can be solved by quadratures, and accordingly the electric conductivity can be found in general form for arbitrary FS and arbitrary electron-phonon interaction.

We give the expression for the function $D(\varphi)$ in the simplest case of a FS with one point of inflection [Fig. 1(b)]:

$$D(\varphi)=\frac{2\pi}{\hbar}\frac{2}{h^6}\frac{1}{24^2}(R_1^{-2}-R_2^{-2})\overline{\left(q_\parallel^2-\frac{\bar{q}_\parallel^2}{1}\right)^2}$$

where

$$\bar{a}=\int d^3q\left(-\frac{\partial N}{\partial\omega}\right)\omega\frac{A_1(\mathbf{q})A_2(\mathbf{q})}{A_1(\mathbf{q})+A_2(\mathbf{q})}q_\parallel q a(\mathbf{q}),$$

$$A_i(\mathbf{q})=\frac{R_i}{v_i^2}M_i(\mathbf{e})|_{\mathbf{en}_i=0}, \quad i=1,2.$$

As can be seen from this expression, relaxation is provided by the interaction between electronic states with parallel velocities by way of phonons distributed over values of $q_\parallel$.

3. THE APPROXIMATION OF INELASTIC ELECTRON-PHONON INTERACTIONS

When inelasticity of the electron-phonon interaction is taken into account, phonon exchange occurs between electrons whose momenta are not strictly symmetrical, so that the isolation of electron-phonon groups is broken (Fig. 2). To study the relaxation processes that occur owing to this we again use the scheme of Sec. 2, subsection 1.

We write the equation analogous to Eq. (10) in the linear approximation in q

$$\mathbf{q}\nabla\chi_p|_{\mathbf{ne}_\parallel=\sin\alpha}=\Phi_\mathbf{q} \tag{15}$$

and examine its solvability in each order in the small parameter s/v. Here $\sin\alpha=s/(v\sin\vartheta)$, where ϑ is the angle between the vector q and the axis p_z, and $q_\parallel=q\sin\vartheta$.

As was pointed out in the Introduction, in this case the energy dependence of $\chi_\mathbf{p}$ is important; therefore, writing it in the form

$$\chi_\mathbf{p}=F(\varphi)+\frac{\varepsilon-\mu}{v}g(\varphi),\quad \sigma R^{-1}\,dF/d\varphi=f(\varphi),$$

we have

$$\mathbf{q}\nabla\chi_\mathbf{p}=q_\parallel[\varkappa\sigma\cos\alpha f(\varphi+\varkappa\sigma\alpha)+\sin\alpha g(\varphi+\varkappa\sigma\alpha)]=\Phi_\mathbf{q},\tag{16}$$

where φ is the coordinate of the point on the FS at which $\mathbf{n}\cdot\mathbf{e}=0$. Keeping terms quadratic in α in this last equation, we get

$$q_\parallel\varkappa\sigma\{f(\varphi)+\varkappa\sigma\alpha[f'(\varphi)+g(\varphi)]-{}^1/{}_2\alpha^2[f(\varphi)-f''(\varphi)-2g'(\varphi)]\}=\Phi_\mathbf{q}.\tag{17}$$

Requiring that the left-hand side in this system take the same value at all points of the FS where $\mathbf{n}\cdot\mathbf{e}=0$, we find the conditions for it to be solvable. In zeroth order in α we have the sell known relation (12). In first order the condition is

$$g(\varphi)+f'(\varphi)=0,\tag{18}$$

because only in this way can the term linear in α (an odd function of p) be the same at the ϑ points p and $-$p. Thus the function $\chi_\mathbf{p}$ which satisfies the condition (18) (local drift) makes the functional (2) (and with it the resistance) vanish to accuracy $(s/v)^2$.[4]

