Scattering of electrons in a strong magnetic field

D. G. Polyakov

A. F. Ioffe Physicotechnical Institute, Academy of Sciences of the USSR, Leningrad
(Submitted 20 January 1982)
Zh. Eksp. Teor. Fiz. **83**, 61–67 (July 1982)

The cross sections σ for scattering by the Coulomb and screened Coulomb potentials in a strong magnetic field (in the sense that the cyclotron energy is greater than the Bohr energy) are computed for an arbitrary energy of the electron motion along the field. It is found that the quantity $\sigma(k)$ (k is the wave vector along the field) tends to infinity as $k \to 0$ in the case of the screened potential, but remains finite in the case of the unscreened attractive potential. These results are generalized to the case of a scattering potential of arbitrary shape.

PACS numbers: 41.70. + t

1. In the present paper we investigate the scattering of electrons by a force center in a magnetic field. The field intensity is assumed to be so high that we can, in considering the scattering of the electrons inside the zeroth Landau band, ignore the presence of all the higher bands. For this purpose, it is sufficient that the electron cyclotron energy be much higher than the characteristic energy of the interaction of the electron with the center.

Let us assume that the potential of the center is axially symmetric about the direction of the field. Then the component M of the angular momentum of the electron along this direction is conserved in the scattering. Therefore, it is convenient to use the representation in which M is a quantum number.[1] For the zeroth Landau level $M = 0, 1, 2, \ldots$. The motion of the free electron along the direction of the magnetic field is characterized by the value of the wave vector k. In our case of scattering inside the zeroth Landau band the quantity k may either be conserved or change sign. We shall be interested in backward scattering, i. e., scattering involving a change in sign of k.

Let the incident electron beam be characterized by the fact that the states with different M are identically populated. In the classical picture of scattering, this corresponds to the situation in which the electron flux is homogeneous. For the scattering cross-section, given by the ratio of the reflected-electron flux to the incident-beam intensity, we then easily obtain the expression

$$\sigma = 2\pi\lambda^2 \sum_{M=0}^{\infty} R_M, \tag{1}$$

where R_M is the coefficient of reflection of an electron with angular-momentum component M and λ is the magnetic length. If the values of M are such that the quantity R_M changes little when M is changed by unity (this, in any event, is the case for large M), the summation over these M can be replaced by integration. Let us represent the contribution to the cross section (1) from such M in the form of an integral:

$$2\pi \int R(\rho)\rho\, d\rho, \quad \rho = (2M)^{1/2}\lambda. \tag{2}$$

This result can be given the following meaning. For large $M \gg 1$ the free electron moves in a plane perpendicular to the direction of the field in a thin ring of radius $(2M)^{1/2}\lambda$ and thickness $\sim\lambda$, so that in the case of large M we can introduce the quasiclassical impact parameters ρ [Eq. (2)]. This expression corresponds then to classical integration over the impact parameter. Notice that the quantity σ determines directly the longitudinal magnetoresistance of an electron gas in the presence of scattering by force centers.

According to the foregoing, the electron motion transverse to the magnetic field is entirely determined by this field (the magnetic field is strong). On the other hand, the longitudinal motion of the electron, which is scattered with a definite value of M, is described by a wave function $F_M(z)$ (the z axis is oriented along the direction of the field) satisfying the equation

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dz^2}+k^2\right)+V_M(z)\right]F_M(z)=0,$$
$$V_M(z) = \langle M|V(\mathbf{r})|M\rangle. \tag{3}$$

Here $V(\mathbf{r})$ is the potential of the center and the matrix element is evaluated with the functions of the transverse motion.

To find the quantity σ, we must find the coefficients of reflection of the electrons from the one-dimensional potentials $V_M(z)$. First, we shall consider in Sec. 2 the solution to Eq. (3) in the particular case of a Coulomb center, then in Sec. 3 we shall generalize the analysis to the case of the screened center, and, finally, in Sec. 4 we shall investigate the character of the cross section $\sigma(k)$ for slow electrons with $k \to 0$ in the case of an arbitrary shape of the scattering potential. In the process we shall distinguish between the cases of attractive and repulsive potentials.

