Scattering of neutrons by rotons in superfluid helium

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We consider scattering of neutrons by rotons in superfluid helium, at the energies and momentum transfers at which production of excitations is forbidden by the conservation laws. The cross section at large momentum transfers is estimated with the aid of the Feynman wave function.

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We consider in this paper the scattering of slow neutrons in superfluid helium at low energy transfers. More accurately speaking, we assume that the point $\varepsilon, \mathbf{q}$ (ε and $\mathbf{q}$ are the transferred energy and momentum) lies much lower than the helium-excitation spectrum curve (shaded region of Fig. 1).

We note first that scattering with such transfers is impossible at absolute zero temperature. Indeed, at $T=0$ a neutron can transfer energy to helium only by producing in it excitations. This makes possible values of ε and q that lie on the spectral curve (production of one excitation) or above it (production of several excitations).[1)]

At finite temperatures the situation changes, since the thermal excitations that make up the normal part of the liquid are present. The neutrons can be scattered by these "ready-made" excitations, and transfer to them practically arbitrary ε and q without changing the number of excitations. The region of relatively low ε is particularly convenient for the observation of this process, since it is not masked here by production processes.

Of course, we are dealing in fact with scattering by rotons. The scattering by phonons is very small. We calculate the probability of neutron scattering by a roton. So long as the momentum transfer q is not too large, i.e., it satisfies the inequality

$$qR_0 \ll \hbar, \tag{1}$$

where R_0 is the characteristic "roton dimension," the roton can be regarded as pointlike, and its interaction with a neutron can be described by the Fermi potential[2)]:

$$U(\mathbf{r}) = 2\pi\hbar^2 \delta N a \delta(\mathbf{r})/M.$$

Here a is the amplitude for scattering of a slow neutron by a helium atom and M is the reduced mass of the neutron and of the helium atom. δN is the excess number of helium atoms located in the region occupied by the roton. This quantity can be calculated by purely thermodynamic means. Indeed, the condition that the superfluid part of the helium be at equilibrium requires that the following quantity be constant[3]

$$\mu(\rho) + m\int \frac{\partial\varepsilon}{\partial\rho} n_p \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} = \text{const},$$

where $\mu(\rho)$ is the chemical potential of liquid helium at $T=0$, ρ is the helium density, m is the mass of the helium atom, ε is the energy of the elementary excitation, and n_p is the distribution function of the elementary excitations. Recognizing that $\varepsilon \approx \Delta$ for rotons, the change of the helium density on account of the presence of the roton is given by the relation

$$\delta\rho \frac{d\mu}{d\rho} = -m\frac{\partial\Delta}{\partial\rho} N_p, \quad N_p = \int n_p \frac{d^3\mathbf{p}}{(2\pi\hbar)^3},$$

where N_p is the number of rotons per unit volume. Integrating over the volume and dividing by the total number of rotons $\int N_p dV$, we obtain for δN

$$\delta N = -\frac{\partial\Delta}{\partial\rho}\frac{d\rho}{d\mu}.$$

Recognizing that $d\mu/d\rho = (m/\rho)dP/d\rho = mu^2/\rho$ (u is the speed of sound at $T=0$), we get ultimately

$$\delta N = -\frac{\partial\Delta}{\partial\rho}\frac{\rho}{mu^2} \approx 0.31. \tag{2}$$

It is easy to calculate with the aid of the potential (1) the differential cross section for scattering of a neutron by a roton. The corresponding formula is

$$d\sigma = \frac{m_n}{P}\frac{(2\pi\hbar)^3}{M^2} a^2 \delta N^2 \delta\left[-\varepsilon + \frac{(p'-p_0)^2 - (p-p_0)^2}{2\mu}\right]\frac{d^3\mathbf{P}'}{(2\pi\hbar)^3}. \tag{3}$$

here m_n is the neutron mass, μ is the effective roton mass, $\mathbf{p}$ and $\mathbf{p}'$ are respectively the initial and final momenta of the roton, p_0 is the momentum at the roton minimum, and $\mathbf{P}$ and $\mathbf{P}'$ are the initial and final neutron momenta. In accord with the initial statements, we are interested in relatively small energy transfers ε:

$$\varepsilon \ll \Delta. \tag{4}$$

Since the initial and final rotons should be located in this case near the minimum of the spectrum, the only rotons that will take part in the scattering are those whose initial momentum $\mathbf{p}$ makes with the momentum-transfer vector $\mathbf{q} = \mathbf{P}' - \mathbf{P}$ and angle α such that $q = 2p_0\cos\alpha$. It is therefore advisable to average first the cross section (3) over the directions of the initial roton momentum. The average cross section with allowance for the smallness of ε is

$$\overline{d\sigma} = \frac{m_n^2}{M^2}\frac{\mu a^2\delta N^2}{2P(2\mu\varepsilon)^{1/2}} d\varepsilon \frac{do}{\sin(\theta/2)}, \tag{5}$$

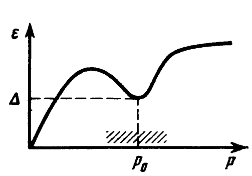

FIG. 1.

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where θ is the angle between $\mathbf{P}$ and $\mathbf{P}'$, and $q \approx 2P\sin(\theta/2)$. Equation (5) solves our problem.

To estimate the order of magnitude of the effect, we integrate (5) with respect to ε from zero to a certain $\varepsilon_{max} \ll \Delta$, and with respect to the angles. The corresponding total cross section is

$$\sigma = 4\pi \frac{m_n^2}{M^2} \frac{(2\mu\varepsilon)^{1/2}}{P} a^2 \delta N^2 \begin{cases} p_0/P, & P > p_0, \\ 1, & P < p_0. \end{cases} \qquad (6)$$

The corresponding neutron range is $l = (N_p \sigma)^{-1}$. The range, of course, decreases with increasing number of rotons, i.e., with rising temperature, and the maximum effect can be estimated by assuming that $T \sim T_\lambda$; then $N_p = 0.1680 \times 10^{22}$ cm^{-3} (Ref. 4). In this case $l \sim 2.9 \times 10^2$ cm (at $P \sim p_0$). Of course, Eq. (6) itself is not valid near the λ point, but this estimate points to the correct order of magnitude. We note for comparison that the range l' for roton production by a neutron is according to the theory $l' = 1.15$ cm, so that[3)]

$$l/l' = 250.$$

Since roton production by neutrons can be easily observed, there is every reason for assuming that the scattering process considered by us is also observable. Some observations of neutron scattering in the region of interest to us are described in Refs. 6 and 7. We, however, cannot explain the results obtained in these references. The effect calculated by us should certainly decrease rapidly with decreasing temperature, something apparently not observed in the cited experiments.

As stated in the beginning of the article, the equations obtained are valid only under condition (1), when the roton can be regarded as pointlike. At $\hbar q^{-1} \sim R_0$ deviations from (5) should be observed, so that a careful check of this formula ensures in principle the possibility of determining the spatial dimension of the roton. Of course, such deviations should certainly take place if $\hbar q^{-1}$ is of the order of the interatomic distances. There exist, however, indications that the roton can be noticeably larger. First, the roton density extrapolated to the λ point is found to be smaller by a factor 13 than the density of the helium atoms, and it is natural to assume that the rotons are densely "packed" near the λ point. Next, the experimental data on the cross section for roton-roton scattering indicate that orbital momenta of the relative roton motion up to $l = 5$ and 6 take part effectively in their interaction,[8] and this can also be interpreted as meaning that the roton is large. It is curious that since the vectors $\mathbf{q}$ and $\mathbf{p}$ are perpendicular at $q \ll p_0$, deviations from (5) should determine the size of the roton in a direction perpendicular to $\mathbf{p}$.