The relaxation of local drift can be secured through scattering of electrons by glancing phonons through small angles $\vartheta\sim s/v$ (we have so far not considered them in Eq. (17), considering $\alpha\sim s/v\ll 1$). Although there are only a few glancing phonons [of the order of $(s/v)^2$ times the total number], and on absorbing such a phonon an electron is shifted on the FS only by an amount $q\sin\vartheta\sim qs/v$, this allows an electron to "perceive" the whole FS at once. A given electron can exchange a glancing phonon with any other electron, which leads to destruction of the local drift. In view of all this, we write the functional F in the form ($\vartheta\ll 1$):

$$\begin{gathered}F(\chi,\Phi)=\frac{2\pi}{\hbar}\frac{2}{h^6}\int d^3q\left(-\frac{\partial N}{\partial\omega}\right)\\ \times\int d^3pG_{\mathbf{pq}}q_\parallel{}^2\left[\frac{\mathbf{e}_\parallel\mathbf{e}_\varphi}{R}\frac{\partial\chi_\mathbf{p}}{\partial\varphi}-\mathbf{e}_\parallel\mathbf{n}\frac{\partial}{\partial\varphi}\left(\frac{1}{R}\frac{\partial\chi_\mathbf{p}}{\partial\varphi}\right)-\Phi_\mathbf{q}\right]^2,\\ G_{\mathbf{pq}}=\omega_\mathbf{q}\delta(\varepsilon_\mathbf{p}-\mu)\delta\left(\mathbf{e}_\parallel\mathbf{n}-\frac{s}{v\sin\vartheta}\right)M_\mathbf{p}(\mathbf{e})\frac{q}{q_\parallel v}.\end{gathered}\tag{19}$$

This expression for F enables us to estimate the resistance: $\rho\sim\rho_B(s/v)^3$. By varying Eq. (19) with respect to $\chi_\mathbf{p}$ and $\Phi_\mathbf{q}$ one easily obtains a fourth-order integrodifferential equation for determining the function $\chi_\mathbf{p}$.

Finally, there is a possible mechanism associated with redistribution of momentum within the thermal layer of the FS, which leads to a resistance $\rho\sim\rho_B(T/\mu)^2$. In fact, keeping terms linear in $(\varepsilon-\mu)$ in the expansion of the left-hand side of Eq. (15) and using Eq. (18), we get an insolvable system of equations

$$q_\parallel\varkappa\sigma\left\{f(\varphi)-\frac{\varepsilon-\mu}{Rv}[f(\varphi)+f''(\varphi)]\right\}\Big|_{\mathbf{en}=0}=\Phi_\mathbf{q}.$$

The last expression can be most simply derived in the case of an isotropic electronic spectrum. The equation of superdiffusion is obtained from the condition for minimizing the functional (7), in which

$$\hat{K}(\mathbf{q},\nabla)=q_\parallel\left\{\frac{\sigma}{R}\frac{\partial}{\partial\varphi}-\frac{(\varepsilon-\mu)\sigma}{vR}\left(1+\frac{\partial^2}{\partial\varphi^2}\right)\frac{1}{R}\frac{\partial}{\partial\varphi}\right\},$$

$$G^{(0)}_{\mathbf{pq}}=[n(\varepsilon)-n(\varepsilon+\omega)]M_\mathbf{p}(\mathbf{e})\delta(\mathbf{qv})q,$$

$$\chi_\mathbf{p}=\chi_\mathbf{p}^{(0)}+\chi_\mathbf{p}^{(1)}T/\mu,$$

and the function $\chi_\mathbf{p}^{(0)}$ satisfies the relation (12).

4. PHONON-PHONON AND ELECTRON-ELECTRON COLLISIONS

Since in a two-dimensional metal electron-phonon collisions are not a very effective mechanism of relaxation (much less effective than in the three-dimensional case), other processes of interaction of quasi-particles can in principle compete with them.

1. Phonon-phonon collisions

At low temperatures the relatively rare phonon-phonon collisions destroy the mutual isolation of electron-phonon groups. This leads to a relaxation which is described by a diffusion equation of the form (1), with the diffusion tensor proportional to the phonon interaction constant. The nonlocal nature of the diffusion is due to the rescattering of phonons that "belong" to electron groups far apart in their directions of **n**.

The characteristic relaxation time is given by

$$\tau_{eff}=\tau_{pp}(p_F/q)^2n_c/n_p.$$

Here $\tau_{pp}\sim T^{-5}$ is the mean free time for phonon-phonon collisions, $n_e\sim T/\mu$ is the number of electrons in the thermal layer, and $n_p\sim T^3$ is the number of phonons. It is clear that $\tau_{pp}n_e/n_p$ is the time during which an electron is displaced by the distance q along the FS as the result of one phonon-phonon collision, and $(p/q)^2$ is the number of Brownian steps needed for displacement across the whole FS. Accordingly, $\rho\sim\tau_{eff}^{-1}\sim T^9$.