2. *The Coulomb potential.* The potential energy of the electron in the case of scattering by a Coulomb center has the form $V(r) = \pm e^2/r$. The criterion for a strong magnetic field implies that the cyclotron energy is much greater than the Bohr energy, i.e., that $\lambda \ll a$ (a is the Bohr radius). As has already been noted above, for $M \gg 1$ the electron moves in a thin cylindrical layer. The effective one-dimensional potential has then the form

$$V_M(z) = \pm e^2/(\rho^2+z^2)^{1/2}, \quad \rho = (2M)^{1/2}\lambda, \quad M \gg 1. \tag{4}$$

a) Attractive potential. Let us find the coefficients of

 0038-5646/82/070033-04$04.00

reflection of incoming electrons with definite values of ρ and k from a potential well described by the function $V_M(z)$, (4), taken with the plus sign. To begin with, let $\rho \gg a$. It is not difficult to see that in this case the potential $V_M(z)$ satisfies at any z and k the inequality

$$|dk_M^{-1}(z)/dz| \ll 1, \quad k_M(z) = (k^2 - 2mV_M(z)/\hbar^2)^{1/2}. \tag{5}$$

Here $k_M(z)$ is the quasiclassical wave vector. This inequality is connected with the long-range nature of the Coulomb potential. The slowness of the decrease of the potential leads to a situation in which, as $|z| \to \infty$, an electron incident with even zero energy (i.e., with $k=0$), and picking up speed in the field of the center, has a wave function that oscillates strongly in a segment of length equal to the distance over which the potential changes significantly. Accordingly, the usual quasiclassical functions are the solutions to Eq. (3) for $\rho \gg a$ (Ref. 1). In the completely classical picture of scattering the electron is not reflected at all from a one-dimensional attractive potential, and in the quasiclassical picture the reflection coefficient for any k is much smaller than unity.

Now let $\rho \lesssim a$. In this case the inequality (5) is no longer fulfilled at all z and k:

$$|dk_M^{-1}(z)/dz| \sim (|z|/a)\,(a^2/(\rho^2+z^2))^{3/4}(\tfrac{1}{2}k^2 a(\rho^2+z^2)^{1/2}+1)^{-3/2}.$$

It is, however, clear that, for $|z| \gg a$, the inequality (5) is satisfied in any case. Therefore, scattering in the region $|z| \gg a$ is quasiclassically weak, and the quantity R_M is entirely determined by the scattering in the region $|z| \lesssim a$. Thus, the reflection coefficient at small k is, unlike in the quasiclassical case $\rho \gg a$, not small. It can be determined for $\rho \ll a$ by using the wave function of the continuous spectrum (see Ref. 2, Appendix 1). In this case it turns out that

$$1 - R_M \sim 1/\ln^2(a/\rho)$$

for $k=0$ and $R_M \ll 1$ for $k \gg k'$. The characteristic quantity

$$k' \sim \ln(a/\rho)/a$$

gives the electron binding energy $\hbar^2 k'^2/m$ in a one-dimensional Coulomb potential cut off in the region of small z at a distance $|z| \sim \rho$ under the condition that $\rho \lesssim a$ (Ref. 3). Perturbation theory is valid in the case of large k, and, applying it to the potential (4), we easily obtain

$$R_M = \frac{4}{(ka)^2} K_0^2(2k\rho), \quad k \gg \frac{\ln(a/\rho)}{a}, \quad \rho \lesssim a, \tag{6}$$

where K_0 is a Macdonald function. This function decreases rapidly (exponentially) with increasing ρ when $k\rho > 1$. Therefore the dominant contribution to the cross section is made by the values $\rho \sim 1/k \ll a$. Integrating (6) with respect to ρ, we find for the cross section the expression

$$\sigma = \pi a^2/(ka)^4, \quad 1/\lambda \gg k \gg 1/a. \tag{7}$$

This result coincides with that obtained in Ref. 4. The upper bound on the value of k is connected with the fact that, for $k \gtrsim 1/\lambda$, the values of $M \sim 1$ are important, and the formula (4) is invalid.