For a theoretical calculation of the scattering at large q that do not satisfy the condition (1), we must specify some concrete model of the roton. We use for this purpose the Feynman wave function,[9] which describes the excitation spectrum perfectly satisfactorily. Normalized to unity, this function is of the form

$$\Psi = (NS(p))^{-1/2} \sum_a \exp\left(\frac{i\mathbf{p}\mathbf{r}_a}{\hbar}\right) \Phi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

The summation is over all the atoms of the liquid. N is the total number of helium atoms, and Φ_0 is the ground-

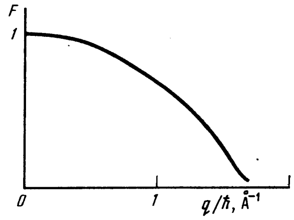

FIG. 2.

state wave function normalized to unity. Calculating the matrix element of the interaction potential

$$\hat{U}(\mathbf{r}) = \frac{2\pi\hbar^2}{M} a \sum_a \delta(\mathbf{r} - \mathbf{r}_a),$$

we obtain in lieu of Eq. (5), which corresponds to a pointlike roton, the expression

$$d\sigma = \frac{m_n^2}{M^2} \frac{\mu a^2}{2P(2\mu\varepsilon)^{1/2} \rho_0^2 S^2(p_0)} \left[\int \rho_3(\mathbf{r}_{13}, \mathbf{r}_{23}) \times \exp\left(\frac{i\mathbf{q}\mathbf{r}_{13} + i\mathbf{p}\mathbf{r}_{23}}{\hbar}\right) d^3\mathbf{r}_{13} d^3\mathbf{r}_{23} \right]^2 d\varepsilon \frac{do}{\sin\theta/2}, \qquad (7)$$

where ρ_3 is the triple correlation function of the helium atoms in the ground state:

$$\rho_3(\mathbf{r}_{13}, \mathbf{r}_{23}) = \sum_{a, a', a''} \int \delta(\mathbf{r}_1 - \mathbf{r}_a) \delta(\mathbf{r}_2 - \mathbf{r}_{a'}) \delta(\mathbf{r}_3 - \mathbf{r}_{a''}) |\Phi_0|^2 d^{3N}\mathbf{r}.$$

A direct experimental determination of the function ρ_3 is impossible and, as is customary in the theory of liquids, it becomes necessary to use some interpolation expression for this function in term of pair correlation functions. The simplest of these interpolation is of the form[10]

$$\int \exp\left(\frac{i\mathbf{q}\mathbf{r}_{13}}{\hbar}\right) \rho_3(\mathbf{r}_{13}, \mathbf{r}_{23}) d^3\mathbf{r}_{13} \approx \rho_0 p_1(r_{23}) \int \exp\left(\frac{i\mathbf{q}\mathbf{r}_{13}}{\hbar}\right) p_1(r_{13}) d^3\mathbf{r}_{13}.$$

The function $p_1(r_{13})$ is connected here with the pair correlation function $\rho_2(\mathbf{r}_{13})$ by the relation

$$\rho_2(\mathbf{r}_{13}) = \rho_0 [\delta(\mathbf{r}_{13}) + p_1(r_{13})].$$

The expression in the square brackets in (7) can then be written in the form

$$[\ldots] = \rho_0 (S(q) - 1)(S(p) - 1). \qquad (8)$$

A numerical calculation shows that allowance for the roton structure in this manner leads to multiplication of the differential cross section by an additional factor $F(q)$, a plot of which is shown in Fig. 2. [The factor $F(q)$ is normalized to unity at small q.] It is seen from the figure that, in the approximation employed, $F(q)$ differs little from unity when $\hbar q^{-1}$ is large compared with the interatomic distances. It is not excluded, however, that this is due to the insufficient accuracy of the Feynman wave function or of the interpolation approximation (8).[4)] The question of the roton size can apparently be answered only by experiment.

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[1)]We disregard here the small region below the initial part of the spectrum, which is subject to excitation decay.[1]

[2)] At sufficiently small q the helium dynamic form factor that determines the scattering can be calculated by general fluctuation theory in the manner used by Mineev[2] for the case of phonons. The direct calculation method used here, however, is simpler and more illustrative.

[3)] We used in the calculation an expression for the scattering dynamic form factor in the "single-pole" approximation: $S(q, \varepsilon) = S(q)\delta([\varepsilon - \varepsilon(q)]/\hbar)$, where $S(q)$ is the helium static form factor (see, e.g., Ref. 5).

[4)] Strictly speaking the roton should be characterized by the distributions of both the density and the velocity of the liquid. The latter distribution does not manifest itself in neutron scattering, but can do so in the interaction. In this sense the roton can have no characteristic spatial dimensions.

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