We note that this last result holds if the frequency of collisions between phonons, $\tau_{pp}^{-1}\sim T^5$ is much smaller than the frequency of their collisions with electrons, $\tau_{pe}^{-1}\sim T$. In the opposite limit ($\tau_{pp}^{-1}\gg\tau_{pe}^{-1}$) the isolated groups are completely broken up and Bloch's law holds, $\rho\sim\rho_B$. In this respect the drag processes in two-dimensional and in three-dimensional systems are decidedly different; in the three-dimensional case the frequent phonon-phonon collisions do not bring about any qualitative change in the result.

2. Electron-electron collisions

First of all we shall show that in a two-dimensional metal with a convex FS normal collisions between electrons are not very effective.

The variational integral analogous to Eq. (2) is

$$F(\chi)=\int d^3p_1\,d^3p_2\,d^3p_3\,d^3p_4\delta^{(4)}(p_1+p_2-p_3-p_4)G(\chi_1+\chi_2-\chi_3-\chi_4)^2.$$

Here p_1, p_2, p_3, p_4 are the four-momenta of the electrons before and after a collision, and the positive function $G=G(p_1,p_2,p_3,p_4)$ falls off rapidly when one of its arguments gets beyond the thermal layer.

If we neglect the thermal smearing of the FS, the quantity $(\chi_1,\chi_2,\chi_3,\chi_4)$ vanishes for any odd function $\chi_\mathbf{p}$

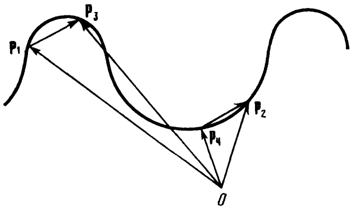

FIG. 4. The electronic states before and after collision in a metal with nonconvex FS may not be the same.

(owing to the conservation laws either $p_1 = p_3, p_2 = p_4$, or else $p_1 + p_2 = p_3 + p_4 = 0$). Putting $(\chi_1 + \chi_2 + \chi_3 + \chi_4) \sim T/\mu$, we have the estimate $\tau_{ee} \sim \tau_0(\mu/T)^2$ where $\tau_0 \sim T^{-2}$ is the characteristic time for interelectronic relaxation in a three-dimensional metal. In the present case, when the FS is rather small and there are no umklapp processes, complete relaxation is obviously impossible and the resistance is zero. Therefore τ_{ee} has the meaning of the time for attaining complete equilibrium, described by a drift distribution function.5)

In the case of a nonconvex (in particular, an open) FS, two-dimensionality does not result in weakening electron-electron relaxation and $\tau_{ee} \sim \tau_0$; only a drift distribution function is in equilibrium for the collisions shown in Fig. 4 (the situation here is analogous to q/p superdiffusion, but its step is not small: $q \sim p_F$). This also applies to convex FSs if they allow umklapp processes for all electronic states (see Fig. 5).

In all of these cases the interelectronic collisions either do not lead to a nonzero resistance, or else give the usual result $\rho_{ee} \sim T^2 \sim \rho_0$. On the other hand, if electron-electron U-processes intermix the momenta only on some part of a convex FS,6) then relaxation is checked on the parts where only normal collisions occur, and $\rho_{ee} \sim \rho_0(T/\mu)^2 \sim T^4$.

5. THE EFFECT OF IMPURITIES

The results we have found of course apply only to very pure metals, in which the mechanisms in question are dominant in certain ranges of temperature. The low effectiveness of these mechanisms (as compared with the case of three-dimensional metals) leads to stronger restrictions on the concentration of impurities. Owing to this it is interesting to see how the two-dimensionality of the electron spectrum affects the temperature corrections to the impurity resistance ρ_i under conditions in which they are dominant. Here widely different situations are possible, depending on the shape of the FS and the relations between the characteristic parameters of the system.

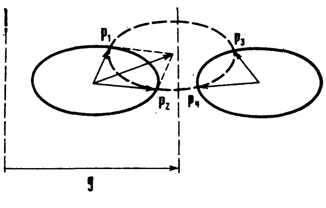

FIG. 5. Electron-electron Umklapp processes. The electronic states p_3 and p_4 after collision are located at the points of intersection of the FS with the surface obtained by shifting the FS by the total momentum $p_1 + p_2$ of the colliding particles.