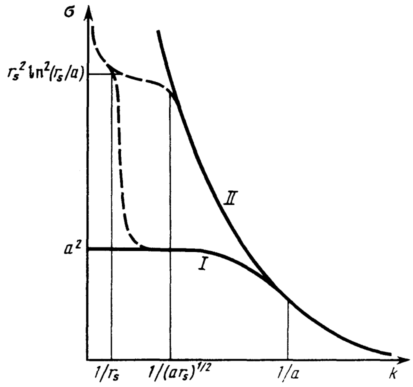

FIG. 1. Dependence of the scattering cross section σ on the magnitude of the electron wave vector k for an attractive (I) and a repulsive (II) Coulomb potential (a is the Bohr radius). The dashed curves are a plot of the dependence $\sigma(k)$ for the screened potential (r_s is the screening distance).

The expression (7) gives the cross section in the region $k \gg 1/a$; for $k \lesssim 1/a$ the cross section is of the order of a^2. Indeed, as shown above, the electrons with impact parameters $\rho \gg a$ fly through the potential almost without being reflected, and therefore do not make a contribution to the cross section. On the other hand, the electrons with $\rho \lesssim a$ and $k \lesssim 1/a$ are reflected with probability ~ 1. Thus, the function $\sigma(k)$ has the form shown in Fig. 1. The exact value of $\sigma(k \to 0)$ is determined by the impact parameters $\rho \sim a$, and must therefore be computed numerically.

b) Repulsive potential. As in the case of the attractive potential, the scattering of electrons with $\rho \gg a$ is quasiclassical. For these ρ values the functions $V_M(z)$ describe large potential barriers. In this case the reflection coefficient R_M drops sharply from unity to a small value when the energy of the longitudinal motion of the electron exceeds the height of the potential barrier. This occurs when $k \approx (2/a\rho)^{1/2}$. As to the width of the k-value region where this change occurs,[1)] it is[1] $\sim 1/\rho$. Accordingly, electrons with ρ such that $k > (2/a\rho)^{1/2}$ pass almost completely through the potential, while those with smaller ρ are almost totally reflected. The foregoing indicates that the cross section is entirely determined precisely by these small ρ values, i.e., $\sigma = \pi\rho^2$, where $\rho = 2/ak^2$:

$$\sigma = 4\pi a^2/(ka)^4, \quad k \ll 1/a. \tag{8}$$

The indicated inequality for k is connected with the fact that, for $k \gtrsim 1/a$, the significant contribution to the cross section is made by the values of $\rho \lesssim a$, for which the quasiclassical treatment is inapplicable. It is easy to see that, for $k \gg 1/a$, perturbation theory is valid and, hence, the cross section is given by the formula (7). It is noteworthy that $\sigma(k)$ for $k \ll 1/a$ and $\sigma(k)$ for $k \gg 1/a$ differ only by a factor of 4: the curve is flatter at large k. The general shape of the function $\sigma(k)$ is shown in Fig. 1. It is significant that, at small $k \ll 1/a$, scattering by repulsive Coulomb centers is much stronger than scattering by attractive ones.

3. *The screened Coulomb potential.* Let us consider the case in which the potential energy of the electron in the field of the center has the form

$$V(r)=\pm(e^2/r)\exp(-r/r_s),$$

where the screening distance $r_s \gg a \gg \lambda$. Then the potentials $V_M(z)$, (3), for $M \gg 1$ are given by the expression

$$V_M(z)=\pm\frac{e^2}{(\rho^2+z^2)^{1/2}}\exp\left(-\frac{(\rho^2+z^2)^{1/2}}{r_s}\right). \quad (9)$$

a) Attractive potential. Let us call the one-dimensional well shallow if it contains only one localized state with binding energy much smaller than the depth of the well. For this to be the case, it is sufficient that the inequality $(mV/\hbar^2)(\Delta z)^2 \ll 1$, where V and Δz are the characteristic depth and dimension of the well, be satisfied. Then the binding energy of the level[1]

$$\hbar^2 k_b^2/2m \sim V(mV/\hbar^2)(\Delta z)^2.$$

If the value of k is not very high (i.e., if $k \ll 1/\Delta z$), then the wave function of the electron changes little over the extension of the potential. This means that the properties of a shallow potential for small k are determined only by the integral of it over all z, i.e., it can be represented in the form of a δ potential. The reflection coefficient R_M then has the well-known form:

$$R_M=\frac{k_{bM}^2}{k^2+k_{bM}^2},\quad k_{bM}=-\frac{m}{\hbar^2}\int_{-\infty}^{\infty}V_M(z)\,dz,\quad k\ll\frac{1}{\Delta z}. \quad (10)$$

It can be seen from the expression (9) that the potentials $V_M(z)$ are shallow for

$$\rho \gg r_s \ln(r_s/a) \quad (mV/\hbar^2 \sim (1/a\rho)\exp(-\rho/r_s),\ \Delta z \sim (\rho r_s)^{1/2}.)$$

The binding energy of the level is then determined by the quantity k_{bM}, which is equal to

$$k_{bM}=\frac{(2\pi)^{1/2}}{a}\left(\frac{r_s}{\rho}\right)^{1/2}\exp\left(-\frac{\rho}{r_s}\right). \quad (11)$$

The formula (10) shows that $R_M=1$ for $k=0$, i.e., the electrons are totally reflected. This is clearly a consequence of the short-range character of the potential: as shown above, in the case of the long-range attractive Coulomb potential electrons with any k have a finite probability of passing through the potential. It is significant that electrons with $k=0$ are reflected in the case of arbitrarily large values of M (corresponding to arbitrarily shallow potentials). This means that, as $k\to 0$, the cross section is determined by ever increasing values of ρ (such that $k_{bM}\sim k$), i.e., it diverges. Substituting the expressions (10) and (11) into the formula (1), and going over to integration, we obtain

$$\sigma=\pi r_s^2\ln^2(1/ka),\quad \ln(1/ka)\gg\ln(r_s/a). \quad (12)$$

The limit of applicability is determined by the fact that, for values of k not satisfying the given inequality, the significant contribution to the cross section is made by the impact parameters $\rho \lesssim r_s\ln(r_s/a)$, for which the corresponding wells are not shallow.

Let us now consider the case $k \gg 1/r_s$. In this case it is convenient to split the contributions to the cross section from the various ρ into two parts: the contributions from the values of $\rho \lesssim a$ and those from $\rho \gg a$. As is easy to verify, the condition (5) for the scattering to be quasiclassical is fulfilled in the first ρ domain for arbitrarily large k and any z. On the basis of the general assumptions of the quasiclassical approach, this implies that the reflection coefficients R_M are exponentially small. We shall not compute these values of R_M here. We only note that the contribution from them to the cross section decreases with increasing k basically in proportion to $\exp(-\alpha k r_s)$ ($\alpha\sim 1$). In the second ρ domain ($\rho \lesssim a$) we can (for $k\gg 1/r_s$) ignore the screening, since in this case the scattering occurs largely in the region $|z|\lesssim a$. The contribution to the scattering from $\rho\lesssim a$ then coincides with the contribution obtained in Sec. 2a.