Let us consider the transition from impurity resistance to the case of q/p superdiffusion in a metal with a convex FS.

At the very lowest temperatures $T \ll T_1$, at which $\tau_i \ll \tau'(p/q)^2 = \tau_d(\tau_i, \tau'$ are the respective mean free times of an electron against collisions with impurities and with phonons, and τ_d is the diffusion time, during which collisions with phonons markedly change the electron momentum), electron-impurity scattering predominates and gives a form of the function χ_p which, generally speaking, does not satisfy Eq. (12). Therefore in this case the contribution of electron-phonon collisions is due not to q/p superdiffusion, but to ordinary diffusion on the FS, corresponding to the temperature-dependent correction to the residual resistance: $\Delta\rho \sim \rho_B$.

At higher temperatures $T \gg T_1$ $(\tau_i \gg \tau_d)$ the rapid diffusion of electrons imposes on the distribution function the structure (12), and is blocked. A stationary flux of electrons is provided either by the impurity mechanism $[T \ll T_3, \tau_i \ll \tau_{sd} = \tau'(p/q)^6]$, or else by the superdiffusion mechanism $(\tau_i \gg \tau_{sd})$. In the former case the resistance is approximately constant and equal to the minimum value of the impurity functional F in the class of functions χ_p that satisfy Eqs. (3) and (12): $\rho \approx \rho_1 > \rho_i$. In the second case $\rho \sim \rho_{sd} \sim T^9$.

It can be shown (for example, by solving the kinetic equation with a perturbation method) that in the intermediate region $T_1 \ll T \ll T_3$

$$\rho-\rho_1 \sim \begin{cases} -\rho_i^2/\rho_B, & T \ll T_2 \\ \rho_{sd}, & T \gg T_2 \end{cases},$$

where the temperature T_2 is determined by the relation $\tau_i \sim \tau_d(p/q)^2$. Accordingly, for temperatures larger than T_1 the resistance curves of two-dimensional and three-dimensional metals are quite different in their shapes.

As a second example let us consider a closed FS without points of inflection. In this case the electron-phonon correction to the resistance is much smaller than ρ_B at all temperatures. Indeed, at the lowest temperatures $(\tau' \gg \tau_i)$ collisions with impurities establish a distribution function of the electrons which in general does not satisfy the condition (18) for local drift, Eq. (18); therefore the electron-phonon contribution is of order of magnitude $\delta\rho \sim \rho_B(s/v)^2$ (see Sec. 2; in the present case $q\nabla\chi - \Phi \sim T/\mu$). In the temperature range $\tau_i \gg \tau'$ collisions with phonons establish local drift, and are blocked in the leading approximation in τ'/τ_i; the amount added to the resistance comes in the next approximation:

$$\delta\rho \sim -(\tau'/\tau_i)^2 (s/v)^2 \rho_B.$$

However, these corrections to the residual resistance are so small that the main correction is the weak temperature dependence of the residual resistance itself:

$$\rho_i(T) = \rho_i + \Delta\rho_i(T), \quad \Delta\rho_i(T) \sim \rho_i(T/\mu)^2$$

(only for $\tau' \approx \tau_i$ do we have $\Delta\rho_i \approx \delta\rho$). Finally, at higher temperatures the s/v mechanisms described in Sec. 2

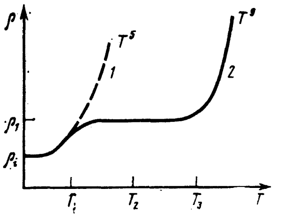

FIG. 6. Resistances of three-dimensional (1) and two-dimensional (2) metals with nonconvex FS in the approximation of elastic electron-phonon collisions.

become important. The boundary of this region can be found from the condition $\Delta\rho_i \approx \rho_{s/v}$, where $\rho_{s/v}$ is the resistance owing to a dominant s/v interaction mechanism.

In conclusion we point out that the relaxation mechanism considered here can affect not only the electric resistance but also other kinetic phenomena for which displacements of electrons over the FS are important. On the other hand, our treatment, based on the simplest model of the phonon spectrum and the interaction of the phonons with the electrons cannot pretend to give a detailed description of the electric conductivity of actual laminar systems, with their extremely varied and complicated properties.