Thus, for $k\sim\ln(r_s/a)/r_s$, the exponentially decreasing—with increasing k—contribution to the cross section from $\rho\gg a$ is comparable to the contribution from $\rho\lesssim a$, and for higher values of k the dependence $\sigma(k)$ is the same as for the unscreened potential. The general shape of the function $\sigma(k)$ for the screened potential is shown in Fig. 1 by the dashed curve.

b) Repulsive potential. For small k the dominant contribution to the cross section is again made by the scattering by the δ-function potentials, and therefore we obtain, as before, the formulas (10) and (12) (here the quantity $\hbar^2k_b^2/2m$ has the meaning of a shallow-virtual-level energy). On the other hand, for large $k\gg 1/r_s\ln^{1/2}(r_s/a)$, the cross section is determined by the scattering of the electrons with impact parameters $\rho\ll r_s\ln(r_s/a)$ (for higher values of ρ, perturbation theory is applicable, and the contribution to the cross section is negligibly small). In this case the potential barriers are large (with the exception of those corresponding to the narrow range $\rho\lesssim a$) in the sense that $(mV/\hbar^2)(\Delta z)^2\gg 1$. Consequently, as in Sec. 2b, the scattering on them is quasiclassical, i.e., the cross section $\sigma=\pi\rho^2$, where ρ is given by the requirement that the electron energy and the energy of the top of the potential barrier be equal:

$$k^2=(2/a\rho)\exp(-\rho/r_s).$$

It can be seen that the screening is insignificant when $k\gg 1/(ar_s)^{1/2}$. But in the region $k\lesssim 1/(ar_s)^{1/2}$ the $\sigma(k)$ dependence is logarithmically weak. The shape of the function $\sigma(k)$ for the screened potential is schematically shown in Fig. 1 by the dashed curve.

4. The considered effective one-dimensionality of the electron motion in a magnetic field allows us to establish easily the character of the dependence $\sigma(k)$ for $k\to 0$ and an axially-symmetric scattering potential of arbitrary shape. As can be seen from the results obtained in Secs. 2 and 3, the law of variation of $\sigma(k\to 0)$ is determined by the asymptotic form of the potential $V(\mathbf{r})$ as $r\to\infty$. Therefore, when speaking of attractive or repulsive potentials, we shall have in mind below the sign of the potential for $r\to\infty$.

a) Let the potential be attractive. If it decreases with increasing r slower than $1/r^2$, then for sufficiently large ρ the product $(mV/\hbar^2)(\Delta z)^2\gg 1$; here V and Δz

are the characteristic magnitude and range of the corresponding one-dimensional potentials. Then, similarly to the case of the Coulomb potential (Sec. 2a), the one-dimensional wells satisfy the quasiclassicality condition (5) for any z and k. This implies that electrons with large impact parameters and any k pass through the well with probability equal to almost unity without being reflected. In other words, the large M do not contribute to the sum (1), and σ is finite at $k \to 0$.

If the potential $V(\mathbf{r})$ falls off faster than $1/r^2$, then at sufficiently large ρ we have $(mV/\hbar^2)(\Delta z)^2 \ll 1$, i.e., the one-dimensional potentials are shallow. Similarly to the case of the screened potential (Sec. 3a) the corresponding values of R_M are given by the formula (10). As $k \to 0$, all the $R_M \to 1$ and $\sigma \to \infty$. It is noteworthy that the quantity σ diverges as $k \to 0$ in the case of short-range potentials, and converge in the long-range case. In other words, a short-range potential in a magnetic field reflects electrons more strongly than a long-range potential when k is sufficiently small. By computing the values of k_{bM}, (10), we can establish the law governing the divergence of $\sigma(k \to 0)$ for any short-range potential. Thus, if the potential decreases in a power-law fashion, i.e., if $V(r) \propto 1/r^l$, then