If, for example, we suppose that the matrix element of the electron-phonon interaction goes to zero for "glancing phonons",[7] $M(\mathrm{e}) \sim \vartheta^2$, then the relaxation of the local drift changes decidedly. In the case of a FS with points of inflection a mechanism appears analogous to q/p superdiffusion and producing a resistance $\rho \sim \rho_B(s/v)^4$. Besides this, the nonlocal mechanism to glancing phonons obviously now gives $\rho \sim \rho_B(s/v)^5$.

In any case, however, the electric conductivity of a laminar conductor is much larger than that of an ordinary three-dimensional metal (with the same density of states). This conclusion may also hold at high temperatures, if the phonons collide with electrons oftener than with each other. In this connection we point out the anomalously large conductivity of certain compounds of graphite, intercalated with various sorts of donors or acceptors (KC_8, KC_{24}, KC_{48}, LiC_6, C_8AsF_5, $C_{16}AsF_5$, etc.).[7] In this paper, Fischer indicates that there is a definite connection between the magnitude of the conductivity and the character of the electron spectrum: The more "two-dimensional" the electrons are, the greater is the electric conductivity along the layers.

Note added in proof (25 May 1982). We note that the thermal conductivity of a two-dimensional metal, unlike the electric conductivity, is governed by momentum relaxation, not energy relaxation. At temperature $T < T_0 = \theta(s/v)^{1/2}$ the transport of heat is mainly by electrons, and for $T > T_0$, by phonons. The thermal conductivity coefficient $\varkappa$ is anomalously large and has an unusual temperature dependence:

$$\varkappa \propto C_e v^2 \tau_{eff} (T/\mu)^2 [1 + (T/T_0)^4].$$

Here C_e is the electronic heat capacity and τ_{eff} is the relaxation time which determines the electric conductivity of a two-dimensional metal.

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1) Qualitative considerations on some relaxation processes in two-dimensional metals are given in Ref. 4.

2) In simplified form a similar method was used earlier[2] to derive the diffusion equation (1).

3) In three-dimensional metals the electric conductivity is usually governed by processes of electron diffusion over the FS (umklapp processes occur relatively easily down to extremely low temperatures, and therefore the "bottleneck" of the relaxation is diffusion[3]). If certain conditions are satisfied, the situation is similar in two-dimensional systems, in which diffusion occurs with much more difficulty than in the three-dimensional case.

4) We remark that our previous[4] estimate of the resistance, $\rho \sim \rho_B(s/v)^2$, associated with the inelasticity of electron-phonon collisions, is erroneous.

5) In the absence of umklapp processes, the contribution $\Delta\rho_{ee} \sim T^4$ of electron-electron collisions can be regarded as a small correction added to the impurity resistance.

6) In the case of a circular FS this occurs for $g/4 < p_F < 2^{1/2}g/4$.

7) This corresponds to the case of a system of rigid conducting layers imbedded in an easily deformed matrix.

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[1] E. M. Lifshitz and L. P. Pitaevskiĭ, Fizicheskaya Kinetika (Physical kinetics), Moscow, Nauka, 1979, Sec. 83. English Translation, Pergamon Press, 1981.

[2] R. N. Gurzhi and A. I. Kopeliovich, Zh. Eksp. Teor. Fiz. **61**, 2514 (1971); **64**, 380 (1973) [Sov. Phys. JETP **34**, 1345 (1972); **37**, 195 (1973)].

[3] R. N. Gurzhi and A. I. Kopeliovich, Usp. Fiz. Nauk **133**, 33 (1981) [Sov. Phys. Uspekhi **24**, 17 (1981)].

[4] R. N. Gurzhi, A. I. Kopeliovich, and S. B. Rutkevich, Pis'ma Zh. Éksp. Teor. Fiz. **32**, 360 (1980) [JETP Lett. **32**, 336 (1980)].

[5] V. L. Gurevich and B. D. Laĭkhtman, Zh. Éksp. Teor. Fiz. **69**, 1230 (1975) [Sov. Phys. JETP **42**, 628 (1976)].

[6] J. M. Ziman, Electrons and Phonons, Oxford University Press, 1960.

[7] John E. Fischer, Physica **99B**, 383 (1980).

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