$$\sigma(k\to 0) \propto k^{-2/(l-1)}, \quad l>2.$$

In a number of papers (see, for example, Ref. 5) a bounded result is obtained for the cross section for scattering by a short-range potential in a magnetic field in the case when $k \to 0$. This is due to the fact that a δ-function potential is assumed in these papers, i.e., it is assumed that it is possible to set $r_0 \to 0$, $ur_0^3 = \mathrm{const}$ (here r_0 and u are the characteristic dimension and magnitude of the three-dimensional potential). At the same time, no matter how small r_0 is, the inequality $k < k_{bM}$ will be satisfied at sufficiently small k (the k_{bM} for small r_0 are small but finite). In this case electrons with given M will be totally reflected. As k decreases, ever increasing values of M become involved in the scattering, and $\sigma \to \infty$. Thus, Skobov and Bychkov's[5] results are valid only for $k > k_{b1}$. In other words, the idea that the smaller k is, the smaller is the contribution from the large M to the cross section for scattering in a magnetic field is incorrect. But all the k_{bM}, except k_{b0}, tend to zero as $r_0 \to 0$, $ur_0^3 = \mathrm{const}$. Therefore, if r_0 is very small, then the Skobov-Bychkov theory[5] is invalid only for very small values of k.

And let us note here without proof that the disregard in Secs. 2 and 3 of the presence of higher Landau bands is wrong when the electron energy is close to the energy of the quasidiscrete levels in the wells "belonging" to the first Landau band (in a narrow region just below the bottom of this band). This is due to the strong resonant scattering into the states corresponding to these levels. The dependence $\sigma(k)$ in this region has the form of a sparse "ridge" of peaks, whose height $\sim \lambda^2$.

b) Now let the potential $V(\mathbf{r})$ be a repulsive one. Then the quantity σ diverges as $k \to 0$ regardless of its dependence on the asymptotic form of $V(r \to \infty)$. For long-range potentials that decrease more slowly than $1/r^2$, the scattering by the one-dimensional potentials is quasiclassical in the case of large ρ. It is borne in mind that reflection coefficients decrease sharply from unity to a small value when the electron energy becomes greater than the height of the potential barrier (see Sec. 2b). Consequently, it is easy to establish that, if the law of decrease of the potential is a power law, i.e., if $V(\mathbf{r}) \propto 1/r^l$, then the cross section

$$\sigma(k\to 0) \propto k^{-4/l}, \quad l<2.$$

If, on the other hand, the potential is a short-range potential, i.e., if it decreases faster than $1/r^2$, then the law of variation of $\sigma(k \to 0)$ is the same as for an attractive potential (this was apparent in Sec. 3b in the particular case of the screened Coulomb potential).

The above-described method of finding $\sigma(k \to 0)$ can be generalized to the case in which the sign of $V(\mathbf{r})$ depends on the direction of the vector $\mathbf{r}$, and therefore the one-dimensional potential is of alternate sign. It may then turn out that the quantity k_{bM} is not given by the formula (10), and we must take into account the change that occurs in the electron wave function $F_M(z)$ in the region of action of the potential. An example is the scattering of an electron by a hydrogen atom in a strong magnetic field.[2]

The author is grateful to V.I. Perel' for useful advice and for a discussion.

[1] The quasiclassical approach is inapplicable only in this narrow region of k values.

[1] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Nauka, Moscow, 1974 (Eng. Transl., Pergamon Press, Oxford, 1977).
[2] V. I. Perel' and D. G. Polyakov, Zh. Eksp. Teor. Fiz. **81**, 1232 (1981) [Sov. Phys. JETP **54**, 657 (1981)].
[3] H. Hasegawa and R. E. Howard, J. Phys. Chem. Solids **21**, 179 (1961).
[4] P. N. Argyres and E. N. Adams, Phys. Rev. **104**, 900 (1956).
[5] V. G. Skobov, Zh. Eksp. Teor. Fiz. **37**, 1467 (1959) [Sov. Phys. JETP **10**, 1039 (1960)]; Yu. A. Bychkov, Zh. Eksp. Teor. Fiz. **39**, 689 (1960) [Sov. Phys. JETP **12**, 483 (1961)].

Translated by A. K. Agyei